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Part 3. Heavy tail phenomena

CHARACTERIZATION OF TAILS THROUGH HAZARD RATE AND CONVOLUTION CLOSURE PROPERTIES

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CHARACTERIZATION OF TAILS THROUGH HAZARD RATE AND CONVOLUTION CLOSURE PROPERTIES

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Abstract

We use the properties of the Matuszewska indices to show asymptotic inequalities for hazard rates. We discuss the relation between membership in the classes of dominatedly or extended rapidly varying tail distributions and corresponding hazard rate conditions. Convolution closure is established for the class of distributions with extended rapidly varying tails.

Keywords: Subexponentiality; extended rapidly varying tail; dominatedly varying tail; Matuszewska index; hazard rate function

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1. Introduction

In this paper we intend to discuss Pitman's criterion for subexponentiality (see [9, Theorem 2]). Some extensions of previous results about the characterization of distribution classes through their hazard rates appeared as byproducts. The motivation was the need for understanding and calculating the monotonicity condition required in these theorems. The ultimate goal is to substitute the monotonicity property with some limit relation.

Consider the Lebesgue convolution for densities f_1 and f_2 on $[0, \infty)$, given by

$$f_1 \star f_2(x) = \int_0^x f_1(y) f_2(x-y)\,\mathrm{d}y,$$

and the convolution formula for the corresponding distributions, given by

$$\overline{F_1 * F_2}(x) = \overline{F}_2(x) + \int_0^x \overline{F}_1(x-y)\,\mathrm{d}F_2(y),$$

where $\overline{F}(u) = 1 - F(u)$ denotes the right tail of any distribution F.

For $u > 1$, write

$$\overline{F}_\star(u) := \liminf_{x\to\infty} \frac{\overline{F}(ux)}{\overline{F}(x)} \quad \text{and} \quad \overline{F}^\star(u) := \limsup_{x\to\infty} \frac{\overline{F}(ux)}{\overline{F}(x)}.$$

We write $m(x) \sim g(x)$ as $x \to \infty$ for the limit relation $\lim_{x\to\infty} m(x)/g(x) = 1$ and introduce the following classes of distributions F.

1. F is said to belong to the class $\mathcal{ER}$ of distribution functions with extended rapidly varying tails if $\overline{F}^\star(u) < 1$ for some $u > 1$.

2. F is said to belong to the subexponential class $\mathcal{S}$ if $\overline{F^{2*}}(x) \sim 2\overline{F}(x)$.
3. F is said to belong to the class $\mathcal{L}$ of long-tailed distributions if $\overline{F}(x-y) \sim \overline{F}(x)$ for $y \in (-\infty, \infty)$.
4. F is said to belong to the class $\mathcal{D}$ of distribution functions with dominatedly varying tails if $\overline{F}_\star(u) > 0$ for all (or, equivalently, for some) $u > 1$ or, equivalently, $\overline{F}^\star(u) < \infty$ for all (or, equivalently, for some) $0 < u < 1$.

Recall that, for a positve function g on $(0, \infty)$, the upper Matuszewska index γ_g is defined as the infimum of those values α for which there exists a constant C such that, for each $U > 1$, as $x \to \infty$,

$$\frac{g(ux)}{g(x)} \leq C(1+o(1))u^\alpha \quad \text{uniformly in } u \in [1, U],$$

and the lower Matuszewska index δ_g is defined as the supremum of those values β for which, for some $D > 0$ and all $U > 1$, as $x \to \infty$,

$$\frac{g(ux)}{g(x)} \geq D(1+o(1))u^\beta \quad \text{uniformly in } u \in [1, U].$$

The classes $\mathcal{D}$ and $\mathcal{ER}$ are linked to the Matuszewska indices of the tails $\overline{F}$ (see [3]). For any distribution F on $(0, \infty)$ with infinite support, $F \in \mathcal{D}$ if and only if $\gamma_{\overline{F}} < \infty$, and $F \in \mathcal{ER}$ if and only if $\delta_{\overline{F}} > 0$. In what follows, we always assume that F has a positive Lebesgue density f. Then it holds (see [1, Theorem 2.1.5]) that

$$\gamma_f = \inf\left\{-\frac{\log f_\star(u)}{\log u} : u > 1\right\} = -\lim_{u\to\infty} \frac{\log f_\star(u)}{\log u},$$

where $f_\star(u) = \liminf_{x\to\infty} f(ux)/f(x)$, and

$$\delta_f = \sup\left\{-\frac{\log f^\star(u)}{\log u} : u > 1\right\} = -\lim_{u\to\infty} \frac{\log f^\star(u)}{\log u},$$

where $f^\star(u) = \limsup_{x\to\infty} f(ux)/f(x)$. Using the Matuszewska indices, we can establish Potter-type inequalities for f; see [1, Proposition 2.2.1]. For example, if $\gamma_f < \infty$ then, for every $\gamma > \gamma_f$, there exist constants $C'(\gamma)$ and $x_0' = x_0'(\gamma)$ such that

$$\frac{f(y)}{f(x)} \geq C'(\gamma)\left(\frac{y}{x}\right)^{-\gamma}, \qquad y \geq x \geq x_0'. \tag{1.1}$$

If $\delta_f > -\infty$ then, for every $\delta < \delta_f$, there exist constants $C(\delta)$ and $x_0 = x_0(\delta)$ such that

$$\frac{f(y)}{f(x)} \leq C(\delta)\left(\frac{y}{x}\right)^{-\delta}, \qquad y \geq x \geq x_0. \tag{1.2}$$

In what follows, we say that the distributions F_1 and F_2 are max-sum equivalent if

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}_1(x) + \overline{F}_2(x)} = 1.$$

2. The class of distributions with extended rapidly varying tails

We say that the distribution F on $(0, \infty)$ is heavy tailed if $\int_0^\infty \mathrm{e}^{sx}\, \mathrm{d}F(x) = \infty$ for all $s > 0$ and light tailed otherwise. The class $\mathcal{ER}$ contains both light- and heavy-tailed distributions. For example, the exponential and Pareto distributions are members of $\mathcal{ER}$. Moreover, the class $\mathcal{ER}$ is not closed under max-sum equivalence. For example, for exponential $F_i = F$, exponential with parameter λ, we have $\lim_{x\to\infty} \overline{F * F}(x)/(2\overline{F}(x)) = \infty$.

In what follows, we will need the hazard rate $h(x) = f(x)/\overline{F}(x)$ for any distribution F on $(0, \infty)$ with positive density f. We also write

$$M_1 = \liminf_{x\to\infty} xh(x) \quad \text{and} \quad M_2 = \limsup_{x\to\infty} xh(x).$$

Whenever we consider a sequence F_i, $i = 1, 2, \ldots$, of such distributions, we will use the corresponding symbols h_i, M_1^i, and M_2^i.

We say that a density has bounded increase if $\delta_f > -\infty$; see [1, p. 71]. Most of the densities of interest in statistics and probability theory satisfy this condition, e.g. the gamma and Weibull densities.

Under the assumption of an eventually nonincreasing density (such that $f(y) \le f(x)$ for all $x \ge y \ge x_0$), the following equivalences were established.

1. $F \in \mathcal{ER}$ if and only if $M_1 > 0$; see [8, Proposition 6].
2. $F \in \mathcal{D}$ if and only if $F \in \mathcal{D} \cap \mathcal{L}$ if and only if $M_2 < \infty$; see [7, Corollary 3.4].

We generalize these results by substituting the condition of an eventually nonincreasing density f by the assumption that f has bounded increase. This allows us to avoid the verification of the monotonicity property of f, and restricts the calculation to that of δ_f through $f^\star(u)$.

Theorem 2.1. *Assume that F is a distribution supported on $(0, \infty)$ with positive Lebesgue density f such that f has bounded increase. Then $F \in \mathcal{ER}$ if and only if $M_1 > 0$.*

Proof. By assumption, we have $\delta_f > -\infty$ and, therefore, inequality (1.2) holds for $\delta < \delta_f$. Let us start with the converse assertion, i.e. we assume that $M_1 > 0$. We write, for any $u > 1$,

$$\frac{\overline{F}(ux)}{\overline{F}(x)} = \overline{F}(ux) \Bigg/ \left(\int_x^{ux} f(t)\, \mathrm{d}t + \overline{F}(ux)\right). \tag{2.1}$$

Now relation (1.2) implies that

$$\frac{f(ux)}{f(t)} \le C(\delta)\left(\frac{ux}{t}\right)^{-\delta}, \qquad ux \ge t > x \ge x_0.$$

Then integration yields

$$\int_x^{ux} f(t)\, \mathrm{d}t \ge \frac{f(ux)}{C(\delta)}(ux)^\delta \int_x^{ux} t^{-\delta}\, \mathrm{d}t =: K(\delta, u)xf(ux). \tag{2.2}$$

We substitute this lower bound into (2.1) to obtain

$$\frac{\overline{F}(ux)}{\overline{F}(x)} \le \frac{\overline{F}(ux)}{K(\delta, u)xf(ux) + \overline{F}(ux)} = \frac{1}{K(\delta, u)xh(ux) + 1}.$$

Since $M_1 > 0$, the latter relation implies that $F \in \mathcal{ER}$.

Now we show the direct implication. From (2.1) we obtain

$$\frac{\overline{F}(ux)}{\overline{F}(x)} = 1 - \int_x^{ux} \frac{f(t)}{\overline{F}(x)}\,\mathrm{d}t, \tag{2.3}$$

and from (1.2) we have

$$\frac{f(t)}{f(x)} \leq C(\delta)\left(\frac{t}{x}\right)^{-\delta}, \qquad t \geq x \geq x_0.$$

An approach similar to that used in inequality (2.2) yields, for some constant $K'(\delta, u)$,

$$\int_x^{ux} f(t)\,\mathrm{d}t \leq C(\delta) f(x) x^{\delta} \int_x^{ux} t^{-\delta}\,\mathrm{d}t =: K'(\delta, u) x f(x). \tag{2.4}$$

Hence, from (2.3) and (2.4), it follows that

$$\frac{\overline{F}(ux)}{\overline{F}(x)} \geq 1 - K'(\delta, u) x h(x).$$

Since $F \in \mathscr{E}\mathscr{R}$, there exists a $u > 1$ such that

$$M_1 = \liminf_{x\to\infty} x h(x) \geq (K'(\delta, u))^{-1} \liminf_{x\to\infty}\left(1 - \frac{\overline{F}(ux)}{\overline{F}(x)}\right) > 0.$$

This completes the proof.

We say that a positive Lebesgue density f is extended rapidly varying if $\delta_f > 1$. In the following result we prove that this property and the bounded increase of f imply that $F \in \mathscr{E}\mathscr{R}$. We find an asymptotic lower bound for the hazard rate using the lower Matuszewska index.

Proposition 2.1. *If f has bounded increase with $\delta_f > 1$ then $F \in \mathscr{E}\mathscr{R}$ and, for any $\delta \in (1, \delta_f)$, there exist positive constants x_0 and $C(\delta)$, defined in (1.2), such that, for all $x \geq x_0$,*

$$x h(x) \geq \frac{\delta - 1}{C(\delta)}. \tag{2.5}$$

Proof. Inequality (1.2) is implied by the assumption that $\delta_f > 1$ for $\delta < \delta_f$. Furthermore, we integrate (1.2) for $\delta \in (1, \delta_f)$:

$$(h(x))^{-1} = \int_x^{\infty} \frac{f(y)}{f(x)}\,\mathrm{d}y \leq C(\delta) x^{\delta} \int_x^{\infty} y^{-\delta}\,\mathrm{d}y = \frac{C(\delta) x}{\delta - 1}.$$

This proves inequality (2.5). The latter relation immediately implies that $M_1 > 0$ and, therefore, by Theorem 2.1, $F \in \mathscr{E}\mathscr{R}$.

We examine the convolution closure of distributions with extended rapidly varying distributions. We write $h_{F_1 * F_2}$ for the hazard rate of $F_1 * F_2$ and

$$M_1^{(1,2)} := \liminf_{x\to\infty} x h_{F_1 * F_2}(x).$$

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We need the fact that if $F_i \in \mathcal{E}\mathcal{R}$ then, for every $0 < \delta < \delta_{\overline{F}_i}$, there exist constants $x_0^i = x_0^i(\delta)$ and $C_i(\delta)$ such that the following Potter-type inequality holds (see [3]):

$$\frac{\overline{F}_i(x)}{\overline{F}_i(y)} \le C_i(\delta)\left(\frac{x}{y}\right)^{-\delta}, \qquad x \ge y \ge x_0^i,\ i = 1, 2.$$

Choosing $y = x_0^i$, the latter relation implies that there exist constants $\Delta_i(\delta)$ such that

$$\overline{F}_i(x) \le \Delta_i(\delta) x^{-\delta}, \qquad x \ge x_0^i,\ i = 1, 2. \tag{2.6}$$

Theorem 2.2. *Assume that $F_1, F_2 \in \mathcal{E}\mathcal{R}$ with positive Lebesgue densities on $(0, \infty)$ and that the following conditions hold.*

1. *The density f_1 has bounded increase with $\delta_{f_1} > 0$.*
2. *$\delta_{\overline{F}_1} < \delta_{\overline{F}_2}$ and $\liminf_{x\to\infty} x^\delta \overline{F}_1(x) > 0$ for some $\delta \in [\delta_{\overline{F}_1}, \delta_{\overline{F}_2})$.*

Then $F_1 * F_2 \in \mathcal{E}\mathcal{R}$.

Proof. We start by proving that $M_1^{(1,2)} > 0$. We have

$$x h_{F_1 * F_2}(x) = \frac{x f_1(x)}{\overline{F}_1(x)} \frac{\overline{F}_1(x)}{\overline{F_1 * F_2}(x)} \frac{f_1 \star f_2(x)}{f_1(x)}.$$

By assumption 1, inequality (1.2) applies for $\delta_0 < \delta_{f_1}$. Therefore, for $x \ge x_0$,

$$\begin{aligned} \frac{f_1 \star f_2(x)}{f_1(x)} &= \int_0^x \frac{f_1(y)}{f_1(x)} f_2(x-y)\,\mathrm{d}y \\ &\ge \int_{x_0}^x \frac{1}{C(\delta_0)}\left(\frac{y}{x}\right)^{-\delta_0} f_2(x-y)\,\mathrm{d}y \\ &\ge \frac{F_2(x - x_0)}{C(\delta_0)}. \end{aligned}$$

Now, from (2.6), for every $\delta_i \in (0, \delta_{\overline{F}_i})$, $i = 1, 2$, sufficiently large x, and some constant $\Delta > 0$,

$$\begin{aligned} \overline{F_1 * F_2}(x) &\le \overline{F}_1\left(\frac{x}{2}\right) + \overline{F}_2\left(\frac{x}{2}\right) \\ &\le \Delta_1(\delta_1)\left(\frac{x}{2}\right)^{-\delta_1} + \Delta_2(\delta_2)\left(\frac{x}{2}\right)^{-\delta_2} \\ &\le \Delta(x^{-\delta_1} + x^{-\delta_2}). \end{aligned}$$

We conclude that there exist $\delta_2 \in [\delta_{\overline{F}_1}, \delta_{\overline{F}_2})$ such that

$$M_1^{(1,2)} \ge \liminf_{x\to\infty} \frac{1}{\Delta C(\delta_0)} \frac{x f_1(x)}{\overline{F}_1(x)} x^{\delta_2} \overline{F}_1(x) \ge \frac{M_1^1}{\Delta C(\delta_0)} \liminf_{x\to\infty} x^{\delta_2} \overline{F}_1(x) > 0.$$

In the last step we used Theorem 2.1 for $M_1^1 > 0$ and assumption 2. Another application of Theorem 2.1 yields the result.

To verify that the two conditions of Theorem 2.2, mentioned above, do not contradict each other, we consider two Pareto distributions with tails $\overline{F}_i(x) = x^{-a_i}$, $i = 1, 2$ and $x \geq 1$. Choose $a_1 = 2$ and $a_2 = 3$. Then it is easy to see that $\delta_{\overline{F}_i} = a_i$, $i = 1, 2$. Therefore, $F_i \in \mathcal{ER}$ and $f_i^\star(u) = u^{-(a_i+1)}$. Hence, condition 1 of Theorem 2.2 holds with $\delta_{f_1} = 3$ and f_1 of bounded increase, and condition 2 of Theorem 2.2 is satisfied for $\delta = 2$. We conclude that the conditions of the theorem hold. The tail of the Pareto distribution F_i belongs to the class $\mathcal{R}_{-a_i}$ of regularly varying functions with index $-a_i$, i.e. $\overline{F_i}(ux) \sim u^{-a_i}\overline{F_i}(x)$ for each $u > 0$. From [5, Lemma 1.3.1] we know that if $\overline{F_i} \in \mathcal{R}_{-a_i}$, $i = 1, 2$, then $\overline{F_1 * F_2} \in \mathcal{R}_{-\min(a_1,a_2)}$. This fact and the definition of regular variation immediately imply that $F_1 * F_2 \in \mathcal{ER}$.

3. The case of subexponential distributions

In this section we present some results on the characterization of the classes $\mathcal{S}$ and $\mathcal{D} \cap \mathcal{L}$ through hazard rates. In the following result we provide an inequality for the hazard rate which is useful for characterizing membership in the class $\mathcal{D} \cap \mathcal{L}$.

Proposition 3.1. *Assume that F has a positive Lebesgue density on $(0, \infty)$ and that $\gamma_f < \infty$. Then $F \in \mathcal{D} \cap \mathcal{L}$ and, for any $\gamma > \gamma_f$, there exist positive constants x_0' and $C'(\gamma)$, defined in (1.1), such that, for all $x \geq x_0'$ and $\lambda > 1$,*

$$xh(x) \leq C'(\gamma)V(\lambda, \gamma), \tag{3.1}$$

where

$$V(\lambda, \gamma) = \begin{cases} \dfrac{\lambda^{-\gamma+1} - 1}{-\gamma + 1} & \text{if } \gamma \neq 1, \\ \log \lambda & \text{if } \gamma = 1. \end{cases}$$

Proof. Since $\gamma_f < \infty$, (1.1) yields

$$\int_x^\infty \frac{f(y)}{f(x)}\,\mathrm{d}y \geq \int_x^{\lambda x} \frac{f(y)}{f(x)}\,\mathrm{d}y \geq C'(\gamma)x^\gamma \int_x^{\lambda x} y^{-\gamma}\,\mathrm{d}y.$$

Then (3.1) holds, $M_2 < \infty$, and from [7, Theorem 3.3] we obtain $F \in \mathcal{D} \cap \mathcal{L}$.

In the next theorem we generalize the statement from [7, Corollary 3.4] by substituting the condition of an eventually nonincreasing density f with the assumption that f has bounded increase. This allows us to avoid the verification of the monotonicity property of f, and restricts the calculation to that of δ_f through $f^\star(u)$.

Theorem 3.1. *Assume that F is supported on $(0, \infty)$ with a positive Lebesgue density f which has bounded increase. Then the following statements are equivalent:*

1. *$F \in \mathcal{D}$,*
2. *$F \in \mathcal{D} \cap \mathcal{L}$, and*
3. *$M_2 < \infty$.*

Proof. We first prove that statement 1 implies statement 3. Observe that

$$\frac{\overline{F}(x/2)}{\overline{F}(x)} = 1 + \int_{x/2}^x \frac{f(y)}{\overline{F}(x)}\,\mathrm{d}y. \tag{3.2}$$

Since f has bounded increase, (1.2) applies for $x \geq y \geq x/2$ and sufficiently large x. Hence, there exists a constant $\Lambda(\delta)$ such that, for large x,

$$\int_{x/2}^{x} f(y)\,\mathrm{d}y \geq \frac{x^{\delta} f(x)}{C(\delta)} \int_{x/2}^{x} y^{-\delta}\,\mathrm{d}y = x f(x)\Lambda(\delta). \tag{3.3}$$

Inequalities (3.2) and (3.3) imply that

$$x h(x) \leq [\Lambda(\delta)]^{-1}\bigg(\frac{\overline{F}(x/2)}{\overline{F}(x)} - 1\bigg).$$

Now, from the assumption that $F \in \mathcal{D}$ we obtain $M_2 < \infty$.

That statement 3 implies statement 2 follows from [7, Theorem 3.3], and it is trivial to show that statement 2 implies statement 1.

In [9, Theorem 2] necessary and sufficient conditions for membership in $\mathcal{S}$ were presented. In [7, Theorem 3.6] a corresponnding result for the important subclass $\mathcal{S}^*$ of $\mathcal{S}$ was given; see also [7, Corollary 3.8]. The previously mentioned results require that the hazard rate be eventually monotone (such that $h(y) \leq h(x)$ for all $y \geq x \geq x_0$). However, a verification of this monotonicity condition is in general not straightforward. In the next result we prove the statement of [9, Theorem 2] under the assumption that $\delta_h > 0$, which might be checked more easily.

Recall the notion of a hazard function $H(x) := -\ln \overline{F}(x)$ with the convention that $H(\infty) = \infty$. An application of [1, Proposition 2.2.1] yields a Potter-type inequality for $g(x) = (h(x))^{-1}$: if $\delta_h > 0$ then, for $0 < \delta < \delta_h$, there exist constants $C(\delta)$ and x_0 such that

$$\frac{h(y)}{h(x)} \leq C(\delta)\bigg(\frac{y}{x}\bigg)^{-\delta}, \qquad y \geq x \geq x_0. \tag{3.4}$$

If $\delta_h > 0$, we say that the hazard rate h has positive decrease; see [1, p. 71].

Theorem 3.2. *Let F be a distribution on $(0, \infty)$ with positive Lebesgue density f. Assume that the hazard rate h has positive decrease. Then $F \in \mathcal{S}$ if and only if*

$$\lim_{x\to\infty} \int_0^x \exp\{\kappa y h(x) - H(y)\} h(y)\,\mathrm{d}y = 1 \quad \textit{for every } \kappa > 0. \tag{3.5}$$

Proof. If F has a positive Lebesgue density, the hazard function H is differentiable and we obtain

$$\begin{aligned}
\frac{\overline{F^{2*}}(x)}{\overline{F}(x)} - 1 &= \int_0^x \exp\{H(x) - H(x-y) - H(y)\} h(y)\,\mathrm{d}y \\
&= \int_0^{x/2} \exp\{H(x) - H(x-y) - H(y)\} h(y)\,\mathrm{d}y \\
&\quad + \int_0^{x/2} \exp\{H(x) - H(x-y) - H(y)\} h(x-y)\,\mathrm{d}y \\
&=: I_1(x) + I_2(x),
\end{aligned}$$

We start by showing the converse implication, i.e. (3.5) implies that $I_1(x) \to 1$ and $I_2(x) \to 0$, and, hence, F is subexponential. For $y \leq x/2$, there exists $\xi \in (x-y, x)$ such that $y h(\xi) = H(x) - H(x-y)$. Then

$$x > \xi > x - y \geq \tfrac{1}{2}x \geq y. \tag{3.6}$$

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An application of (3.4) yields, for large x and $\delta < \delta_h$,

$$\max\left(\frac{h(\xi)}{h(x/2)}, \frac{h(x)}{h(\xi)}\right) \le C(\delta). \tag{3.7}$$

Hence, for any x and y satisfying (3.6),

$$\frac{yh(x)}{C(\delta)} \le H(x) - H(x-y) \le C(\delta) y h\left(\frac{x}{2}\right). \tag{3.8}$$

Since $H(x) - H(x-y) \ge 0$, we have the trivial bound $I_1(x) \ge F(x/2)$, and from the right-most inequality in (3.8) we conclude that

$$F\left(\frac{x}{2}\right) \le I_1(x) \le \int_0^{x/2} \exp\left\{C(\delta) y h\left(\frac{x}{2}\right) - H(y)\right\} h(y)\,\mathrm{d}y. \tag{3.9}$$

Together with (3.5), this implies the desired relation $I_1(x) \to 1$.

It remains to show that $I_2(x) \to 0$. From (3.8) we obtain

$$I_2(x) \le \left(\int_0^{x_0} + \int_{x_0}^{x/2}\right) \exp\left\{C(\delta) y h\left(\frac{x}{2}\right) - H(y)\right\} h(x-y)\,\mathrm{d}y.$$

The first integral converges to 0 as $x \to \infty$ since $h(x) \to 0$. By (3.4) and (3.6), there exists some x_0 such that

$$\frac{h(x-y)}{h(y)} \le C(\delta), \qquad x_0 \le y \le x/2,$$

and, therefore, up to a constant multiple, the second integral is bounded by the right-hand expression in (3.9), which converges to 1. Moreover, the integrand on the right-hand side of (3.9) converges for every y as $x \to \infty$. The integrand in the second integral above converges to 0 for every y. An application of Pratt's lemma [10, Theorem 1] shows that the second integral converges to 0 as $x \to \infty$. Hence, $I_2(x) \to 0$ and the converse implication of the result is proved.

For the direct part, assuming that F is subexponential, we obtain, from (3.6) and (3.7),

$$\frac{\overline{F^{2*}}(x)}{\overline{F}(x)} - 1 \ge I_1(x) \ge \int_0^{x/2} \exp\{-H(y)\} h(y)\,\mathrm{d}y = \int_0^{x/2} f(y)\,\mathrm{d}y.$$

Since the left- and right-hand sides converge to 1 as $x \to \infty$, the proof of (3.5) is complete.

We introduce the class $\mathcal{A} = \mathcal{S} \cap \mathcal{ER}$ of heavy-tailed distributions on $(0, \infty)$. In what follows, we will frequently use the relation $\mathcal{D} \cap \mathcal{A} = \mathcal{D} \cap \mathcal{L} \cap \mathcal{ER}$, which is a consequence of [6, Theorem 1] and the definition of the class $\mathcal{A}$. Also, note that the inclusion $\mathcal{D} \cap \mathcal{A} \subset \mathcal{D} \cap \mathcal{L}$ is strict since the distributions with slowly varying tails are not contained in $\mathcal{ER}$, but belong to $\mathcal{D} \cap \mathcal{L}$. Next we show that $\mathcal{D} \cap \mathcal{A}$ is closed under convolution.

Proposition 3.2. *Assume that $F_i \in \mathcal{D} \cap \mathcal{A}$, $i = 1, 2$. Then $F_1 * F_2 \in \mathcal{D} \cap \mathcal{A}$ and*

$$\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x) \quad \textit{as } x \to \infty. \tag{3.10}$$

Proof. Relation (3.10) follows from $F_i \in \mathcal{D} \cap \mathcal{A} \subset \mathcal{D} \cap \mathcal{L}$, $i = 1, 2$, and [2, Theorem 2.1]. Furthermore, from [4, Proposition 2] (or [2, Theorem 2.1]) we conclude that $F_1 * F_2 \in \mathcal{D} \cap \mathcal{L}$. Let us use relation (3.10) to show that $F_1 * F_2 \in \mathcal{ER}$. Then, for some $u > 1$, we have

$$\limsup_{x\to\infty} \frac{\overline{F_1 * F_2}(ux)}{\overline{F_1 * F_2}(x)} = \limsup_{x\to\infty} \frac{\overline{F}_1(ux) + \overline{F}_2(ux)}{\overline{F}_1(x) + \overline{F}_2(x)} \le \max\{\overline{F}_1^\star(u), \overline{F}_2^\star(u)\}.$$

Taking into account the fact that $F_i \in \mathcal{ER}$, $i = 1, 2$, we obtain the result. This completes the proof.

In the next statement a characterization of the class $\mathcal{D} \cap \mathcal{A}$ with respect to the hazard rate and the limits $\overline{F}_\star(u)$ and $\overline{F}^\star(u)$ for all $u > 1$ is presented, and in this way a generalization of [8, Theorems 3.3 and 3.7] is provided. The generalization is achieved by substituting the condition of an eventually nonincreasing density f by the assumption that f is of bounded increase. This allows us to avoid the verification of the monotonicity property of f, and restricts the calculation to that of δ_f through $f^\star(u)$.

Corollary 3.1. *Assume that F is a distribution supported on $(0, \infty)$ with positive Lebesgue density f such that f has bounded increase. Then $F \in \mathcal{D} \cap \mathcal{A}$ if and only if one of the following statements holds:*

1. $0 < M_1 \le M_2 < \infty$,
2. $0 < \overline{F}_\star(u) \le \overline{F}^\star(u) < 1$.

Proof. 1. Let us begin with the direct implication. From the assumption that $F \in \mathcal{D}$ and Theorem 3.1, we obtain $M_2 < \infty$. From the assumption that $F \in \mathcal{ER}$ and Theorem 2.1, we find that $M_1 > 0$. Next, for the inverse part, we directly apply Theorems 2.1 and 3.1.

2. The condition $F \in \mathcal{D} \cap \mathcal{A}$ is equivalent to $F \in \mathcal{D} \cap \mathcal{ER}$. Because of $F \in \mathcal{D}$ from Theorem 3.1, we obtain $F \in \mathcal{D} \cap \mathcal{L}$. Hence, $F \in \mathcal{D} \cap \mathcal{A}$. This completes the proof.

Corollary 3.2. *Let F be a distribution on $(0, \infty)$ with positive Lebesgue density f. Assume that the hazard rate h has positive decrease. Then $F \in \mathcal{A}$ if and only if $M_1 > 0$ and*

$$\lim_{x\to\infty} \int_0^x \exp\{\kappa y h(x) - H(y)\} h(y)\, \mathrm{d}y = 1 \quad \textit{for every } \kappa > 0.$$

Proof. From the decreasing property of the tail $\overline{F}$ we obtain

$$\frac{f(ux)}{f(x)} \le \frac{h(ux)}{h(x)} \quad \text{for every } u > 1.$$

Hence, $\delta_f \ge \delta_h > 0$. From the last inequality and Theorems 2.1 and 3.2, we conclude the result. This completes the proof.

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