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HANKEL MEASURES FOR FOCK SPACE

ERMIN WANG

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Abstract

Inspired by Xiao's work on Hankel measures for Hardy and Bergman spaces ['Pseudo-Carleson measures for weighted Bergman spaces'. *Michigan Math. J.* **47** (2000), 447–452], we introduce Hankel measures for Fock space F^p_α. Given $p \geq 1$, we obtain several equivalent descriptions for such measures on F^p_α.

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1. Introduction

Let $\mathbb{C}$ be the complex plane and let $H(\mathbb{C})$ be the family of all holomorphic functions on $\mathbb{C}$. Given any $\alpha > 0$, set

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}\, dA(z),$$

where dA is the Euclidean area measure on $\mathbb{C}$. For $1 \leq p < \infty$, the Fock space F^p_α consists of those functions $f \in H(\mathbb{C})$ for which

$$\|f\|^p_{p,\alpha} = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)e^{-\alpha|z|^2/2}|^p\, dA(z) < \infty.$$

Similarly, for $p = \infty$, we use the notation F^∞_α to denote the space of holomorphic functions f on $\mathbb{C}$ such that

$$\|f\|_{\infty,\alpha} = \operatorname{ess\,sup}\{|f(z)|e^{-\alpha|z|^2/2} : z \in \mathbb{C}\} < \infty.$$

The Fock space F^2_α is a Hilbert space with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)}\, d\lambda_\alpha(z).$$

For any fixed $w \in \mathbb{C}$, the mapping $f \mapsto f(w)$ is a bounded linear functional on F^2_α. By the Riesz representation theorem in functional analysis, there exists a unique function $K_w \in F^2_\alpha$ such that $f(w) = \langle f, K_w\rangle_\alpha$ for all $f \in F^2_\alpha$. The function $K_\alpha(z, w) = K_w(z)$ is

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called the reproducing kernel of F^2_α. It is well known that $K_\alpha(z,w) = e^{\alpha z\overline{w}}$ (see, for example, [8]). Let $k_w(z) = e^{\alpha z\overline{w}-\alpha|w|^2/2}$ be the normalised reproducing kernel at w. The orthogonal projection $P_\alpha : L^2(\mathbb{C}, d\lambda_\alpha) \to F^2_\alpha$ can be represented as

$$P_\alpha f(z) = \int_{\mathbb{C}} f(w)K_\alpha(z,w)\, d\lambda_\alpha(w).$$

With this expression, P_α can be extended to a bounded linear operator from $L^p(\mathbb{C}, d\lambda_\alpha)$ to F^p_α for $1 \le p \le \infty$, and $P_\alpha f = f$ for $f \in F^p_\alpha$. (See [4, 8] for more details of the theory of Fock spaces.)

A positive Borel measure μ on $\mathbb{C}$ is called a Carleson measure for F^p_α if there exists a positive constant C such that

$$\int_{\mathbb{C}} |f(z)e^{-\alpha|z|^2}|^p\, d\mu(z) \le C\|f\|^p_{p,\alpha}$$

for all $f \in F^p_\alpha$. Carleson measure plays a fundamental role in the theory of Toeplitz operators and interpolation (see, for example, [3, 8]).

The concept of Hankel measures was first proposed by Xiao in [6, 7]. Motivated by the study of Hankel operators on the weighted Bergman space and Hardy space, as well as the parallel to the notion of Carleson measures, Xiao introduced the notion of Hankel measures. More precisely, a complex Borel measure μ on the open unit disk $\mathbb{D}$ is called a Hankel measure (or pseudo-Carleson measure) for the weighted Bergman space A^2_α if there exists a positive constant C such that

$$\left|\int_{\mathbb{D}} f(z)^2\, d\mu(z)\right| \le C\|f\|^2_{A^2_\alpha}$$

for all $f \in A^2_\alpha$. In [6], Xiao characterised the properties of the measure μ in order that μ be a Hankel measure for A^2_α. The similar problem for Hardy space was also considered in [7]. Later, Hankel measure was studied in some other contexts (see [1, 2]).

Inspired by Xiao's work, we extend the theory of Hankel measures to Fock space. A complex Borel measure μ on $\mathbb{C}$ is called a Hankel measure for F^p_α ($p \ge 1$) if there exists a positive constant C such that

$$\left|\int_{\mathbb{C}} f(z)^p e^{-p\alpha|z|^2/2}\, d\mu(z)\right| \le C\|f\|^p_{p,\alpha}$$

for all $f \in F^p_\alpha$. It is clear that every Carleson measure for F^p_α must be a Hankel measure for F^p_α, but not conversely. The main result of this paper characterises the Hankel measure for F^p_α.

THEOREM 1.1 (Main Theorem). *Let $\alpha > 0, p \ge 1$ and let μ be a complex Borel measure on $\mathbb{C}$. Then the following statements are equivalent:*

(1) *μ is a Hankel measure for F^p_α;*
(2) *for any $f \in F^1_\alpha$,*

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$$\left|\int_{\mathbb{C}} f(z)e^{-\alpha|z|^2/2}\,d\mu(z)\right| \le C\|f\|_{1,\alpha};$$

(3) $\sup_{w\in\mathbb{C}} |\int_{\mathbb{C}} e^{p\alpha z\overline{w}-p\alpha(|z|^2+|w|^2)/2}\,d\mu(z)| < \infty$;
(4) $P\overline{\mu}(z) = \int_{\mathbb{C}} e^{\alpha z\overline{w}-\alpha|z|^2/2}\,d\overline{\mu}(z)$ *defines a function in* F^∞_α;
(5) $K_\mu : f \mapsto \int_{\mathbb{C}} e^{\alpha zw} f(w)\,d\mu(z)$ *exists as a bounded operator on* F^p_α *for each* $p \ge 1$.

Throughout this paper, we use C to denote positive constants whose value may change from line to line, but do not depend on the functions being considered. We call two quantities A and B equivalent, denoted by $A \simeq B$, if there exists some C such that $C^{-1}A \le B \le CA$. Given some exponent $s \ge 1$, we always use s' to denote the conjugate of s, defined by $s^{-1} + s'^{-1} = 1$.

2. Proof of the main theorem

We begin by stating some known results which are used in the proof of the main theorem.

Given some $a \in \mathbb{C}$ and $r > 0$, write $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ for the Euclidean ball centred at a with radius r. A sequence $\{a_k\}$ in $\mathbb{C}$ is called an r-lattice if the following conditions are satisfied:

(1) $\bigcup_{k=1}^\infty D(a_k, r) = \mathbb{C}$;
(2) $\{D(a_k, r/4)\}_{k=1}^\infty$ are mutually disjoint.

Given $r > 0$, it is easy to pick $a_k \in \mathbb{C}$ such that $\{a_k\}$ is an r-lattice.

The atomic decomposition for Fock spaces is a powerful result in the theory.

THEOREM 2.1 [4]. *Let* $1 \le p \le \infty$. *There exists a positive constant* r_0 *such that for any* r *with* $0 < r < r_0$, *the space* F^p_α *consists exactly of the functions*

$$f(z) = \sum_{k=1}^\infty \lambda_k e^{\alpha z\overline{a}_k-\alpha|a_k|^2/2}, \tag{2.1}$$

where $\{\lambda_k\} \in l^p$ *and* $\{a_k\}$ *is an r-lattice. Moreover, there exists a positive constant C (independent of f) such that*

$$C^{-1}\|f\|_{p,\alpha} \le \inf \|\{\lambda_k\}\|_{l^p} \le C\|f\|_{p,\alpha}$$

for all $f \in F^p_\alpha$, *where the infimum is taken over all sequences* $\{\lambda_k\}$ *that give rise to the decomposition in (2.1).*

We also need the duality theorem.

THEOREM 2.2 [4]. *Suppose* $\alpha > 0, 1 \le p < \infty$. *Then the dual space of* F^p_α *can be identified with* $F^{p'}_\alpha$ *under the integral pairing*

$$\langle f, g\rangle_\alpha = \lim_{R\to\infty} \int_{|z|<R} f(z)\overline{g(z)}e^{-\alpha|z|^2}\,dA(z).$$

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PROOF OF THE MAIN THEOREM. We first show the equivalence of (1), (2) and (3).

(2)⇒(1). Suppose (2) is valid. Replacing $f(z)e^{-\alpha|z|^2/2}$ by $(f(z)e^{-\alpha|z|^2/2})^p$ gives

$$\left|\int_{\mathbb{C}} f(z)^p e^{-p\alpha|z|^2/2}\,d\mu(z)\right| \le C\int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2/2}\,dA(z) = C\|f\|_{p,\alpha}^p.$$

Thus, μ is a Hankel measure for F_α^p.

(1)⇒(3). Suppose (1) is true. Let $f(z) = k_w(z)$. Since $\|k_w\|_{p,\alpha} = 1$ (see [8, Lemma 2.33]), we obtain

$$\left|\int_{\mathbb{C}} k_w(z)^p e^{-p\alpha|z|^2/2}\,d\mu(z)\right| \le C\|k_w\|_{p,\alpha}^p = C.$$

From this, the statement (3) follows.

(3)⇒(2). For $f \in F_\alpha^1$, fix $0 < r < r_0$ where r_0 is as in Theorem 2.1, and let $\{a_k\}$ be an r-lattice. By Theorem 2.1, for any $\{\lambda_k\} \in l^1$, the function f admits the decomposition

$$f(z) = \sum_{k=1}^{\infty} \lambda_k e^{\alpha z\bar{a}_k - \alpha|a_k|^2/2}$$

with $\|\{\lambda_k\}\|_{l^1} \le C\|f\|_{1,\alpha}$. Therefore,

$$\begin{aligned}\left|\int_{\mathbb{C}} f(z)e^{-\alpha|z|^2/2}\,d\mu(z)\right| &= \left|\int_{\mathbb{C}} \sum_{k=1}^{\infty} \lambda_k e^{\alpha z\bar{a}_k - \alpha(|a_k|^2+|z|^2)/2}\,d\mu(z)\right| \\ &\le \sum_{k=1}^{\infty} |\lambda_k| \left|\int_{\mathbb{C}} e^{\alpha z\bar{a}_k - \alpha(|a_k|^2+|z|^2)/2}\,d\mu(z)\right| \\ &\le \|\{\lambda_k\}\|_{l^1}\cdot \sup_j \left|\int_{\mathbb{C}} e^{\alpha z\bar{a}_k - \alpha(|a_k|^2+|z|^2)/2}\,d\mu(z)\right| \\ &\le C\|f\|_{1,\alpha},\end{aligned}$$

which gives (2).

Next we prove the equivalence of (2), (4) and (5).

(2)⇔(4). The reproducing formula of F_α^1 implies

$$\begin{aligned}\int_{\mathbb{C}} f(z)e^{-\frac{\alpha}{2}|z|^2}\,d\mu(z) &= \int_{\mathbb{C}}\left(\int_{\mathbb{C}} f(w)e^{\alpha z\bar{w}}\,d\lambda_\alpha(w)\right)e^{-\alpha|z|^2/2}\,d\mu(z) \\ &= \int_{\mathbb{C}} f(w)\left(\int_{\mathbb{C}} e^{\alpha z\bar{w} - \alpha|z|^2/2}\,d\mu(z)\right)d\lambda_\alpha(w) \\ &= \int_{\mathbb{C}} f(w)\overline{P\overline{\mu}(w)}\,d\lambda_\alpha(w) \\ &= \langle f, P\overline{\mu}\rangle_\alpha.\end{aligned}$$

Since $(F_\alpha^1)^* \simeq F_\alpha^\infty$ under the pairing $\langle\cdot,\cdot\rangle_\alpha$, we obtain the equivalence between (2) and (4).

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(4)⇔(5). Following [5, 6], we call K_μ a (small) Hankel operator associated with the symbol μ. For $p \geq 1$, we claim that for $f \in F^p_\alpha, g \in F^{p'}_\alpha$,

$$\langle K_\mu, g\rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(\overline{z})}\, d\mu(z).$$

In fact, by the reproducing formula of F^1_α,

$$\begin{aligned}\langle K_\mu, g\rangle_\alpha &= \int_{\mathbb{C}} K_\mu f(w)\overline{g(w)}\, d\lambda_\alpha(w)\\ &= \int_{\mathbb{C}}\Big(\int_{\mathbb{C}} e^{\alpha z w} f(z)\, d\mu(z)\Big)\overline{g(w)}\, d\lambda_\alpha(w)\\ &= \int_{\mathbb{C}} f(z)\Big(\overline{\int_{\mathbb{C}} e^{\alpha \overline{z}\overline{w}} g(w)\, d\lambda_\alpha(w)}\Big)\, d\mu(z)\\ &= \int_{\mathbb{C}} f(z)\overline{g(\overline{z})}\, d\mu(z).\end{aligned}$$

Notice that $\overline{g(\overline{z})} \in F^{p'}_\alpha$ whenever $g \in F^{p'}_\alpha$, so $\overline{g(\overline{z})}$ is analytic on $\mathbb{C}$. From the reproducing formula,

$$f(z)\overline{g(\overline{z})} = \int_{\mathbb{C}} f(u)\overline{g(\overline{u})}e^{\alpha z\overline{u}}\, d\lambda_\alpha(u).$$

Set $h(u) = \int_{\mathbb{C}} e^{\alpha z\overline{u}}\, d\mu(z)$. By Fubini's theorem,

$$\langle K_\mu, g\rangle_\alpha = \int_{\mathbb{C}}\int_{\mathbb{C}} f(u)\overline{g(\overline{u})}e^{\alpha z\overline{u}}\, d\lambda_\alpha(u)\, d\mu(z) = \int_{\mathbb{C}} f(u)\overline{g(\overline{u})}h(u)\, d\lambda_\alpha(u).$$

This shows that K_μ is a bounded operator on F^p_α if and only if $\overline{h} \in F^\infty_\alpha$. Indeed, if $\overline{h} \in F^\infty_\alpha$, then for any $f \in F^p_\alpha, g \in F^{p'}_\alpha$, the dual relation $(F^1_\alpha)^* \simeq F^\infty_\alpha$ under the pairing $\langle\cdot,\cdot\rangle_\alpha$, together with Hölder's inequality, gives

$$|\langle K_\mu, g\rangle_\alpha| \leq C\|f\|_{p,\alpha}\|g\|_{p',\alpha}\|\overline{h}\|_{\infty,\alpha}.$$

Since $(F^p_\alpha)^* \simeq F^{p'}_\alpha$ relative to $\langle\cdot,\cdot\rangle_\alpha$, it follows that K_μ is bounded on F^p_α. Conversely, if K_μ is bounded on F^p_α, for a sequence $\{a_k\} \subset \mathbb{C}$, as in Theorem 2.1, let

$$f_k(z) = (e^{\alpha z\overline{a}_k - \alpha|a_k|^2/2})^{1/p}, \quad g_k(z) = (e^{\alpha z a_k - \alpha|a_k|^2/2})^{1/p'}.$$

When $F \in F^1_\alpha$, one can find a sequence $\{\lambda_k\} \in l^1$ such that

$$F(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)\overline{g_k(\overline{z})}$$

with $\|\{\lambda_k\}\|_{l^1} \leq C\|F\|_{1,\alpha}$. Consequently,

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$$
\begin{aligned}
|\langle F, \overline{h}\rangle_\alpha| &= \Big| \int_{\mathbb{C}} \sum_{k=1}^{\infty} \lambda_k f_k(z) \overline{g_k(\overline{z})} h(z)\, d\lambda_\alpha(z) \Big| \\
&\leq \sum_{k=1}^{\infty} |\lambda_k|\, |\langle K_\mu f_k, g_k\rangle_\alpha| \\
&\leq C \|\{\lambda_k\}\|_{l^1} \cdot \sup_j \|K_\mu f_k\|_{p,\alpha} \cdot \|g_k\|_{p',\alpha} \\
&\leq C \|F\|_{1,\alpha}.
\end{aligned}
$$

This shows $\overline{h} \in F_\alpha^\infty$ and completes the proof of the main theorem. □

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ERMIN WANG, School of Mathematics and Statistics,
Lingnan Normal University, Zhanjiang, Guangdong 524048, PR China
e-mail: wem0913@sina.com

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