COMPOSITIO MATHEMATICA

Topological complexity of collision-free motion planning on surfaces

Daniel C. Cohen and Michael Farber

Compositio Math. **147** (2011), 649–660.

doi:10.1112/S0010437X10005038

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 147 (2011) 649–660
doi:10.1112/S0010437X10005038

Topological complexity of collision-free motion planning on surfaces

Daniel C. Cohen and Michael Farber

Abstract

The topological complexity $\mathsf{TC}(X)$ is a numerical homotopy invariant of a topological space X which is motivated by robotics and is similar in spirit to the classical Lusternik–Schnirelmann category of X. Given a mechanical system with configuration space X, the invariant $\mathsf{TC}(X)$ measures the complexity of motion planning algorithms which can be designed for the system. In this paper, we compute the topological complexity of the configuration space of n distinct ordered points on an orientable surface, for both closed and punctured surfaces. Our main tool is a theorem of B. Totaro describing the cohomology of configuration spaces of algebraic varieties. For configuration spaces of punctured surfaces, this is used in conjunction with techniques from the theory of mixed Hodge structures.

1. Introduction

Let X be a path-connected topological space, with the homotopy type of a finite CW complex. Viewing X as the space of configurations of a mechanical system, the motion planning problem consists of constructing an algorithm which takes as input pairs of configurations $(x_0, x_1) \in X \times X$, and produces a continuous path $\gamma : [0, 1] \to X$ from the initial configuration $x_0 = \gamma(0)$ to the terminal configuration $x_1 = \gamma(1)$. The motion planning problem is a central theme of robotics, see, for example, Latombe [Lat91] and Sharir [Sha97].

A topological approach to this problem was developed by the second author [Far03]. Let PX denote the space of all continuous paths $\gamma : [0, 1] \to X$, equipped with the compact-open topology. The map $\pi : PX \to X \times X$ defined by sending a path to its endpoints, $\pi : \gamma \mapsto (\gamma(0), \gamma(1))$, is a fibration, with fiber ΩX, the based loop space of X. The motion planning problem asks for a section of this fibration, a map $s : X \times X \to PX$ satisfying $\pi \circ s = \mathrm{id}_{X\times X}$. It would be desirable for the motion planning algorithm to depend continuously on the input. However, one can show that there exists a globally continuous section $s : X \times X \to PX$ if and only if X is contractible, see [Far03, Theorem 1].

Definition. The *topological complexity* of X, $\mathsf{TC}(X)$, is the smallest positive integer k for which $X \times X = U_1 \cup \cdots \cup U_k$, where U_i is open and there exists a continuous section $s_i : U_i \to PX$ satisfying $\pi \circ s_i = \mathrm{id}_{U_i}$ for $1 \leqslant i \leqslant k$. In other words, the topological complexity of X is the Schwarz genus (or sectional category) of the path space fibration,

$$\mathsf{TC}(X) = \mathsf{genus}(\pi : PX \to X \times X).$$

Received 30 January 2010, accepted in final form 5 May 2010, published online 27 September 2010.
2000 Mathematics Subject Classification 55R80 (primary), 20F36, 55M30 (secondary).
Keywords: surface configuration space, topological complexity.

The relevance of the topological invariant $\mathsf{TC}(X)$ to the motion planning problem from robotics has been investigated in a number of publications. In [Far06, Theorem 13.1], four different quantities measuring the instability of motion planning algorithms (including the random algorithms) are all shown to coincide with $\mathsf{TC}(X)$. Additionally, a relation between a relative version of $\mathsf{TC}(X)$ and the standard technique of navigation functions from robotics is provided by [Far08, Theorem 4.31]. In practical situations, the effects measured by $\mathsf{TC}(X)$ are significant only when the configuration space X has large dimension (due to the known inequality $\mathsf{TC}(X) \leqslant 2\dim(X)+1$). For instance, the topological complexity $\mathsf{TC}(X)$ is large when one simultaneously controls many moving objects; see [Far08, Theorem 4.56] and also the main results of this paper stated below.

When faced with the problem of the motion of n distinct particles in X with the condition that no collision may occur during the motion, one is led to consider the topological complexity of the space

$$F(X,n)=\{(x_1,\dots,x_n)\in X^{\times n} \mid x_i\neq x_j \text{ for } i\neq j\},$$

the configuration space of n distinct ordered points in X, where $X^{\times n}=X\times\cdots\times X$ is the Cartesian product of n copies of X. In the case where $X=\mathbb{R}^m$ is a Euclidean space, the topological complexity of the corresponding configuration space was found in [FG09, FY04]. See also [Coh10] in the case $X=\mathbb{R}^2$. In this paper, we determine $\mathsf{TC}(F(X,n))$ in the case where $X=\Sigma_g$ is an orientable surface.

Theorem A. *The topological complexity of the configuration space of n distinct ordered points on an orientable surface Σ_g of genus g is*

$$\mathsf{TC}(F(\Sigma_g,n))=\begin{cases}3 & \text{if } g=0 \text{ and } n\leqslant 2,\\ 2n-2 & \text{if } g=0 \text{ and } n\geqslant 3,\\ 2n+1 & \text{if } g=1 \text{ and } n\geqslant 1,\\ 2n+3 & \text{if } g\geqslant 2 \text{ and } n\geqslant 1.\end{cases}$$

We also consider the problem of the collision-free motion of n distinct particles on Σ_g in the presence of $m\geqslant 1$ obstacles (also assumed to be particles). In other words, we consider collision-free motion of n particles on the punctured surface $\Sigma_g\smallsetminus Q_m$. Here, and throughout the paper, we use the symbol Q_m to denote a collection of $m\geqslant 1$ distinct points in a space X.

Theorem B. *For $m\geqslant 1$, the topological complexity of the configuration space of n distinct ordered points on the punctured surface $\Sigma_g\smallsetminus Q_m$ is*

$$\mathsf{TC}(F(\Sigma_g\smallsetminus Q_m,n))=\begin{cases}1 & \text{if } g=0,\ m=1, \text{ and } n=1,\\ 2n-2 & \text{if } g=0,\ m=1, \text{ and } n\geqslant 2,\\ 2n & \text{if } g=0,\ m=2, \text{ and } n\geqslant 1,\\ 2n+1 & \text{otherwise.}\end{cases}$$

As indicated, the topological complexity of the configuration space varies depending on the genus of the surface and the number of punctures, but in most cases depends only on the number of particles. After recalling a number of necessary properties and discussing technical tools we will use throughout the paper in § 2, we analyze the configuration space of the sphere in § 3, the configuration space of the torus in § 4, and the configuration space of a surface of higher genus in § 5. The topological complexity of the configuration space of n ordered points on a punctured surface is determined in § 6.

For small values of n, our results summarize known facts concerning the topological complexity of orientable surfaces themselves. If $n=1$, we have $F(X,1)=X$ for any space X. Let S^2 be the two-dimensional sphere, $T=S^1\times S^1$ the torus, and denote a surface of genus $g\geqslant 2$ by $\Sigma=\Sigma_g$. Then, $\mathsf{TC}(S^2)=3$, $\mathsf{TC}(T)=3$, and $\mathsf{TC}(\Sigma)=5$ as indicated above. Furthermore, there is a homotopy equivalence $F(S^2,2)\simeq S^2$, so $\mathsf{TC}(F(S^2,2))=3$ as well. Regarding punctured surfaces, the complement of m points on a surface has the homotopy type of either a point $*$ (with $\mathsf{TC}(*)=1$), a circle S^1 (with $\mathsf{TC}(S^1)=2$), or a bouquet $\bigvee_r S^1$ of $r\geqslant 2$ circles (with $\mathsf{TC}(\bigvee_r S^1)=3$). Refer to the survey [Far06] for a discussion of these and other relevant results.

It is interesting to compare the results stated in Theorem A above with the topological complexity of the Cartesian product $(\Sigma_g)^{\times n}$. Using the arguments described in [Far08, pp. 115–116], one easily obtains

$$\mathsf{TC}((\Sigma_g)^{\times n})=\begin{cases}2n+1 & \text{if } g=0 \text{ or } g=1,\\ 4n+1 & \text{if } g\geqslant 2.\end{cases}$$

Thus, on a surface of high genus, the complexity of the collision-free motion planning problem for n distinct points is roughly half of the complexity of the similar problem when the points are allowed to collide. This seems surprising and contradicts our intuitive understanding of the relative complexity of the two problems. To explain this apparent paradox, we observe that if one wants to apply the motion planning algorithm A_n designed for n distinct particles to the general case (where particles may collide), then one must combine all the algorithms $A_1, A_2, \dots, A_n$ with obvious switches from A_i to A_j. These switches naturally increase the complexity of the global algorithm. This example provides an illustration of the fact that the concept $\mathsf{TC}(X)$ reflects only the *topological* complexity, which is just a part of the *total* complexity of the problem.

2. Preliminaries

In this section, we record a number of relevant properties of topological complexity, and discuss some relevant results on the cohomology of configuration spaces.

The topological complexity $\mathsf{TC}(X)$ depends only on the homotopy type of X, and satisfies the inequality

$$\mathsf{TC}(X)\leqslant 2\dim X+1, \tag{2.1}$$

where $\dim(X)$ denotes the covering dimension of X, see [Far03, §§ 2–3].

Recall that the spaces we consider in this paper are, up to homotopy type, finite CW complexes. For two such spaces X and Y, as shown in [Far03, Theorem 11], the topological complexity of the Cartesian product admits the upper bound

$$\mathsf{TC}(X\times Y)\leqslant \mathsf{TC}(X)+\mathsf{TC}(Y)-1. \tag{2.2}$$

Let $A=\bigoplus_{i=0}^{\ell} A^i$ be a graded algebra over a field $\Bbbk$, with A^i finite-dimensional for each i. All the algebras considered in this paper are graded commutative and connected ($A^0=\Bbbk$). Define $\mathsf{cl}\,A$, the cup length of A, to be the largest integer q for which there are homogeneous elements $a_1,\dots,a_q$ of positive degree in A such that $a_1\cdots a_q\neq 0$.

The tensor product $A\otimes A$ has natural graded algebra structure, with multiplication given by $(u_1\otimes v_1)\cdot(u_2\otimes v_2)=(-1)^{|v_1|\cdot|u_2|}u_1u_2\otimes v_1v_2$. Multiplication in A defines an algebra homomorphism $\mu: A\otimes A\to A$. The zero-divisor cup length of A, denoted by $\mathsf{zcl}\,A$, is the cup length of the ideal $Z=\ker(\mu)$ of zero-divisors. It is a straightforward exercise to verify that the zero-divisor cup length has the properties listed below.

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press

Lemma 2.1. *Let $\Bbbk$ be a field, and let A and B be graded, graded commutative, connected, unital algebras over $\Bbbk$.*

(i) *If B is a subalgebra of A, then $\mathsf{zcl}\, A \geqslant \mathsf{zcl}\, B$.*
(ii) *If B is an epimorphic image of A, then $\mathsf{zcl}\, A \geqslant \mathsf{zcl}\, B$.*
(iii) *The tensor product $A \otimes B$ satisfies $\mathsf{zcl}\, A \otimes B \geqslant \mathsf{zcl}\, A + \mathsf{zcl}\, B$.*

By [Far03, Theorem 7], for any field $\Bbbk$, the topological complexity of X is larger than the zero-divisor cup length of the cohomology algebra $A = H^*(X;\Bbbk)$,

$$\mathsf{TC}(X) \geqslant \mathsf{zcl}\, H^*(X;\Bbbk) + 1. \tag{2.3}$$

For configuration spaces of ordered points on surfaces, topological considerations, together with the inequalities (2.1) and (2.2), yield upper bounds on the topological complexity. These bounds are shown to be sharp using the cohomological lower bound (2.3). This requires some understanding of the structure of the cohomology ring of the configuration space.

The cohomology of the configuration space $F(X,n)$ of n distinct ordered points in X has been the object of a great deal of study, particularly for X an oriented manifold. See, for instance, [Tot96] and the references therein. In the case where X is a Euclidean space, the structure of the ring $H^*(F(\mathbb{R}^m,n);\mathbb{Z})$ was determined by Arnol'd [Arn69] (for $n=2$) and Cohen [Coh76]. This ring has generators $\alpha_{i,j}$, $1 \leqslant i,j \leqslant n$, $i \neq j$, of degree $m-1$, and relations $\alpha_{i,j} = (-1)^m \alpha_{j,i}$, $\alpha_{i,j}^2 = 0$, and $\alpha_{i,j}\alpha_{i,k} + \alpha_{j,k}\alpha_{j,i} + \alpha_{k,i}\alpha_{k,j} = 0$ for distinct i,j,k. Recall that $X^{\times n} = X \times \cdots \times X$ (n times) denotes the Cartesian product. The structure of $H^*(F(\mathbb{R}^m,n);\mathbb{Z})$ plays a significant role in the Leray spectral sequence of the inclusion $F(X,n) \hookrightarrow X^{\times n}$, analyzed by Cohen and Taylor [CT78] and Totaro [Tot96].

Let X be an oriented real manifold of dimension m, and let $\Delta \in H^m(X \times X;\Bbbk)$ be the cohomology class dual to the diagonal, where $\Bbbk$ is a field. If X is closed, $\omega \in H^m(X;\Bbbk)$ is a fixed generator, and $\{\beta_i\}$ and $\{\beta_i^*\}$ are dual bases for $H^*(X;\Bbbk)$ satisfying $\beta_i \cup \beta_j^* = \delta_{i,j}\omega$, where $\delta_{i,j}$ is the Kronecker symbol, then the diagonal cohomology class may be expressed as

$$\Delta = \sum (-1)^{|\beta_i|} \beta_i \times \beta_i^*,$$

see [MS74, Theorem 11.11].

Let $p_i : X^{\times n} \to X$ and $p_{i,j} : X^{\times n} \to X \times X$ denote the natural projections, $p_i(x_1,\ldots,x_n) = x_i$ and $p_{i,j}(x_1,\ldots,x_n) = (x_i,x_j)$, where $1 \leqslant i,j \leqslant n$ and $i \neq j$. Then, as shown by Cohen and Taylor [CT78] and Totaro [Tot96], the inclusion $F(X,n) \hookrightarrow X^{\times n}$ determines a Leray spectral sequence which converges to $H^*(F(X,n);\Bbbk)$. The initial term is the quotient of the algebra $H^*(X^{\times n};\Bbbk) \otimes H^*(F(\mathbb{R}^m,n);\Bbbk)$ by the relations $(p_i^*(x) - p_j^*(x)) \otimes \alpha_{i,j}$ for $i \neq j$ and $x \in H^*(X)$. The first non-trivial differential $d = d_m$ is given by $d\alpha_{i,j} = p_{i,j}^*\Delta$.

The case where X is a smooth, complex projective variety (specifically, a curve) is of particular interest to us. In this instance, with rational coefficients $\Bbbk = \mathbb{Q}$, Totaro [Tot96, Theorem 4] shows that the differential in the spectral sequence described above is the only non-trivial differential, and that the E_∞ term is isomorphic to the cohomology ring $H^*(F(X,n);\mathbb{Q})$ as a $\mathbb{Q}$-algebra. It follows that $H^*(F(X,n);\mathbb{Q})$ is determined by $H^*(X;\mathbb{Q})$. Specifically, if X is of real dimension m, the ring $H^*(F(X,n);\mathbb{Q})$ is isomorphic to the cohomology of the algebra

$$H^*(X^{\times n};\mathbb{Q}) \otimes H^*(F(\mathbb{R}^m,n);\mathbb{Q}) / \langle (p_i^*(x) - p_j^*(x)) \otimes \alpha_{i,j} \mid 1 \leqslant i,j \leqslant n, i \neq j \rangle$$

with respect to the differential induced by $d = d_m$ described above. This fact implies the following result, which we will use extensively.

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press

PROPOSITION 2.2. *Let X be a smooth projective variety over $\mathbb{C}$ of real dimension m, let $H = H^*(X^{\times n}; \mathbb{Q})$, and let I be the ideal in H generated by the elements*

$$\Delta_{i,j} = p_{i,j}^* \Delta \in H^m(X^{\times n}; \mathbb{Q})$$

for all $1 \leqslant i < j \leqslant n$. Then the quotient H/I is a subalgebra of the rational cohomology ring of the configuration space $F(X, n)$. Thus, using Lemma 2.1, one obtains

$$\mathsf{TC}(F(X, n)) \geqslant \mathsf{zcl}\, H/I + 1.$$

We will also make frequent use of the classical Fadell–Neuwirth theorem [FN62, Theorem 3], which shows that, for $m < n$, the projection $F(X, n) \to F(X, m)$ onto the first m coordinates is a fibration, with fiber $F(X \smallsetminus Q_m, n - m)$. Recall that the symbol Q_m denotes a collection of $m \geqslant 1$ distinct points in the topological space X.

3. Genus zero

In this section, we determine the topological complexity of the configuration space of n ordered points on the sphere S^2. Since $F(S^2, 1) = S^2$ and $F(S^2, 2)$ is equivalent to the tangent bundle over S^2, hence has the homotopy type of S^2, we have $\mathsf{TC}(F(S^2, n)) = 3$ for $n \leqslant 2$. For larger n, the following holds.

THEOREM 3.1. *For $n \geqslant 3$, the topological complexity of the configuration space of n distinct ordered points on the sphere is*

$$\mathsf{TC}(F(S^2, n)) = 2n - 2.$$

Proof. Let $\mathrm{PSL}(2, \mathbb{C})$ be the group of Möbius transformations acting on the sphere $S^2 = \mathbb{P}^1$. As shown by Feichtner and Ziegler [FZ00, Theorem 2.1], there are homeomorphisms $F(S^2, 3) \to \mathrm{PSL}(2, \mathbb{C})$, and, for $n \geqslant 4$,

$$F(S^2, n) \to \mathrm{PSL}(2, \mathbb{C}) \times F(S^2 \smallsetminus Q_3, n - 3).$$

Since $\mathrm{PSL}(2, \mathbb{C})$ deformation retracts onto $\mathrm{SO}(3)$ and $F(S^2 \smallsetminus Q_3, n-3) = F(\mathbb{R}^2 - Q_2, n-3)$, we obtain homotopy equivalences $F(S^2, 3) \simeq \mathrm{SO}(3)$, and, for $n \geqslant 4$,

$$F(S^2, n) \simeq \mathrm{SO}(3) \times F(\mathbb{R}^2 \smallsetminus Q_2, n - 3). \tag{3.1}$$

The topological complexity of the connected Lie group $\mathrm{SO}(3)$ is $\mathsf{TC}(\mathrm{SO}(3)) = \mathrm{cat}(\mathrm{SO}(3)) = 4$, see [Far04, Lemma 8.2]. This finishes the proof for $n = 3$. For $n \geqslant 4$, it is also known that $\mathsf{TC}(F(\mathbb{R}^2 \smallsetminus Q_2, n - 3)) = 2n - 5$, see [FGY07, Theorem 6.1]. These facts and the product inequality (2.2) imply

$$\mathsf{TC}(F(S^2, n)) \leqslant \mathsf{TC}(\mathrm{SO}(3)) + \mathsf{TC}(F(\mathbb{R}^2 \smallsetminus Q_2, n - 3)) - 1 = 4 + 2n - 5 - 1 = 2n - 2.$$

To complete the proof, by (2.3) it suffices to show that $\mathsf{zcl}\, H^*(F(S^2, n); \mathbb{Z}_2) \geqslant 2n - 3$. As indicated, we will consider cohomology with $\mathbb{Z}_2$ coefficients. From the homotopy equivalence (3.1), we have

$$H^*(F(S^2, n); \mathbb{Z}_2) \cong H^*(\mathrm{SO}(3); \mathbb{Z}_2) \otimes H^*(F(\mathbb{R}^2 \smallsetminus Q_2, n - 3); \mathbb{Z}_2).$$

As noted in Lemma 2.1, it follows that

$$\mathsf{zcl}\, H^*(F(S^2, n); \mathbb{Z}_2) \geqslant \mathsf{zcl}\, H^*(\mathrm{SO}(3); \mathbb{Z}_2) + \mathsf{zcl}\, H^*(F(\mathbb{R}^2 \smallsetminus Q_2, n - 3); \mathbb{Z}_2).$$

It is readily checked that the zero-divisor cup length of $H^*(\mathrm{SO}(3); \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^4)$ is equal to 3. For the configuration space, the zero-divisor cup length of the integral cohomology

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press

ring $H^*(F(\mathbb{R}^2 \smallsetminus Q_2, n-3); \mathbb{Z})$ was computed in [FGY07, § 6]. Repeating this computation with $\mathbb{Z}_2$ coefficients yields the same result,

$$\mathsf{zcl}\, H^*(F(\mathbb{R}^2 \smallsetminus Q_2, n-3); \mathbb{Z}_2) = \mathsf{zcl}\, H^*(F(\mathbb{R}^2 \smallsetminus Q_2, n-3); \mathbb{Z}) = 2(n-3).$$

It follows that $\mathsf{zcl}\, H^*(F(S^2, n)) \geqslant 3 + 2(n-3) = 2n-3$, as required. □

Remark 3.2. For $n \geqslant 3$, the topological complexity of $F(S^2, n)$ coincides with that of $F(\mathbb{R}^2, n)$, the configuration space of n distinct ordered points in the plane. See Farber and Yuzvinsky [FY04] for the calculation of the latter.

4. Genus one

Theorem 4.1. *The topological complexity of the configuration space of n distinct ordered points on the torus is*

$$\mathsf{TC}(F(T, n)) = 2n + 1.$$

Proof. For $n = 1$, since $F(T, 1) = T$, we have $\mathsf{TC}(F(T, 1)) = 3$ as noted previously. So assume that $n \geqslant 2$.

Since $T = S^1 \times S^1$ is a group, we have

$$F(T, n) \cong T \times F(T \smallsetminus Q_1, n-1).$$

Explicitly, view S^1 as the set of complex numbers of length one, and let $Q_1 = \{(1, 1)\} \in T$. It is then readily checked that the map $T \times F(T \smallsetminus Q_1, n-1) \to F(T, n)$ defined by

$$((u, v), ((z_1, w_1), \ldots, (z_{n-1}, w_{n-1}))) \mapsto ((u, v), (uz_1, vw_1), \ldots, (uz_{n-1}, vw_{n-1}))$$

is a homeomorphism.

Using Fadell–Neuwirth fibrations, one can show that $F(T \smallsetminus Q_1, n-1)$ is an Eilenberg–Mac Lane space of type $K(G, 1)$, where $G = \pi_1(F(T \smallsetminus Q_1, n-1))$ is the pure braid group of $T \smallsetminus Q_1$. Since the group G is an $(n-1)$-fold iterated semidirect product of free groups [Bel04, GG03], the space $F(T \smallsetminus Q_1, n-1)$ has the homotopy type of a cell complex of dimension $n-1$, see [CS98, § 1.3]. So $\mathsf{TC}(F(T \smallsetminus Q_1, n-1)) \leqslant 2n-1$ by (2.1), and the product inequality (2.2) yields

$$\mathsf{TC}(F(T, n)) \leqslant \mathsf{TC}(T) + \mathsf{TC}(F(T \smallsetminus Q_1, n-1)) = 3 + 2n - 1 - 1 = 2n + 1.$$

By (2.3), it suffices to show that $\mathsf{zcl}\, H^*(F(T, n); \mathbb{Q}) \geqslant 2n$. We establish this using the Leray spectral sequence of the inclusion $F(T, n) \hookrightarrow T^{\times n}$ developed by Totaro [Tot96]. Since we use rational coefficients throughout the argument, we subsequently suppress coefficients and write $H^*(X) = H^*(X; \mathbb{Q})$ for brevity. Let $a, b \in H^1(T)$ be the generators of $H^*(T)$. Note that the diagonal class $\Delta \in H^2(T \times T)$ is given by

$$\Delta = ab \times 1 + 1 \times ab + b \times a - a \times b = (1 \times a - a \times 1)(1 \times b - b \times 1).$$

Let $H_T = H^*(T^{\times n}) = [H^*(T)]^{\otimes n}$. Note that H_T is an exterior algebra. Denote the generators of H_T by a_i, b_i, $1 \leqslant i \leqslant n$, where $u_i = 1 \times \cdots \times \overset{i}{u} \times \cdots \times 1$. Let I_T be the ideal in H_T generated by the elements $\Delta_{i,j} = p_{i,j}^* \Delta$, $1 \leqslant i < j \leqslant n$, and observe that, in this notation, we have

$$\Delta_{i,j} = (a_j - a_i)(b_j - b_i).$$

Realizing T as a smooth, complex projective curve, Proposition 2.2 implies that the algebra

$$A_T = H_T/I_T \tag{4.1}$$

is a subalgebra of $H^*(F(T,n))$. Since $\mathsf{zcl}\, H^*(F(T,n)) \geqslant \mathsf{zcl}\, A_T$ by Lemma 2.1, it is enough to show that $\mathsf{zcl}\, A_T \geqslant 2n$.

Introduce a new basis for H_T as follows:

$$x_j = \begin{cases} a_1 & \text{if } j=1, \\ a_j - a_1 & \text{if } 2 \leqslant j \leqslant n, \end{cases} \qquad y_j = \begin{cases} b_1 & \text{if } j=1, \\ b_j - b_1 & \text{if } 2 \leqslant j \leqslant n. \end{cases}$$

In this basis, $\Delta_{1,j} = x_j y_j$ and $\Delta_{i,j} = x_j y_j - x_j y_i - x_i y_j + x_i y_i$ for $i > 1$. Consequently, the ideal I_T is given by

$$I_T = \langle x_j y_j \ (2 \leqslant j \leqslant n), x_j y_i + x_i y_j \ (2 \leqslant i < j \leqslant n)\rangle. \tag{4.2}$$

Since I_T is generated in degree two, we abuse notation and denote the generators of $A_T = H_T/I_T$ by x_i, y_i, $1 \leqslant i \leqslant n$. From the description (4.2) of the ideal I_T, monomials in A_T cannot have any repetition of indices (at least 2). Additionally, the presence of $x_j y_i + x_i y_j$ in I_T implies that any monomial in A_T may be expressed, up to sign, as a monomial in which all x-indices are smaller than all y-indices (with the exception of 1). Since such expressions are unique and non-zero in A_T, this algebra has basis

$$\{x_1^{\epsilon_x} y_1^{\epsilon_y} x_J y_K \mid \epsilon_x, \epsilon_y \in \{0,1\},\, J, K \subset [2,n], \max J < \min K\}, \tag{4.3}$$

where $[2,n] = \{2,3,\dots,n\}$ and, for instance, $x_J = x_{j_1} \cdots x_{j_p}$ if $J = \{j_1, \dots, j_p\}$.

We now complete the proof by showing that the zero-divisor cup length of A_T is $2n$. Consider the zero-divisors $\bar{x}_j = x_j \otimes 1 - 1 \otimes x_j$ and $\bar{y}_j = y_j \otimes 1 - 1 \otimes y_j$, $1 \leqslant j \leqslant n$, in $A_T \otimes A_T$. We claim that their product is non-zero. Note that $\bar{x}_j \bar{y}_j = y_j \otimes x_j - x_j \otimes y_j$ if $2 \leqslant j \leqslant n$, while $\bar{x}_1 \bar{y}_1 = x_1 y_1 \otimes 1 + y_1 \otimes x_1 - x_1 \otimes y_1 + 1 \otimes x_1 y_1$. So we have

$$\prod_{j=1}^{n} \bar{x}_j \bar{y}_j = \bar{x}_1 \bar{y}_1 \prod_{j=2}^{n} (y_j \otimes x_j - x_j \otimes y_j) = \bar{x}_1 \bar{y}_1 \sum_{J \subset [2,n]} \epsilon_J y_J x_{J^c} \otimes y_{J^c} x_J,$$

where $\epsilon_J \in \{1, -1\}$ and $J^c = [2,n] \smallsetminus J$. In particular, the above sum includes the summand $(-1)^n y_2 y_3 \cdots y_n \otimes x_2 x_3 \cdots x_n$ which cannot arise when other summands are expressed in terms of the specified basis (4.3) for A_T. Consequently, expanding the product $\prod_{j=1}^{n} \bar{x}_j \bar{y}_j$ yields a summand $\pm y_1 y_2 y_3 \cdots y_n \otimes x_1 x_2 x_3 \cdots x_n$, and no other summand in the expansion involves this tensor product of basis elements. Thus, $\prod_{j=1}^{n} \bar{x}_j \bar{y}_j$ is non-zero in $A_T \otimes A_T$, as asserted. □

Remark 4.2. The subalgebra $A_T = H_T/I_T$ is not isomorphic to $H^*(F(T,n))$. One can check, for instance, that the differential $d = d_2 : E_2^{1,1} \to E_2^{3,0}$ has non-trivial kernel, where $E_2^{1,1}$ is the quotient of $H^1(F(T,n)) \otimes H^1(F(\mathbb{R}^2,n))$ by the relations $(p_i^*(x) - p_j^*(x)) \otimes \alpha_{i,j}$ for $x \in H^1(T)$ and $E_2^{3,0} = H^3(F(T,n))$. However, A_T and $H^*(F(T,n))$ do have the same zero-divisor cup length. Theorem 4.1 implies that $\mathsf{zcl}\, A_T = \mathsf{zcl}\, H^*(F(T,n)) = 2n$.

Remark 4.3. The algebra $A_T = H_T/I_T$ is Koszul. A straightforward application of the Buchberger criterion (see [AHH97, Theorem 1.4]) reveals that the generating set (4.2) of the defining ideal I_T is a Gröbner basis. Since all of these generators are of degree two, the Koszulity of A_T follows (see, for instance, [Yuz01, Theorem 6.16]).

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press

5. Higher genus

Theorem 5.1. *The topological complexity of the configuration space of n distinct ordered points on a surface Σ of genus $g \geqslant 2$ is*

$$\mathsf{TC}(F(\Sigma, n)) = 2n + 3.$$

Proof. For $n = 1$, since $F(\Sigma, 1) = \Sigma$, we have $\mathsf{TC}(F(\Sigma, 1)) = 5$ as noted previously. So assume that $n \geqslant 2$.

The configuration space $F(\Sigma, n)$ is an Eilenberg–Mac Lane space of type $K(G, 1)$, where $G = \pi_1(F(\Sigma, n))$ is the pure braid group of Σ. Since the Fadell–Neuwirth fibration $F(\Sigma, n) \to \Sigma$ has a section, the group $G \cong \pi_1(F(\Sigma \smallsetminus Q_1, n-1)) \rtimes \pi_1(\Sigma)$ is a semidirect product. As in the genus one case, the group $\pi_1(F(\Sigma \smallsetminus Q_1, n-1))$ is an $(n-1)$-fold iterated semidirect product of free groups. It follows that the cohomological dimension of G is equal to $n + 1$, as is the geometric dimension. Consequently, $F(\Sigma, n)$ has the homotopy type of a cell complex of dimension $n + 1$. So $\mathsf{TC}(F(\Sigma, n)) \leqslant 2n + 3$.

By (2.3), it suffices to show that $\mathsf{zcl}\, H^*(F(\Sigma, n); \mathbb{Q}) \geqslant 2n + 2$. We again use the Leray spectral sequence of the inclusion $F(\Sigma, n) \hookrightarrow \Sigma^{\times n}$ following [Tot96], and write $H^*(\Sigma) = H^*(\Sigma; \mathbb{Q})$. Let $a(p)$, $b(p)$, $1 \leqslant p \leqslant g$, be the generators of $H^1(\Sigma)$, satisfying, for $p \neq q$, $a(p)b(p) = a(q)b(q) = \omega$ and $a(p)a(q) = b(p)b(q) = a(p)b(q) = 0$, where ω generates $H^2(\Sigma)$. Then the diagonal class $\Delta \in H^2(\Sigma \times \Sigma)$ may be expressed as

$$\Delta = \omega \times 1 + 1 \times \omega + \sum_{p=1}^{g} (b(p) \times a(p) - a(p) \times b(p)).$$

Let $H_\Sigma = H^*(\Sigma^{\times n}) = [H^*(\Sigma)]^{\otimes n}$, and let I_Σ be the ideal in H_Σ generated by the elements $\Delta_{i,j} = p_{i,j}^* \Delta$, $1 \leqslant i < j \leqslant n$. Realizing Σ as a smooth, complex projective curve, by Proposition 2.2 and Lemma 2.1, it is enough to show that the subalgebra $A_\Sigma = H_\Sigma / I_\Sigma$ of $H^*(F(\Sigma, n))$ satisfies $\mathsf{zcl}\, A_\Sigma \geqslant 2n + 2$. Annihilating all generators of H_Σ of the form $1 \times \cdots \times u \times \cdots \times 1$, where $u \in \{a(q), b(q) \mid 3 \leqslant q \leqslant n\}$, it suffices to consider the genus $g = 2$ case.

For a genus two surface Σ, denote the generators of $H^1(\Sigma)$ by a, b, c, d, with $ab = cd = \omega$, and other cup products equal to zero. Let a_i, b_i, c_i, d_i be the generators of H_Σ, where for $1 \leqslant i \leqslant n$, $u_i = 1 \times \cdots \times \overset{i}{u} \times \cdots \times 1$ as before. In this notation, we have

$$\Delta_{i,j} = \omega_i + \omega_j - a_i b_j + b_i a_j - c_i d_j + d_i c_j.$$

Consider the ideal $J_\Sigma = \langle c_i c_j, c_i d_j, d_i d_j \mid 1 \leqslant i, j \leqslant n, i \neq j \rangle$ in H_Σ. Observe that

$$\Delta_{i,j} = a_i b_i + a_j b_j - a_i b_j + b_i a_j = (a_j - a_i)(b_j - b_i) \mod J,$$

and that

$$B_\Sigma = H_\Sigma / (I_\Sigma + J_\Sigma) \cong (H_\Sigma / I_\Sigma) / ((I_\Sigma + J_\Sigma) / I_\Sigma)$$

is a quotient of A_Σ. Consequently, $\mathsf{zcl}\, B_\Sigma \leqslant \mathsf{zcl}\, A_\Sigma$. The subalgebra of B_Σ generated by $\{a_i, b_i \mid 1 \leqslant i \leqslant n\}$ is isomorphic to the algebra A_T arising in the genus one case, see (4.1). Letting $x_j = a_j - a_1$ and $y_j = b_j - b_1$ for $j \geqslant 2$ as in that case, it follows that the set

$$\{x_J y_K \mid J, K \subset [2, n], \max J < \min K\}$$

is linearly independent in B_Σ. From this, it follows easily that the zero-divisor cup length of B_Σ is (at least) $2n + 2$. Indeed, writing $\bar{u} = u \otimes 1 - 1 \otimes u \in B_\Sigma \otimes B_\Sigma$ for $u \in B_\Sigma$, a calculation

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press

reveals that $\bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1 = 2\omega_1 \otimes \omega_1$. Then, expanding the product $\bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1 \prod_{j=2}^{n} \bar{x}_j\bar{y}_j$ of $2n+2$ zero-divisors yields a summand

$$\pm 2\omega_1 y_2 y_3 \cdots y_n \otimes \omega_1 x_2 x_3 \cdots x_n,$$

and no other summand in the expansion involves this (non-zero) tensor product. Thus, $2n+2 \leqslant \mathsf{zcl}\, B_\Sigma \leqslant \mathsf{zcl}\, A_\Sigma \leqslant \mathsf{zcl}\, H^*(F(\Sigma, n))$. □

6. Punctured surfaces

In this section, we determine the topological complexity of the configuration space of n ordered points on a punctured surface. Observe that a punctured surface is not a smooth projective variety, so Proposition 2.2 does not apply directly. In the high genus case, we use this result in conjunction with other tools, specifically mixed Hodge structures.

Recall that $X \smallsetminus Q_m$ denotes the complement of a set Q_m of m distinct points in X.

Theorem 6.1. *For $m \geqslant 1$, the topological complexity of the configuration space of n distinct ordered points on $S^2 \smallsetminus Q_m$ is*

$$\mathsf{TC}(F(S^2 \smallsetminus Q_m, n)) = \begin{cases} 1 & \text{if } m = 1 \text{ and } n = 1, \\ 2n-2 & \text{if } m = 1 \text{ and } n \geqslant 2, \\ 2n & \text{if } m = 2 \text{ and } n \geqslant 1, \\ 2n+1 & \text{if } m \geqslant 3 \text{ and } n \geqslant 1. \end{cases}$$

Proof. Note that $F(S^2 \smallsetminus Q_1, n) = F(\mathbb{R}^2, n)$, that $F(S^2 \smallsetminus Q_2, n) = F(\mathbb{R}^2 \smallsetminus Q_1, n) \simeq F(\mathbb{R}^2, n+1)$, and that, for $m \geqslant 3$, $F(S^2 \smallsetminus Q_m, n) = F(R^2 \smallsetminus Q_{m-1}, n)$ is the configuration space of n points in the complement of at least two points in $\mathbb{R}^2$. For $n = 1$, $F(S^2 \smallsetminus Q_m, 1)$ has the homotopy type of a bouquet of $m-1$ circles (where a bouquet of zero circles is a point), and the theorem follows easily. For $n \geqslant 2$, in light of the above observations, the theorem follows from results of Farber and Yuzvinsky [FY04] for $m \leqslant 2$, and from results of Farber *et al.* [FGY07] for $m \geqslant 3$. □

Remark 6.2. For $m \geqslant 1$, the configuration space $F(S^2 \smallsetminus Q_m, n)$ is an Eilenberg–Mac Lane space of type $K(\pi, 1)$, where $\pi = \pi_1(F(S^2 \smallsetminus Q_m, n))$ is the pure braid group of $S^2 \smallsetminus Q_m$. Since these groups are almost-direct products of free groups, the above result may also be obtained using the methods of [Coh10].

Theorem 6.3. *Let Σ be a surface of genus $g \geqslant 1$. For $m \geqslant 1$, the topological complexity of the configuration space of n distinct ordered points on $\Sigma \smallsetminus Q_m$ is*

$$\mathsf{TC}(F(\Sigma \smallsetminus Q_m, n)) = 2n+1.$$

Proof. For $n = 1$, since $F(\Sigma \smallsetminus Q_m, 1) = \Sigma \smallsetminus Q_m$ has the homotopy type of a bouquet of $r \geqslant 2$ circles, we have $\mathsf{TC}(F(\Sigma \smallsetminus Q_m, 1)) = 3$. So assume that $n \geqslant 2$.

The configuration space $F(\Sigma \smallsetminus Q_m, n)$ is an Eilenberg–Mac Lane space of type $K(G, 1)$, where $G = \pi_1(F(\Sigma \smallsetminus Q_m, n))$ is the pure braid group of $\Sigma \smallsetminus Q_m$. As in previous cases, the group G is an n-fold iterated semidirect product of free groups, and the geometric dimension of G is equal to n. Consequently, $F(\Sigma \smallsetminus Q_m, n)$ has the homotopy type of a cell complex of dimension n. So $\mathsf{TC}(F(\Sigma \smallsetminus Q_m, n)) \leqslant 2n+1$.

By (2.3), it suffices to show that $\mathsf{zcl}\, H^*(F(\Sigma \smallsetminus Q_m, n); \Bbbk) \geqslant 2n$. We will use complex coefficients $\Bbbk = \mathbb{C}$, and write $H^*(X) = H^*(X; \mathbb{C})$.

First, assume that $m=1$. Let $p \in \Sigma$ and $Q_1 = \{p\}$. The configuration space $F(\Sigma \smallsetminus Q_1, n)$ may be realized as $F(\Sigma \smallsetminus Q_1, n) = X \smallsetminus D = X \smallsetminus \bigcup_{i=1}^n D_i$, where $X = F(\Sigma, n)$ and $D_i = \{(x_1, \ldots, x_n) \in X \mid x_i = p\}$. Note that $D_i \cong F(\Sigma \smallsetminus Q_1, n-1)$ is closed in X, and $D_i \cap D_j = \emptyset$ if $i \neq j$. Consider the corresponding Gysin sequence

$$\cdots \longrightarrow H^{k-2}(D) \xrightarrow{\delta} H^k(X) \xrightarrow{j^*} H^k(X \smallsetminus D) \xrightarrow{R} H^{k-1}(D) \xrightarrow{\delta} H^{k+1}(X) \longrightarrow \cdots \tag{6.1}$$

where j^* is induced by the inclusion $j : X \smallsetminus D \hookrightarrow X$, R is the residue map, and δ is the connecting homomorphism. Using this sequence, we observe that the map $j^* : H^1(X) \to H^1(X \smallsetminus D)$, that is, $j^* : H^1(F(\Sigma, n)) \to H^1(F(\Sigma \smallsetminus Q_1, n))$, is injective.

Let $H_\Sigma = H^*(\Sigma^{\times n}) = [H^*(\Sigma)]^{\otimes n}$, and let I_Σ be the ideal in H_Σ generated by the elements $\Delta_{i,j} = p_{i,j}^* \Delta$, $1 \leqslant i < j \leqslant n$. Then, by Proposition 2.2, $A_\Sigma = H_\Sigma / I_\Sigma$ is a subalgebra of $H^*(F(\Sigma, n)) = H^*(X)$. Since I_Σ is generated in degree two, the generators of H_Σ are among the generators of $H^*(X)$. Consider the generators $x_1 = a_1$, $y_1 = b_1$, $x_i = a_i - a_1$, $y_i = b_i - b_1$, $2 \leqslant i \leqslant n$, which arose in the proofs of Theorems 4.1 and 5.1. The above observation implies that their images under the map j^* are among the generators of $H^1(X \smallsetminus D) = H^1(F(\Sigma \smallsetminus Q_1, n))$.

Write $u_i = j^*(x_i)$ and $v_i = j^*(y_i)$, and let $\bar{u}_i = u_i \otimes 1 - 1 \otimes u_i$ and $\bar{v}_i = v_i \otimes 1 - 1 \otimes v_i$ be the corresponding zero-divisors in $H^*(X \smallsetminus D) \otimes H^*(X \smallsetminus D)$. We will show that the product of these $2n$ zero-divisors is non-zero using mixed Hodge structures. References include [Dim92, GS75, PS08].

Realizing Σ as a smooth projective variety, the Hodge structure on $H^*(\Sigma)$ is pure. Assume without loss that the elements a and b of $H^1(\Sigma)$ are of types $(1,0)$ and $(0,1)$ respectively. Then, for each i, x_i and y_i are of types $(1,0)$ and $(0,1)$. Since the map $j^* : H^*(X) \to H^*(X \smallsetminus D)$ preserves type, the elements u_i and v_i of $H^1(X \smallsetminus D)$ are of types $(1,0)$ and $(0,1)$. Furthermore, each of these elements is of weight 1.

Since $X = F(\Sigma, n)$ is smooth, for each m, the weight filtration on $H^m(X)$ satisfies $0 = W_{m-1}(H^m(X)) \subset W_m(H^m(X)) = f^*(H^m(\bar{X}))$, where $f : X \hookrightarrow \bar{X}$ is any compactification, see [Dim92, Theorem C24]. Taking $\bar{X} = \Sigma^{\times n}$, we have $W_m(H^m(X)) = f^*(H^m(\Sigma^{\times n}))$. It follows that $A_\Sigma^m \subset W_m(H^m(X))$. Recall that the divisor D has disjoint components $D_i = \{(x_1, \ldots, x_n) \in X \mid x_i = p\}$, let $\Sigma_i^{\times n-1} = \{(x_1, \ldots, x_n) \in \Sigma^{\times n} \mid x_i = p\}$, and let $f_i : D_i \hookrightarrow \Sigma_i^{\times n-1}$. Then, $W_m(H^m(D_i)) = f_i^*(H^m(\Sigma^{\times n-1}))$. Note that the diagram

$$\begin{array}{ccc} D_i & \longrightarrow & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma_i^{\times n-1} & \longrightarrow & \Sigma^{\times n} \end{array}$$

commutes, where the horizontal maps are inclusions.

The Gysin mapping $H^{k-2}(\Sigma_i^{\times n-1}) \to H^k(\Sigma^{\times n})$ is obtained by applying Poincaré duality to the map $H_{2n-k}(\Sigma_i^{\times n-1}) \to H_{2n-k}(\Sigma^{\times n})$ induced by inclusion, see [GS75, § 5]. The ring $H^*(\Sigma_i^{\times n-1})$ is generated by the (images of the) generators u_j of $H^*(\Sigma^{\times n})$ which do not involve the index i. In terms of these generators, one can check that this map is, up to sign, multiplication by $\omega_i = a_i b_i$. As noted in [Dim92, Remark C30], the connecting homomorphism $\delta : H^{n-2}(D) \to H^n(X)$ in the Gysin sequence (6.1) is a morphism of mixed Hodge structures of type $(1,1)$. It follows, by functoriality, that the restriction $\delta : W_{m-2}(H^{m-2}(D_i)) \to W_m(H^m(X))$ is also multiplication by $a_i b_i$.

These considerations imply that the image of $\delta : W_{n-2}(H^{n-2}(D)) \to W_n(H^n(X))$ is contained in the ideal $\langle a_i b_i \mid 1 \leqslant i \leqslant n \rangle$. In terms of the generators x_i and y_i, this ideal is generated

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press

by x_1y_1 and $x_iy_1 + x_1y_i$ for $2 \leqslant i \leqslant n$. Observe that the basis elements $x_1 \cdots x_k y_{k+1} \cdots y_n$ of A_Σ from (4.3) are non-trivial modulo this ideal. Consequently, the corresponding elements $u_1 \cdots u_k v_{k+1} \cdots v_n = j^*(x_1 \cdots x_k y_{k+1} \cdots y_n)$ of $H^n(X \setminus D)$ are non-zero, and are linearly independent since they are of distinct types $(k, n-k)$.

It follows easily that the product $\prod_{i=1}^n \bar{u}_i \bar{v}_i$ is non-zero in $H^*(X \setminus D) \otimes H^*(X \setminus D)$. Expanding this product yields a linear combination of terms of the form $\alpha \otimes \beta$, where α and β are among the independent elements $u_1 \cdots u_k v_{k+1} \cdots v_n$ noted above. In particular, there is a single summand $\pm u_1 u_2 \cdots u_n \otimes v_1 v_2 \cdots v_n$, and no other summand in the expansion involves this (non-zero) tensor product. Thus, $\mathsf{zcl}\, H^*(X \setminus D) = \mathsf{zcl}\, H^*(F(\Sigma \setminus Q_1, n)) \geqslant 2n$ in the case $m = 1$.

For $m > 1$, assume inductively that the image of the product $\prod_{i=1}^n \bar{x}_i \bar{y}_i$ is non-zero in $H^*(F(\Sigma \setminus Q_{m-1}, n)) \otimes H^*(F(\Sigma \setminus Q_{m-1}, n))$. Write $F(\Sigma \setminus Q_m, n) = F(\Sigma \setminus Q_{m-1}, n) \setminus D$ in a manner analogous to the case $m = 1$. Then, arguing as above reveals that the image of this product is non-zero in $H^*(F(\Sigma \setminus Q_m, n)) \otimes H^*(F(\Sigma \setminus Q_m, n))$ as well. It follows that $\mathsf{zcl}\, H^*(F(\Sigma \setminus Q_m, n)) \geqslant 2n$ as required. □

Acknowledgements

Portions of this work were carried out during the Fall of 2008, when the first author visited the Department of Mathematical Sciences at the University of Durham. He thanks the Department for its hospitality, and for providing a productive mathematical environment.

References

AHH97 A. Aramova, J. Herzog and T. Hibi, *Gotzmann theorems for exterior algebras and combinatorics*, J. Algebra **191** (1997), 174–211; MR 1444495.

Arn69 V. I. Arnol'd, *The cohomology ring of the group of dyed braids*, Math. Notes **5** (1969), 138–140; MR 0242196.

Bel04 P. Bellingeri, *On presentations of surface braid groups*, J. Algebra **274** (2004), 543–563; MR 2043362.

Coh76 F. R. Cohen, *The homology of $\mathcal{C}_{n+1}$-spaces, $n \geqslant 0$*, in *The homology of iterated loop spaces*, Lecture Notes in Mathematics, vol. 533 (Springer, Berlin, 1976), 207–352; MR 0436146.

Coh10 D. Cohen, *Cohomology rings of almost-direct products of free groups*, Compositio Math. **146** (2010), 465–479.

CS98 D. Cohen and A. Suciu, *Homology of iterated semidirect products of free groups*, J. Pure Appl. Algebra **126** (1998), 87–120; MR 1600518.

CT78 F. R. Cohen and L. Taylor, *Computations of Gelfand–Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, in *Geometric applications of homotopy theory, I*, Lecture Notes in Mathematics, vol. 657 (Springer, Berlin, 1978), 106–143; MR 0513543.

Dim92 A. Dimca, *Singularities and topology of hypersurfaces*, Universitext (Springer, New York, 1992); MR 1194180.

FN62 E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 119–126; MR 0141126.

Far03 M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), 211–221; MR 1957228.

Far04 M. Farber, *Instabilities of robot motion*, Topology Appl. **140** (2004), 245–266; MR 2074919.

Far06 M. Farber, *Topology of robot motion planning*, in *Morse theoretic methods in non-linear analysis and in symplectic topology*, NATO Science Series II: Mathematics, Physics and Chemistry, vol. 217 (Springer, Dordrecht, 2006), 185–230; MR 2276952.

Far08 M. Farber, *Invitation to topological robotics*, Zurich Lectures in Advanced Mathematics (Eur. Math. Soc., Zürich, Switzerland, 2008); MR 2455573.

FG09 M. Farber and M. Grant, *Topological complexity of configuration spaces*, Proc. Amer. Math. Soc. **137** (2009), 1841–1847; MR 2470845.

FGY07 M. Farber, M. Grant and S. Yuzvinsky, *Topological complexity of collision free motion planning algorithms in the presence of multiple moving obstacles*, in *Topology and robotics*, Contemporary Mathematics, vol. 438 (American Mathematical Society, Providence, RI, 2007), 75–83; MR 2359030.

FY04 M. Farber and S. Yuzvinsky, *Topological robotics: subspace arrangements and collision free motion planning*, in *Geometry, topology, and mathematical physics*, American Mathematical Society Translations, Series 2, vol. 212 (American Mathematical Society, Providence, RI, 2004), 145–156; MR 2070052.

FZ00 E. Feichtner and G. Ziegler, *The integral cohomology algebras of ordered configuration spaces of spheres*, Doc. Math. **5** (2000), 115–139; MR 1752611.

GG03 D. Gonçalves and J. Guaschi, *On the structure of surface pure braid groups*, J. Pure Appl. Algebra **182** (2003), 33–64; MR 1977999.

GS75 P. Griffiths and W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, in *Discrete subgroups of Lie groups and applicatons to moduli* (Oxford University Press, Bombay, 1975), 31–127; MR 0419850.

Jam78 I. M. James, *On category, in the sense of Lusternik–Schnirelmann*, Topology **17** (1978), 331–348; MR 0516214.

Lat91 J.-C. Latombe, *Robot motion planning* (Kluwer, Dordrecht, 1991).

MS74 J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76 (Princeton University Press, Princeton, NJ, 1974); (University of Tokyo Press, Tokyo, 1974); MR 0440554.

PS08 C. Peters and J. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 52 (Springer, Berlin, 2008); MR 2393625.

Sch66 A. Schwarz, *The genus of a fiber space*, Amer. Math. Soc. Transl. **55** (1966), 49–140; MR 0154284.

Sha97 M. Sharir, *Algorithmic motion planning*, in *Handbook of discrete and computational geometry*, CRC Press Series on Discrete Mathematics and its Applications (CRC, Boca Raton, FL, 1997), 733–754; MR 1730196.

Tot96 B Totaro, *Configuration spaces of algebraic varieties*, Topology **35** (1996), 1057–1067; MR 1404924.

Yuz01 S. Yuzvinsky, *Orlik–Solomon algebras in algebra and topology*, Russian Math. Surveys **56** (2001), 293–364; MR 1859708.

Daniel C. Cohen cohen@math.lsu.edu
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

Michael Farber Michael.Farber@durham.ac.uk
Department of Mathematics, University of Durham, Durham DH1 3LE, UK

https://doi.org/10.1112/S0010437X10005038 Published online by Cambridge University Press