Canadian Mathematical Bulletin
doi:10.4153/CMB-2011-050-6

Classification of Inductive Limits of Outer Actions of $\mathbb{R}$ on Approximate Circle Algebras

Andrew J. Dean

Abstract. In this paper we present a classification, up to equivariant isomorphism, of C^*-dynamical systems $(A, \mathbb{R}, \alpha)$ arising as inductive limits of directed systems $\{(A_n, \mathbb{R}, \alpha_n), \varphi_{nm}\}$, where each A_n is a finite direct sum of matrix algebras over the continuous functions on the unit circle, and the α_ns are outer actions generated by rotation of the spectrum.

1 Introduction

There are now several results classifying certain kinds of actions of $\mathbb{R}$ on C^*-algebras (*cf.*, [1–3]). In each of these cases, there exists an increasing sequence of homogeneous sub-C^*-algebras that are each globally invariant under the action, which is inner when restricted to each of these sub-C^*-algebras and that has dense union. In this paper, we shall give a classification result for a class of actions that do not have this property.

Let a be an irrational multiple of 2π, and let β denote the automorphism of $\mathbb{T}$ given by rotation by a. Let $\varphi_n \colon M_{n!}(C(\mathbb{T})) \to M_{(n+1)!}(C(\mathbb{T}))$ be given by $[\varphi_n(f)](z) = \operatorname{diag}(f(z), \beta f(z), \dots, \beta^n f(z))$, and let B denote the inductive limit of the inductive system $\{A_n, \varphi_{nm}\}$, where $A_n = M_{n!}(C(\mathbb{T}))$. There is a unique automorphism, which we shall also denote β, on B that restricts to $id \otimes \beta$ on each A_n (since all of the maps in the inductive system are injective, we view them as inclusions). Suppose τ is a tracial state on B, and let τ_n denote its restriction to A_n. We have that $\tau_n(f) = \tau_{n+1}(\varphi_n(f))$, and that $|\tau_{n+1}(\varphi(f)) - \tau_{n+1}(\varphi(\beta(f)))| \leq 2 \parallel f \parallel /(n+1)$. Pushing forward in the system, we see that $\tau(f) = \tau(\beta(f))$ for all $f \in A_n$. Since the orbit of β is dense in the rotations of the circle, it follows that τ_n is invariant under rotations, and we have $\tau_n = Tr \otimes dm$, where m is the Haar measure on the circle. Thus B has a unique trace, and since it is simple, it follows from classification theory that it is a Bunce–Deddens algebra.

Rotation on the circle with unit speed gives a one parameter automorphism group on each A_n, and the connecting maps are equivariant. So in this way we get an action α of the reals on B. Observe that it has eigen-unitaries, *i.e,* unitaries u such that for some non-zero real number r we have $\alpha_t(u) = e^{irt}u$. We will show that this cannot occur in the type of C^*-dynamical system discussed in previous results.

Received by the editors January 9, 2009.
Published electronically March 24, 2011.
Research funded by NSERC
AMS subject classification: **46L57**, 46L35.
Keywords: classification, C^*-dynamical system.

Let (B,β) be a C^*-dynamical system and suppose that $1 \in B_n \subseteq B_{n+1}$ and $B = \overline{\cup_{n=1}^{\infty} B_n}$, where each B_n is a homogeneous C^*-algebra such that β restricts to an action β_n on B_n generated by an inner derivation given by a self-adjoint element $h_n \in B_n$, and that u is an eigen-unitary for β in B with eigenvalue $a \neq 0$. Then, for some large enough n, there exists a unitary $v \in B_n$ such that v is norm close to u. Consider the mapping $P\colon B \to B$ given by $P(x) = (1/2\pi)\int_0^{2\pi} e^{-iat}\beta_t(x)\,dt$. Then P is a norm decreasing linear map such that $P(u) = u$. Also, $P(B_n) = B_n$ for each n, and P commutes with equivariant $*$-homomorphisms. Let π be the canonical $*$-homomorphism onto one of the minimal quotients of B_n. Then π is equivariant when the action on $\pi(B_n)$ is that given by $\pi(h_n)$. Applying P repeatedly to v, we get a sequence $v_m = P^m(v)$ such that $\|v_m - u\|$ is decreasing. Also, each v_m, being norm close to u, is invertible. Hence $\pi(v_m)$ is invertible in $\pi(B_n)$. In a finite dimensional C^*-dynamical system, each operator is a finite linear combination of eigen-operators. Eigen-operators with eigenvalues distinct from a will be driven to zero by repeated application of the map P, so for large enough m, $\pi(v_m) = P^m(\pi(v))$ may be approximated by an invertible element x that is an eigen-operator in $\pi(B_n)$ with eigenvalue a. But then $x|x|^{-1}$ is an eigen-unitary with eigenvalue a in $\pi(B_n)$. It is easy to see that eigen-unitaries are not possible in a finite dimensional C^*-dynamical system (see [1]).

Thus, the action α does not fall into the domain of any of the classification theorems for C^*-dynamical systems found before. We shall give below a classification for a class of C^*-dynamical systems that includes this one. We refer the reader to [6] for general results about actions of $\mathbb{R}$ on C^*-algebras.

2 The Invariant

We shall take the view that a classification for a certain category consists of a functor, called the invariant, into another category such that isomorphisms from the target category lift to isomorphisms in the original category and that some class of automorphisms in the original category that it is natural to regard as trivial are each sent to an identity map. In our case, we shall regard equivariant inner automorphisms coming from unitaries in the fixed point subalgebras as trivial (see [5] for a discussion of classification). In this section, we describe the objects we are going to classify and our invariant.

Definition 2.1 We shall call a C^*-dynamical system (A,α) of special form if A is a finite direct sum of full matrix algebras over the continuous functions on the unit circle, and the action α on each summand is given by $\alpha_t(f)(z) = f(e^{iat}z)$ for some non-zero $a \in \mathbb{R}$ and all $t \in \mathbb{R}, z \in \mathbb{T}$.

Lemma 2.2 *Let (A,α) and (B,β) be C^*-dynamical systems of special form, each having one minimal direct summand, and let $\varphi\colon A \to B$ be a unital equivariant $*$-homomorphism. Suppose the dimension of an irreducible representation of B is k times that of an irreducible representation of A. Then there exist a natural number n and complex numbers $\lambda_1,\dots,\lambda_k$ of modulus 1 such that, with suitable choices of matrix units for A^α and B^β, $\varphi(f)(z) = \operatorname{diag}(f((\lambda_1 z)^n),\dots,f((\lambda_k z)^n))$ for all $f \in A$ and $z \in \mathbb{T}$.*

Proof A unital *-homomorphism $\varphi\colon A \to B$ is determined by what it does to A^α and the canonical generator $1\otimes z$ of the centre of A. Since $1\otimes z$ is an eigen-unitary, so is its image, so we have $\varphi(1\otimes z) = u\otimes z^n$ for some unitary $u \in B^\beta$ and $n \in \mathbb{N}$.If v is a partial isometry in $\varphi(A^\alpha)' \cap B^\beta$, then v commutes with $u \otimes 1$. Since $\varphi(A^\alpha)' \cap B^\beta \cong M_k$, we see that $u = w \oplus \cdots \oplus w$ (k copies) for some unitary w, with an appropriate choice of matrix units for B^β. The lemma follows. ∎

Definition 2.3 Let (A, α) be a C^*-dynamical system. An element $u \in A$ is called a partial unitary if and only if $uu^* = u^*u = (u^*u)^2$. A partial unitary u is called an eigen-partial-unitary if and only if there exists $a \in \mathbb{R}$ such that $\alpha_t(u) = e^{iat}u$ for all $t \in \mathbb{R}$. In the case that $u \neq 0$, the a is clearly unique and we call it the eigenvalue of u. (We say for convenience that 0 is an eigen-partial-unitary with eigenvalue zero.)

Define a relation $\sim$ on the set of eigen-partial-unitaries of A as follows. Say $u \sim w$ if and only if there exists a sequence partial isometries $v_n \in A^\alpha$ such that $v_n w v_n^* \to u$ and $v_n^* u v_n \to w$. Then $\sim$ is an equivalence relation. Write $\widetilde{D}(A, \alpha)$ for the set of equivalence classes of partial unitaries. If u and w are eigen-partial-unitaries with the same eigenvalue, and $u \perp w$, then $u + w$ is also an eigen-partial-unitary. Let $G(A, \alpha)$ denote the abelian group generated by $\widetilde{D}(A, \alpha)$ with the relations $[u+v] = [u]+[v]$ if v and u are orthogonal eigen-partial-unitaries with the same eigenvalue. Let $G^+(A, \alpha)$ denote the sub-semi-group of $G(A, \alpha)$ generated by the image of $\widetilde{D}(A, \alpha)$ under the canonical inclusion, and write $D(A, \alpha)$ for this image.

Next, we define a collection of additive maps $\pi_k\colon G(A, \alpha) \to G(A, a)$, $k \in \mathbb{Z}$ as follows. If u is an eigen-partial-unitary, set

$$\pi_k([u]) = \begin{cases} [u^k] & \text{if } k \geq 1, \\ [u^*u] & \text{if } k = 0, \\ [(u^*)^{|k|}] & \text{if } k \leq 0. \end{cases}$$

It follows from the universal property of $G(A, \alpha)$ that these maps extend uniquely to additive maps on all of $G(A, \alpha)$. Notice that if $m, n \in \mathbb{Z}$, then $\pi_m \circ \pi_n = \pi_{mn}$. We define an action of $\mathbb{R}$ on $(G(A, \alpha), G^+(A, \alpha), D(A, \alpha), \{\pi_k\}_{k\in\mathbb{Z}})$ by $\widetilde{\alpha}_t([u]) = [\alpha_t(u)]$ for any eigen partial unitary $u \in A$. If (B, β) is another C^*-dynamical system and $\varphi\colon A \to B$ is a *-homomorphism, then φ defines a homomorphism φ_* of groups with distinguished positive cones and subsets by $\varphi_*([u]) = [\varphi(u)]$ for any eigen-partial-unitary $u \in A$. Furthermore, this map intertwines the maps π_k. By equivariance we have $\varphi_* \circ \widetilde{\alpha}_t = \widetilde{\beta}_t \circ \varphi_*$ for all $t \in \mathbb{R}$. We call $(G(A, \alpha), G^+(A, \alpha), D(A, \alpha), \{\pi_k\}_{k\in\mathbb{Z}}, \widetilde{\alpha})$ the invariant of (A, α), and write $\operatorname{Inv}(A, \alpha)$ for it.

We define a target category C as follows. An object in C is a 5-tuple $(G, G^+, D, \{\pi_k\}, \alpha)$, where G is an abelian group, D is a distinguished generating subset of G, G^+ is the sub-semi-group of G generated by D, $\{\pi_k\}$ is a collection of additive maps from G to G indexed by the integers with $\pi_m \circ \pi_n = \pi_{mn}$ for all n and m, and α is an action of $\mathbb{R}$ on G preserving these other structures. A morphism in C is a group homomorphism from the group in the first object to that in the second that preserves all of these other structures. With φ_* defined as above, the invariant is then a functor into the target category C.

Lemma 2.4 *Let $\{(A_n,\alpha_n),\varphi_{nm}\}$ be an inductive system of C^*-dynamical systems of special form with equivariant connecting maps, and let (A,α) denote the limit C^*-dynamical system. Then*

$$A^\alpha = \overline{\bigcup_{n=1}^{\infty} \varphi_{n\infty}(A_n^{\alpha_n})}.$$

Proof Let $\{(A_n,\alpha_n),\varphi_{nm}\}$ and (A,α) be as above. We may assume without loss of generality that the connecting maps are injective. Let $a \in A^\alpha$, and let $a_n \in A_n$ be such that $\varphi_{n\infty} \to a$ in A. Let $r_1,\dots,r_k$ be the distinct eigenvalues of the canonical generators of the centres of the minimal direct summands of A_n, and consider the maps $P_m\colon A \to A$ given by

$$P_m(x) = \frac{r_1}{2\pi}\int_0^{2\pi/r_1} \alpha_{t_1}\left(\frac{r_2}{2\pi}\int_0^{2\pi/r_2} \alpha_{t_2}(\,\cdots)\, d_{t_2}\right) d_{t_1}.$$

Each $P_m\colon A \to A$ is a norm decreasing map that fixes A^α and maps A_m to $A_m^{\alpha_m}$. Thus, $P_m(a_m) \to a$, so $a \in \overline{\cup_{n=1}^{\infty}\varphi_{n\infty}(A_n^{\alpha_n})}$. ■

Lemma 2.5 *Let (A_1,α_1) and (A_2,α_2) be two C^*-dynamical systems. Then we have* $\mathrm{Inv}(A_1,\alpha_1)\oplus \mathrm{Inv}(A_2,\alpha_2) = \mathrm{Inv}(A_1\oplus A_2,\alpha_1\oplus\alpha_2)$.

Proof This is immediate from the definition. ■

Lemma 2.6 *Let $A = M_n(C(\mathbb{T}))$ and let $\alpha_t(f)(z) = f(e^{iat}z)$ for $f \in A$, $t\in\mathbb{R}$, $z\in\mathbb{T}$, where $a \neq 0$. Then $\mathrm{Inv}(A,\alpha)$ is singly generated as an object in C by $x=[p\otimes z]$, where p is a minimal projection in M_n, and has the following universal mapping property: If $(G,G^+,D,\{\pi_k\},\beta_t)$ is an object in C and g is an element of D with $\beta_{2\pi/a}(g)=g$, then there exists a morphism $\varphi\colon\ \mathrm{Inv}(A,\alpha)\to(G,G^+,D,\{\pi_k\},\beta_t)$ such that $\varphi(x)=g$.*

Proof Any eigen-operator in A is of the form $b\otimes z^m$ for some $b\in A^\alpha \cong M_n$ and $m\in\mathbb{Z}$. If the eigen-operator is a partial unitary, then b is a partial unitary. Thus, any class in $\widetilde{D}(A,\alpha)$ is a sum of finitely many classes $[e^{iamt_0}p\otimes z^m]$ where p is a minimal projection in A^α. We have $[e^{iamt_0}p\otimes z^m] = \alpha_{t_0}(\pi_m([p\otimes z]))$, so $\mathrm{Inv}(A,\alpha)$ is generated by $[p\otimes z]$ as described.

Let h be an element of $\widetilde{D}(A,\alpha)$. Then we have that h may be written in the form $h = \alpha_{t_1}(\pi_{m_1}(x))+\dots+\alpha_{t_k}(\pi_{m_k}(x))$ for some $t_1,\dots,t_k\in\mathbb{R}$ and $m_1,\dots,m_k\in\mathbb{Z}$. The mapping property will follow if we show that this expression for h is unique up to the periodicity of α. Let τ denote the trace at 1, and the additive functional on $\widetilde{D}(A,\alpha)$ it defines. We have that $\tau(\pi_0(h))=k$, so the number of terms in the expression for h is unique. In general, $\tau(\pi_n(h)) = \sum_i t_i^n$. By a straight forward induction argument, this gives that the symmetric functions in the t_is, and so the t_is themselves, are uniquely determined up to reordering. Consider the additive function $\varphi_n\colon \widetilde{D}(A,\alpha)\to\mathbb{R}$ given by $\varphi_n([u]) = \int_{\mathbb{T}} tr(u(z)z^{-n})\,dm$. We have that $\varphi_n(\pi_l(x)) = \delta_{nl}$. This shows that the m_is are determined with their multiplicities. Finally, we may decompose our expression for h into terms with the same m_i and use the argument with τ above on each term separately, so the t_is and m_is must match. ■

Lemma 2.7 *Let $\{(A_n, \alpha_n), \varphi_{nm}\}$ be an inductive system of C^*-dynamical systems of special form with equivariant connecting maps, and let (A, α) denote the limit C^*-dynamical system. Then* $\operatorname{Inv}(A, \alpha) = \lim\{\operatorname{Inv}(A_n, \alpha_n), \operatorname{Inv}(\varphi_{nm})\}$ *in the category C.*

Proof It will suffice to show that all of the canonical generators are in the images of the invariants of the finite stages, and that the canonical relations are also realised at finite stages. We begin by showing the first statement. Let u be an eigen-partial-unitary in A. Then we may approximate u by a partial unitary w in A_n for some n, and since u^*u and w^*w are equivalent projections in A^α, we may suppose $u^*u = w^*w$.

Suppose that $\alpha_t(u) = e^{iat}u$, where $a \neq 0$. Consider the map $P\colon A \to A$ given by $P(x) = (1/2\pi)\int_0^{2\pi} \alpha_t(x)e^{-iat}\,dt$. Then P is a norm decreasing linear map such that $P(u) = u$. We can approximate the components of w in each minimal direct summand of A_n by a finite linear combination of the form $a_1 \otimes z^{n_1} + \cdots + a_k \otimes z^{n_k}$. Applying P repeatedly, we send all the elementary tensors with eigenvalues distinct from a to zero, while improving the approximation of u. Call the new approximation, with eigenvalue a, w_1. The components in each minimal direct summand of A_n of this element w_1 are of the form $b \otimes z^m$. It follows that b is approximately a partial unitary, and since being a partial unitary is a stable relation, we may assume that b, and hence w_1, is a partial unitary. Since $w_1^*u \in A^\alpha$, we can choose a sequence v_m with $v_m \in A_m$ such that $v_m \to w_1^*u$. It follows that $[u] = [w_1]$ in $\widetilde{D}(A, \alpha)$. Finally, we deal with the case where $u \in A^\alpha$. This follows easily from the observation that being a partial unitary is a stable relation and from Lemma 2.4.

Next, we show that the relations come from the finite stages. Suppose that $w_1, w_2 \in A_m$, are eigen-partial-unitaries, and that $[w_1] = [w_2]$ in $\widetilde{D}(A, \alpha)$. Then there exists a sequence $\{v_n\}$ of partial isometries in A^α with $v_n w_1 v_n^* \to w_2$. Using Lemma 2.4, we may suppose $v_n \in A_n^{\alpha_n}$. Also, we have that w_1 and w_2 have the same eigenvalue. Each of w_1 and w_2 may be written as a finite linear combination of terms of the form $a_i \otimes z^{n_i}$, with the a_i partial unitaries with finite spectrum having orthogonal support projections for different is. Moving further along in the inductive system if necessary, $v_n w_1 v_n^* \to w_2$ implies that the eigenvalues of these a_is match for the two unitaries, and that the corresponding spectral projections are equivalent. It follows that $[w_1] = [w_2]$ in $\operatorname{Inv}(A_n, \alpha_n)$ for some large enough n. ∎

3 Classification

The proof of our main theorem (Theorem 3.3) follows the pattern known as Elliott's intertwining argument (see [4] for a discussion of this type of argument).

Lemma 3.1 *Let (A, α) and (B, β) be two C^*-dynamical systems of special form, and let $\varphi\colon \operatorname{Inv}(A, \alpha) \to \operatorname{Inv}(B, \beta)$ be a morphism in the category C. Then there exists an equivariant *-homomorphism $\widetilde{\varphi}\colon A \to B$ such that $\varphi = \widetilde{\varphi}_*$.*

Proof Let (A, α), (B, β), and $\varphi\colon \operatorname{Inv}(A, \alpha) \to \operatorname{Inv}(B, \beta)$ be as above. Consider first the case where A and B each have one minimal direct summand and $\varphi([1_A]) = [1_B]$. Let A and B be identified with $M_n(C(\mathbb{T}))$ and $M_{nk}(C(\mathbb{T}))$ in such a way as to express them as C^*- dynamical systems of special form with eigenvalues a and b, respectively. Let x denote the canonical generator of $[e_{11} \otimes z]$ of $\operatorname{Inv}(A, \alpha)$ as described

https://doi.org/10.4153/CMB-2011-050-6 Published online by Cambridge University Press

in Lemma 2.6, and let $y = [v_{11} \otimes z]$ be one for $\mathrm{Inv}(B, \beta)$. From Lemma 2.6, since $\varphi(x) \in \widetilde{D}(B, \beta)$, $\varphi(x) = \beta_{t_1}(\pi_n(y)) + \cdots + \beta_{t_k}(\pi_n(y))$ for some k, n, and $t_1, \dots, t_k$. The $*$-homomorphism $\widetilde{\varphi}(f)(z) = \mathrm{diag}(f((e^{ibt_1}z)^n), \dots, f((e^{ibt_k})^n))$ thus meets our requirements. ■

Now we consider the general case. From Lemma 2.5, we may reduce to the case where both A and B have one minimal direct summand. Cutting down B by a projection in B^β with class $\varphi([1])$in $\widetilde{D}(B, \beta)$ then brings us to the situation dealt with above.

Lemma 3.2 *Let (A, α) and (B, β) be two C^*-dynamical systems of special form, and let φ and ψ be two equivariant $*$-homomorphisms from A to B such that* $\mathrm{Inv}(\varphi) = \mathrm{Inv}(\psi)$. *Then there exists a unitary $u \in B^\beta$ such that $\psi = Ad\, u \circ \varphi$.*

Proof Let (A, a), (B, β), φ, and ψ be as above. Notice that any tracial state τ on A defines an additive homomorphism, which we shall also call τ, from $G(A, \alpha)$ to $\mathbb{C}$ such that $\tau([v]) = \tau(v)$ for any eigen-partial-unitary u.

Consider first that case in which A and B have one minimal direct summand and the maps are unital. Then φ and ψ are in the form given by Lemma 2.2. Suppose that k is the K_0 multiplicity of the maps, and let $\lambda_1, \dots, \lambda_k$ be the complex parameters of modulus 1 associated with φ, and let $\gamma_1, \dots, \gamma_k$ be those for ψ. Let x denote a canonical generator of $\mathrm{Inv}(A, \alpha)$ in $G(A, a)$ as described in Lemma 2.6, and let y be one for $\mathrm{Inv}(B, \beta)$. Then for each of φ and ψ, the integer parameter for each (the winding number) is given by the ratio of the period of $\widetilde{\beta}$ on the image of x to the period of y, so these two numbers are the same, n say. Now fix identifications of A with $M_m(C(\mathbb{T}))$ and of B with $M_{mk}(C(\mathbb{T}))$ that express the systems in special form. Let τ_1 denote the trace associated with the fibre at 1. We then have $\tau_1(\varphi(\pi_l(x)) = \sum_{j=1}^{k}(\lambda_j)^l = \tau_1(\psi(\pi_l(x)) = \sum_{j=1}^{k}(\gamma_j)^l$ for all integers l. It is now straightforward to show that the symmetric functions in the λs are equal to those in the γs, so that after reordering, we have $\lambda_1 = \gamma_1, \lambda_2 = \gamma_2, \dots, \lambda_k = \gamma_k$. It is then immediate that there exists a unitary $u \in B^\beta$ such that $\psi = Ad\, u \circ \varphi$.

Now consider the general case. We may use Lemma 2.5 to reduce to the case where both A and B have only one minimal direct summand. Since $\psi([1]) = \varphi([1])$, we may conjugate by a unitary in B^β to get $\varphi(1) = \psi(1)$, and cutting B down by this invariant projection brings us to the case discussed above. ■

Theorem 3.3 *Let (A, α) and (B, β) be two C^*-dynamical systems that arise as inductive limits of C^*-dynamical systems of special form, and suppose that $\widetilde{\eta}\colon$* $\mathrm{Inv}(A, \alpha) \to \mathrm{Inv}(B, \beta)$ *is an isomorphism of invariants. Then there exists an equivariant isomorphism $\eta\colon A \to B$ such that $\widetilde{\eta} = \mathrm{Inv}(\eta)$.*

Proof Let $\{(A_n, \alpha_n), \varphi_{nm}\}$, $\{(B_n, \beta_n), \psi_{nm}\}$, $(A, \alpha), (B, \beta)$, and $\eta\colon \mathrm{Inv}(A, \alpha) \to \mathrm{Inv}(B, \beta)$ be as above, and suppose that ζ is the inverse of η.

Using Lemmas 2.5, 2.6, and 2.7, and a standard argument, we may pull back the pair of inverse isomorphisms η, ζ to a commutative diagram (possibly after passing

https://doi.org/10.4153/CMB-2011-050-6 Published online by Cambridge University Press

to subsequences and renumbering):

$$\begin{tikzcd}
\mathrm{Inv}(A_1,\alpha_1) \arrow[r] \arrow[d, "\gamma_1"] & \mathrm{Inv}(A_2,\alpha_2) \arrow[r] \arrow[d, "\gamma_2"] & \cdots \arrow[r] & \mathrm{Inv}(A,\alpha) \arrow[d, shift left, "\gamma"] \\
\mathrm{Inv}(B_1,\beta_1) \arrow[r] \arrow[ur, "\zeta_1"'] & \mathrm{Inv}(B_2,\beta_2) \arrow[r] \arrow[ur, "\zeta_2"'] & \cdots \arrow[r] & \mathrm{Inv}(B,\beta). \arrow[u, shift left, "\zeta"]
\end{tikzcd}$$

By Lemma 3.1, we may lift each of the homomorphisms between the invariants at the finite stages to equivariant *-homomorphisms between the C^*-algebras, so we have a not necessarily commutative diagram

$$\begin{tikzcd}
(A_1,\alpha_1) \arrow[r] \arrow[d, "\gamma_1"] & (A_2,\alpha_2) \arrow[r] \arrow[d, "\gamma_2"] & \cdots \arrow[r] & (A,\alpha) \\
(B_1,\beta_1) \arrow[r] \arrow[ur, "\zeta_1"'] & (B_2,\beta_2) \arrow[r] \arrow[ur, "\zeta_2"'] & \cdots \arrow[r] & (B,\beta),
\end{tikzcd}$$

where the images of the maps in this diagram under the invariant are the corresponding maps in the diagram above.

Now consider the first triangle in the diagram above:

$$\begin{tikzcd}
(A_1,\alpha_1) \arrow[r] \arrow[d, "\gamma_1"] & (A_2,\alpha_2) \ . \\
(B_1,\beta_1). \arrow[ur, "\zeta_1"'] &
\end{tikzcd}$$

Using Lemma 3.2, we may choose a unitary $u \in A_2^{\alpha_2}$ such that replacing ζ_1 by $Ad\, u \circ \zeta_1$ makes the triangle commute. This will not change the image of this map under the invariant. Proceeding left to right through the diagram above, we may alter each of the vertical maps in turn, so that each triangle commutes. Thus we may replace the diagram above with one that is commutative. This gives rise to a pair of inverse isomorphisms between the limit algebras, and since all of the vertical maps are equivariant, these will be equivariant as well. That they induce the given maps on the invariant follows from functoriality. ∎

4 Closing Remarks

It should be noted that, since they are not simple or of real rank zero in general, the C^*-algebras discussed in this paper do not fall into the classes that have been classified by the Elliott program.

References

[1] A. J. Dean, *Classification of AF flows.* Canad. Math. Bull. **46**(2003), no. 2, 164–177. doi:10.4153/CMB-2003-018-0

[2] ———, *Inductive limits of inner actions on approximate interval algebras generated by elements with finite spectrum.* J. Ramanujan Math. Soc. **24**(2009), no. 4, 323–339.
[3] ———, *Classification of certain inductive limit type actions of* $\mathbb{R}$ *on* C^**-algebras.* J. Operator Theory **61**(2009), no. 2, 439–457.
[4] G. A. Elliott, *On the classification of* C^**-algebras of real rank zero.* J. Reine Angew. Math. **443**(1993), 179–219. doi:10.1515/crll.1993.443.179
[5] ———, *Towards a theory of classification.* Adv. Math. **223**(2010), no. 1, 30–48. doi:10.1016/j.aim.2009.07.018
[6] S. Sakai, *Operator algebras in dynamical systems. The theory of unbounded derivations in* C^**-algebras.* Encyclopedia of Mathematics and its Applications, 41, Cambridge University Press, Cambridge, 1991.

Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON, P7B 5E1
e-mail: andrew.j.dean@lakeheadu.ca