THE NUMBER OF VERY REDUCED $4 \times n$ LATIN RECTANGLES

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To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction. Two permutations (displayed in the two rows)

$$\begin{array}{c} a_1\, a_2\, a_3 \ldots a_n, \\ b_1\, b_2\, b_3 \ldots b_n, \end{array}$$

of the integers $1, 2, \ldots, n$ are called discordant if $a_i \neq b_i$, $i = 1, 2, \ldots, n$. Let $v(4, n)$, $n \geqslant 4$, be the number of permutations discordant with the three permutations

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \ldots & n, \\ n & 1 & 2 & 3 & \ldots & n-1, \\ n-1 & n & 1 & 2 & \ldots & n-2. \end{array}$$

The purpose of this note is to establish that

$$v(4, n) = \sum_{i=0}^{n} (-1)^i g(n, 3; i)(n-i)!, \qquad n \geqslant 4, \tag{1}$$

where

$$g(n, 3; i) = \begin{cases} \displaystyle\sum_{\alpha=0}^{[i/2]} \sum_{\beta=0}^{m} \frac{n}{n-i} \binom{n+\alpha-i-1}{\alpha} \binom{n-i}{\beta} 2^{\beta} \binom{n-\alpha-1}{i-2\alpha-\beta}, & 0 \leqslant i < n, \\ \displaystyle 3 + \sum_{\alpha=1}^{[n/2]} \frac{n}{\alpha} \binom{n-\alpha-1}{\alpha-1}, & i = n, \end{cases} \tag{2}$$

where $m = \min(n-i, i-2\alpha)$.

The numbers $v(4, n)$ arose in an unsuccessful attempt to obtain an expression for the number of $4 \times n$ Latin rectangles. They are treated in (**4**, p. 231; **6**), but no explicit expression for them is given.

A $k \times n$ Latin rectangle, $1 \leqslant k \leqslant n$, is a rectangular array of k mutually discordant permutations (the rows of the array) of degree n. If $2 \leqslant k \leqslant n$ and the first row is $1\ 2\ 3 \ldots n$, we call the rectangle reduced. If $3 \leqslant k \leqslant n$ and the first $k - 1$ rows are

$$\begin{array}{ccccccccc} 1 & 2 & 3 & \ldots & k-2 & k-1 & k & \ldots & n, \\ n & 1 & 2 & \ldots & k-3 & k-2 & k-1 & \ldots & n-1, \\ n-1 & n & 1 & \ldots & k-4 & k-3 & k-2 & \ldots & n-2, \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & & \cdot \\ n-k+3 & n-k+4 & & \ldots & n & 1 & 2 & \ldots & n-k+2, \end{array} \tag{3}$$

Received May 11, 1967.

we call the rectangle very reduced. Let $l(k, n)$, $r(k, n)$, and $v(k, n)$ denote, respectively, the number of $k \times n$ Latin rectangles, $k \times n$ reduced Latin rectangles, and $k \times n$ very reduced Latin rectangles. Although these numbers are of considerable interest and have been much investigated, very little is known about them. Erdös and Kaplansky **(1)** proved that

$$l(k, n) \sim n!^k e^{-\binom{k}{2}} \tag{4}$$

where $k < (\log n)^{3/2}$. The validity of (4) when $k < n^{1/3}$ was proved by Yamamoto **(5)**.

Exact expressions are known only for $k = 1, 2, 3$, namely:

$$r(2, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!, \qquad n \geqslant 1 \tag{5}$$

(4, p. 59**)**,

$$v(3, n) = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i} \binom{2n-i}{i} (n-i)!, \qquad n \geqslant 1 \tag{6}$$

(2), and

$$r(3, n) = \sum_{i=0}^{[n/2]} r(2, i) r(2, n-i) v(3, n-2i) \tag{7}$$

where $v(3, 0) = 1$ **(4**, p. 206**)**. Of course

$$l(1, n) = n!, \qquad l(k, n) = n! r(k, n), \quad k = 2, 3, \ldots . \tag{8}$$

The number $r(2, n)$ is the solution to the "*problème des recontres*," which, in its simplest form, asks for the number of permutations discordant with $1\ 2\ 3 \ldots n$. The number $2n!\, v(3, n)$ is the solution to the "*problème des ménages*," which asks for the number of ways of seating n married couples at a circular table, men and women in alternate positions, so that no man sits next to his wife. In this spirit, $2n!\, v(k, n)$ is the number of ways of seating n married couples at a circular table, men and women in alternate positions, so that every man finds that none of the $[k/2]$ closest women on his left and none of the $[(k-1)/2]$ closest women on his right is his wife.

2. The number $v(k, n)$**.** A permutation is said to contain the event $[s\ t]$ if, in the permutation, s is in position t. Thus, the permutation $a_1 a_2 \ldots a_n$ contains the events $[a_1\ 1], [a_2\ 2], \ldots, [a_n\ n]$. Two distinct events $[s\ t]$, $[p\ q]$ are inconsistent if $s = p$ or $t = q$; otherwise they are consistent. A collection of events is consistent if every two of them are consistent. Now suppose i consistent events are chosen. Then it is not difficult to see that the number of permutations (of degree n), each containing all of the i chosen events, is $(n-i)!$ (The number depends only on the number of events, and is the same for any choice of i consistent events **(3)**!)

Now consider the evaluation of the number $v(k, n)$, $3 \leqslant k \leqslant n$. A permutation is discordant with the $k - 1$ permutations (3) if it contains none of the events

$$
(9)\qquad
\begin{array}{ccccc}
[1\ 1] & [1\ 2] & [1\ 3] & \dots & [1\ k-1] \\
[2\ 2] & [2\ 3] & [2\ 4] & \dots & [2\ k] \\
[3\ 3] & [3\ 4] & [3\ 5] & \dots & [3\ k+1] \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
[n-1\ n-1] & [n-1\ n] & [n-1\ 1] & \dots & [n-1\ k-3] \\
[n\ n] & [n\ 1] & [n\ 2] & \dots & [n\ k-2]
\end{array}
$$

This array has $k - 1$ events in each row and $k - 1$ events in each "diagonal" ("rising circular diagonal"). That is, the ith row is

$$[i\ i]\,[i\ i+1]\,[i\ i+2]\dots[i\ i+k-2], \qquad i = 1, \dots, n-k+2;$$
$$[i\ i]\,[i\ i+1]\dots[i\ n-1]\,[i\ n]\,[i\ 1]\dots[i\ i+k-n-2], \qquad i = n-k+3, \dots, n.$$

The ith "diagonal" is

$$[i\ i]\,[i-1\ i]\,[i-2\ i]\dots[1\ i]\,[n\ i]\,[n-1\ i]\dots[n-k+i+2\ i], \qquad i = 1, \dots, k-2;$$
$$[i\ i]\,[i-1\ i]\dots[i-k+2\ i], \qquad i = k-1, \dots, n.$$

The "diagonals" can be visualized by thinking of the array (9) as written on a cylinder so that the first and last rows are adjacent. Now it is easy to see that two events in (9) are consistent if and only if they are not in the same row and not in the same "diagonal." Let $g(n, k-1; i)$ denote the number of ways of choosing i consistent events from (9). Then, by the Principle of Inclusion and Exclusion,

$$(10)\qquad v(k, n) = \sum_{i=0}^{n} (-1)^i g(n, k-1; i)(n-i)!, \qquad 4 \leqslant k \leqslant n,$$

where $g(n, k-1; 0) = 1$, and the problem becomes that of evaluating $g(n, k-1; i)$, $4 \leqslant k \leqslant n$, $1 \leqslant i \leqslant n$. More conveniently, $g(n, k; i)$ is the number of ways of choosing i "consistent" entries from the array

$$
(11)\qquad
\begin{array}{ccccc}
1 & \dots & 2 & \dots & k \\
k+1 & \dots & k+2 & \dots & 2k \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
(n-2)k+1 & \dots & (n-2)k+2 & \dots & (n-1)k \\
(n-1)k+1 & \dots & (n-1)k+1 & \dots & nk
\end{array},
$$

(the last row is considered adjacent to the first row): no two chosen entries come from the same row or from the same "diagonal."

https://doi.org/10.4153/CJM-1967-092-5 Published online by Cambridge University Press

3. The numbers $g(n, 1; i)$, $g(n, 2; i)$, $g(n, 3; i)$. In the case $k = 1$, array (11) becomes the array

$$(12)\qquad \begin{array}{c} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ n \end{array}$$

Every row and every "diagonal" consists of just one entry, so any i entries are consistent. Hence

$$g(n, 1; i) = \binom{n}{i}$$

and this, with (10) when $k = 2$, yields (5).

In the case $k = 2$, array (11) becomes the array

$$(13)\qquad \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2n-3 & 2n-2 \\ 2n-1 & 2n \end{array}$$

If we rearrange these $2n$ entries to form a circle, as in

$$(14)\qquad 1\ 2\ 3\ \ldots\ 2n-2\ \ 2n-1\ \ 2n$$

(where 1 and 2n are considered adjacent), we see that two entries from (13) are consistent if they are not adjacent in (14). Thus, $g(n, 2; i)$ is the number of ways of choosing i objects, no two consecutive, from $2n$ (unlike) objects arranged in a circle, that is **(5)**

$$g(n, 2; i) = \frac{2n}{2n-i}\binom{2n-i}{i}.$$

This, with (10) when $k = 3$, yields (6).

In the case $k = 3$, array (11) becomes the array

$$(15)\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 3n-5 & 3n-4 & 3n-3 \\ 3n-2 & 3n-1 & 3n \end{array}$$

https://doi.org/10.4153/CJM-1967-092-5 Published online by Cambridge University Press

Arranging the $3n$ entries in a circle, as in

$$(16) \qquad 1\ 2\ 3\ 4\ 5\ 6\ \ldots\ 3n-2\ \ 3n-1\ \ 3n$$

(where the entries 1 and $3n$ are considered adjacent), we see that in order to choose i consistent entries from (15), we must choose them so that in (16):

A. Two adjacent entries are chosen only if they are one of the pairs of $(3\ 4)$, $(6\ 7)$, $(9\ 10), \ldots, (3n-3\ \ 3n-2)$, $(3n\ \ 1)$.

B. Not both of the entries in any of the pairs $(1\ 3)$, $(2\ 4)$, $(3\ 5), \ldots$, $(3n-2\ \ n)$, $(3n-1\ \ 1)$, $(3n\ \ 2)$ are chosen.

C. Not both of the entries in any of the pairs $(3\ 7)$, $(6\ 10)$, $(9\ 13), \ldots$, $(3n-6\ \ 3n-2)$. $(3n-3\ \ 1)$, $(3n\ \ 4)$ are chosen.

Note that any choice of i entries from (16) can be represented by a circular array of i symbols 1 and $3n-i$ symbols 0 (the 1's representing the chosen entries, the 0's representing the entries not chosen) with one of these $3n$ symbols marked with an asterisk to indicate that it represents the entry 1 of the array (16).

Now we describe how to arrange i 1's and $3n-i$ 0's in a circle, and mark one of them with an asterisk so that the arrangement represents a choice of i $(0 \leqslant i < n)$ entries from (16) which satisfy conditions A,B,C.

Suppose the choice contains precisely α $(0 \leqslant \alpha \leqslant i/2)$ of the pairs listed in condition A. In the circular arrangement there will be α pairs 11; indeed, because of condition B these pairs will each appear in a sequence 001100. Let X denote such a sequence. The $i-2\alpha$ remaining 1's will each appear in a sequence 010; let Y denote such a sequence. There are 4α 0's in the X's, $2(i-2\alpha)$ 0's in the Y's, and so there are $3n-i-4\alpha-2(i-2\alpha) = 3(n-i)$ 0's not in the X's or Y's. Let Z denote a sequence 000.

Step 1. Place $n-i$ symbols Z in a circle. They determine $n-i$ cells—the spaces between the Z's. Label one of the Z's with a distinguishing mark, say $\bar{Z}$, so that henceforth any cell (even those yet to be determined) can be distinguished by its position relative to $\bar{Z}$.

Step 2. Place α symbols X, without restriction, into the $n-i$ cells. This (placing α like objects into $n-i$ different cells) can be done in

$$\binom{\alpha+n-i-1}{\alpha}$$

ways.

Now we shall place the $i-2\alpha$ symbols Y. First note that we have $n-i+\alpha$ symbols placed in a circle, determining $n-i+\alpha$ cells. Each of these is a permissible place for the Y's. Furthermore, each Z determines two cells Z: $0 \downarrow 0 \downarrow 0$ (indicated by the arrows) and, because of conditions B and C, not both of them can be occupied by Y's. Distinguish the two cells determined in each Z by calling one of them an R-cell and the other an L-cell. Suppose that precisely β $(0 \leqslant \beta \leqslant \min(n-i, i-2\alpha))$ of these (the $2(n-i)$ R- and L-cells) are each occupied by one or more Y's.

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Step 3. Choose β of the Z's. This can be done in

$$\binom{n-i}{\beta}$$

ways.

Step 4. Replace each Z by 000 (replace $\bar{Z}$ by $\bar{0}00$; we still need that distinguishing bar) and, in each of the β Z's chosen in Step 3, choose either the R-cell or the L-cell. This can be done in 2^β ways.

Place a single Y into each of the β R- and L-cells chosen in Step 4.

Step 5. Place, without restriction, the remaining $i - 2\alpha - \beta$ Y's into the $n - i + \alpha + \beta$ permissible cells (the $n - i + \alpha$ cells described in Step 1 and the β cells already occupied by single Y's). This can be done in

$$\binom{i-2\alpha-\beta+n-i+\alpha+\beta-1}{i-2\alpha-\beta} = \binom{n-\alpha-1}{i-2\alpha-\beta}$$

ways.

Replace each X by 001100 and each Y by 010. We have, so far, constructed

$$\binom{\alpha+n-i-1}{\alpha}\binom{n-i}{\beta}2^\beta\binom{n-\alpha-1}{i-2\alpha-\beta}$$

circular arrangements of i 1's and $3n - i$ 0's with one 0 topped by a bar. Now focus attention on this $\bar{0}$ and every third entry, counting clockwise around the circle and starting with $\bar{0}$.

Step 6. Mark one of the n entries described above with an asterisk. This can be done in n ways.

Erase the bar, and the arrangements now fall into sets of $n - i$ each which are alike by rotation.

Summing over β, we have counted

$$\sum_{\beta=0}^{m} \frac{n}{n-i}\binom{\alpha+n-i-1}{\alpha}\binom{n-i}{\beta}2^\beta\binom{n-\alpha-1}{i-2\alpha-\beta},$$

where $m = \min(n - i, i - 2\alpha)$, circular arrangements, of i 1's and $3n - i$ 0's, representing (as the reader can justify) precisely those choices of i entries from (16) which satisfy conditions A, B, C, and containing exactly α of the pairs listed in condition A. Summing over α, we have the first part of (2).

There remains to determine the number $g(n, 3; n)$. Here there are no symbols Z. When $0 < \alpha \leqslant n/2$, we place α X's in a circle, label the α cells so they can be distinguished, place the $n - 2\alpha$ Y's into the α cells in

$$\binom{n-2\alpha+\alpha-1}{n-2\alpha} = \binom{n-\alpha-1}{\alpha-1}$$

ways, replace the X's by 001100, replace the Y's by 010, mark one of n permissible entries by an asterisk, erase the labels which distinguish the cells (so the arrangements fall into sets of α each), and find

$$\frac{n}{\alpha}\binom{n-\alpha-1}{\alpha-1} \tag{17}$$

such arrangements.

When $\alpha = 0$ there are no Z's and no X's, s we place the n Y's in a circle, label the n cells, replace each Y by 010, mark one of the $3n$ (!!) entries by an asterisk, erase the labels which distinguish the cells (so the arrangements fall into sets of n each), and find $3n/n = 3$ arrangements.

Summing (17) over $\alpha = 1, 2, \ldots, [n/2]$ and adding 3 (for the case $\alpha = 0$) we have the second part of (2).

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