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Explicit birational geometry of 3-folds and 4-folds of general type, III

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Abstract

Nonsingular projective 3-folds V of general type can be naturally classified into 18 families according to the *pluricanonical section index* $\delta(V) := \min\{m \mid P_m \geqslant 2\}$ since $1 \leqslant \delta(V) \leqslant 18$ due to our previous series (I, II). Based on our further classification to 3-folds with $\delta(V) \geqslant 13$ and an intensive geometrical investigation to those with $\delta(V) \leqslant 12$, we prove that $\mathrm{Vol}(V) \geqslant \frac{1}{1680}$ and that the pluricanonical map Φ_m is birational for all $m \geqslant 61$, which greatly improves known results. An optimal birationality of Φ_m for the case $\delta(V) = 2$ is obtained. As an effective application, we study projective 4-folds of general type with $p_g \geqslant 2$ in the last section.

1. Introduction

One of the fundamental aspects of birational geometry is to understand the behavior of the natural pluricanonical map Φ_m of any variety for any $m \in \mathbb{Z}_{>0}$. The induced fibrations possibly reduce the studies to lower-dimensional situations. Varieties of general type, which are those with birational pluricanonical maps Φ_m for sufficiently large m, are therefore considered as the basic building blocks of varieties.

For varieties of general type, a key problem is to find an effective integer $m > 0$ so that Φ_m is birational. The remarkable theorem of Hacon and McKernan [HM06], Takayama [Tak06], and Tsuji [Tsu06] says that there is a constant $c(n)$ so that Φ_m is birational for all n-dimensional varieties of general type and for all $m \geqslant c(n)$. However, these constants are explicitly known only when $n \leqslant 3$.

In fact, the problem is almost equivalent to finding a practical lower bound of the canonical volume which computes the rate of growth of plurigenera, or equivalent to find m_0 such that plurigenus P_{m_0} is sufficiently large. One may also refer to the nice survey article by Hacon and McKernan [HM10] for various boundedness results in birational geometry.

The motivation of this series is to study birational geometry of 3-folds and higher-dimensional varieties of general type. The main purpose is to investigate the following open problem.

Open problem 1.1. Find optimal constants $v_3 \in \mathbb{Q}_{>0}$ and $b_3 \in \mathbb{Z}_{>0}$ so that, for all nonsingular projective 3-folds V of general type:

(i) $\mathrm{Vol}(V) \geqslant v_3$; and
(ii) Φ_m is birational for all $m \geqslant b_3$.

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Recall that we have proved the following theorem.

Theorem 1.2 [CC10b, Theorems 1.1, 1.2]. *Let V be a nonsingular projective 3-fold of general type. Then:*

(1) $\mathrm{Vol}(V) \geqslant \frac{1}{2660}$;
(2) *there exists a positive integer $m_0(V) \leqslant 18$ so that $P_{m_0} \geqslant 2$;*
(3) *the pluricanonical map Φ_m is birational onto its image for all $m \geqslant 73$.*

For more results on explicit birational geometry of 3-folds of general type, one may refer to our previous papers [CC10a, CC10b].

In order to formulate our main statements of this article, we need to recall some general results and introduce some definition. Given a projective variety V of general type, there exists a minimal model X birational to V (cf. [BCHM10]). Thanks to the Riemann–Roch formula and vanishing theorem, $\mathrm{Vol}(V) = K_X^{\dim X}$. Note that in dimension three or higher, a minimal model may have singularities. Hence, $K_X^{\dim X}$ is just a positive rational number.

A minimal model has at worst terminal singularities. In dimension three, terminal singularities were classified by Mori. A three-dimensional terminal singularity is one of the following: a terminal quotient singularity of type $(1/r)(1, -1, b)$ for some b relatively prime to r which we usually denote it as (b, r) for short, an isolated cDV point, a quotient of an isolated cDV point. It is well known to experts that a three-dimensional terminal point can be deformed into a collection of terminal quotient singularities, which is called *basket of singularities*. An important feature of three-dimensional birational geometry is the singular Riemann–Roch formula due to Reid [Rei87]:

$$\chi(X, mK_X) = \frac{m(m-1)(2m-1)K_X^3}{12} + (1-2m)\chi(X, \mathcal{O}_X) + l_m,$$

where l_m denotes the contribution of singularities which can be computed by baskets. It follows that all plurigenera and hence canonical volume of a minimal 3-fold X are completely determined by $P_2(X)$, $\chi(X, \mathcal{O}_X)$ and baskets of singularities B_X, of which we called such a triple *the weighted basket* of X. For the basic properties of weighted baskets, one may refer to [CC10a, § 3]. Since our problems are birational in nature, the studies of nonsingular threefold V is equivalent to the studies of its minimal model X. In particular, we may and do consider the weighted basket of V as the weighted basket of its minimal model X.[1]

Next, we would like to define the *pluricanonical section index* (or, in short, the *ps-index*)

$$\delta(V) := \min\{m \mid m \in \mathbb{Z}_{>0}, P_m(V) \geqslant 2\},$$

which is clearly a birational invariant. By Theorem 1.2, we have $\delta(V) \leqslant 18$ for any 3-fold V of general type. Note that 3-folds V with $\delta(V) = 1$ (i.e. $p_g(V) \geqslant 2$) have been studied intensively in [Che03, Che07] where optimal results are realized. Threefolds of general type with $\delta(V) \geqslant 2$ are far from being clear. Sometimes we use the symbol $\delta(X)$ directly since X is birationally equivalent to V.

Example 1.3. The 'worst' known minimal 3-fold is the weighted hyper-surface $X := X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$ (cf. [Ian00]) which has the invariants: $\delta(X) = 10$ and $\mathrm{Vol}(X) = K_X^3 = \frac{1}{420}$. Also Φ_{26} is not birational.

[1] Even though minimal models are not necessarily unique, it is known that two birational minimal models are connected by flops (cf. [Kaw08]). Together with the fact that a three-dimensional flop preserves singularity types (cf. [Kol89]), it follows that baskets of V are independent of choices of minimal models.

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In this paper, we mainly investigate projective 3-folds of general type with $\delta(V) \geqslant 2$. Our main results are as follows.

Theorem 1.4 (Theorem 5.1). *Let V be a nonsingular projective 3-fold of general type with $\delta(V) \geqslant 13$. Then its weighted basket $\mathbb{B} = \{B_V, P_2(V), \chi(\mathcal{O}_V)\}$ belongs to one of the types in Tables F0, F1 and F2 in Appendix A and the following is true:*

(1) $\delta(V) = 18$ *if and only if* $\mathbb{B}(V) = \{B_{2a}, 0, 2\}$;
(2) $\delta(V) \neq 16, 17$;
(3) $\delta(V) = 15$ *if and only if* $\mathbb{B}(V)$ *belongs to one of the types in Table F1;*
(4) $\delta(V) = 14$ *if and only if* $\mathbb{B}(V)$ *belongs to one of the types in Table F2;*
(5) $\delta(V) = 13$ *if and only if* $\mathbb{B}(V) = \{B_{41}, 0, 2\}$;

where B_{2a} and B_{41} can be found in Table F0.

Some other results for 3-folds with large $\delta(V)$ are given in § 4. For example, one has the following corollary.

Corollary 1.5 (Corollary 4.8). *Let V be a nonsingular projective 3-fold of general type with* $\mathrm{Vol}(V) < \frac{1}{336}$. *Then* $\delta(V) \geqslant 8$.

We also prove the following result.

Theorem 1.6. *Let V be a nonsingular projective 3-fold of general type. Then:*

(1) Φ_m *is birational for all* $m \geqslant 61$;
(2) $\mathrm{Vol}(V) \geqslant \frac{1}{1680}$; *furthermore,* $\mathrm{Vol}(V) = \frac{1}{1680}$ *if and only if* $\mathbb{B}(V) = \{B_{7a}, 0, 2\}$ *or* $\{B_{36a}, 0, 2\}$, *where B_{7a} and B_{36a} can be found in Table F2.*

A direct by-product of our method is the following.

Corollary 1.7. *Let V be a nonsingular projective 3-fold of general type with $p_g(V) = 1$. Then:*

(1) $\mathrm{Vol}(V) \geqslant \frac{1}{75}$;
(2) Φ_m *is birational for all* $m \geqslant 18$.

In the second part of this paper we prove some optimal results on 3-folds with $\delta(V) = 2$.

Theorem 1.8. *Let V be a nonsingular projective 3-fold of general type with $\delta(V) \leqslant 2$. Then:*

(1) Φ_m *is birational for all* $m \geqslant 11$;
(2) *if Φ_{10} is not birational, then $0 \leqslant \chi(\mathcal{O}_V) \leqslant 3$ and $|2K_V|$ is composed of a rational pencil of $(1, 2)$ surfaces; furthermore, $\#\{\mathbb{B}(V)\} < +\infty$ and the initial basket B^0 of B_V belongs to one of the types in Tables II1, II2 and II3 in Appendix A.*

The following examples show that our results in Theorem 1.8 are optimal.

Example 1.9 (Iano-Fletcher [Ian00, pp. 151–153]). (1) General weighted complete intersections $X_{22} \subset \mathbb{P}(1, 2, 3, 4, 11)$ and $X_{6,18} \subset \mathbb{P}(2, 2, 3, 3, 4, 9)$ both have ps-index $\delta = 2$. Since both X_{22} and $X_{6,18}$ have non-birational 10-canonical map, Theorem 1.8(1) is optimal.

(2) The 3-fold X_{22} corresponds to No. 1 in Table II1 with $\chi = 0$ and $X_{6,18}$ belongs to No. 11 (with $t = 1$) in Table II1.

Remark 1.10. Theorem 1.8 is parallel to the main results in [Che03]. We have similar statements to Theorem 1.8 for 3-folds with $\delta(V) \geqslant 3$. We omit them since we are not sure whether they are optimal or not.

In the last part we study projective 4-folds. The main result is the following theorem.

Theorem 1.11 (Theorem 8.2). *Let V be a nonsingular projective 4-fold of general type. Then:*

(i) *when $p_g(V) \geqslant 2$, $\Phi_{|mK_V|}$ is birational for all $m \geqslant 35$;*
(ii) *when $p_g(V) \geqslant 19$, $\Phi_{|mK_V|}$ is birational for all $m \geqslant 18$.*

This paper is organized as follows. In § 2, we start with general setting on rational maps on varieties of general type and review some known useful inequalities. Then we list several basic lemmas on 3-folds. In § 3, we improve our technique used in [CC10b] to bound K_X^3 from below. Applying our basket analysis developed in [CC10a], we obtain an effective function $v(x)$ in § 4 so that $K_X^3 \geqslant v(\delta(X))$ for any given minimal 3-fold X. Section 5 is devoted to compiling the clean list for $\mathbb{B}(X)$ with $\delta(X) \geqslant 13$. Then, in § 6, we are able to study the birationality of Φ_m. Section 7 is dedicated to classifying 3-folds with $\delta = 2$. Finally, we study nonsingular projective 4-folds of general type with $p_g \geqslant 2$ in § 8. All subsidiary tables are presented in Appendix A.

Throughout we work over any algebraically closed field k of characteristic 0. We are in favor of the following symbols:

- '$\sim$' denotes linear equivalence or $\mathbb{Q}$-linear equivalence;
- '$\equiv$' denotes numerical equivalence;
- '$|A| \preceq |B|$' means that $|B| \supseteq |A|$ + fixed effective divisors.

2. Preliminaries

We begin with the general setting on rational maps defined by some sub-linear system of the pluricanonical system $|mK|$ on varieties of general type. Let V be any nonsingular projective variety of general type with dimension $n \geqslant 3$. According to the Minimal Model Program, V has a minimal model (see, for example, [KMM87, KM98, BCHM10, Siu08]). From the point of view of birational geometry, we may always consider the rational map on minimal varieties of general type. A minimal model X is a normal projective variety with a nef canonical divisor K_X and with $\mathbb{Q}$-factorial terminal singularities.

2.1 The rational map Φ_Λ for $\Lambda \subset |m_0K|$

Let X be a minimal projective variety of general type on which $P_{m_0}(X) \geqslant 2$ for a positive integer m_0. Let $\Lambda \subset |m_0K_X|$ be a positive dimensional linear system. Fix an effective Weil divisor $K_{m_0} \sim m_0K_X$ on X. Take successive blow-ups $\pi : X' \to X$ along nonsingular centers, such that the following conditions are satisfied:

(i) X' is smooth;
(ii) the moving part of $\pi^*(\Lambda)$ is base point free and so that $g := \Phi_\Lambda \circ \pi$ is a non-constant morphism;
(iii) $\pi^*(K_{m_0}) \cup \{\pi - \text{exceptional divisors}\}$ has simple normal crossing supports.

Sometimes we will take further blow-ups so that π satisfies some more conditions, which will be specified explicitly.

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We have a morphism $g : X' \longrightarrow \overline{\Phi_\Lambda(X)} \subseteq \mathbb{P}^N$. Let $X' \xrightarrow{f} \Gamma \xrightarrow{s} \overline{\Phi_\Lambda(X)}$ be the Stein factorization of g. We have the following commutative diagram.

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & \Gamma \\ {\scriptstyle\pi}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle s} \\ X & \overset{\Phi_\Lambda}{\dashrightarrow} & \overline{\Phi_\Lambda(X)} \end{array}$$

We may write $m_0K_{X'} =_{\mathbb{Q}} \pi^*(m_0K_X) + E_{\pi,m_0}$ where E_{π,m_0} is an effective π-exceptional $\mathbb{Q}$-divisor. Denote by M_{m_0} (respectively M_Λ) the movable part of $|m_0K_{X'}|$ (respectively $\pi^*\Lambda$). Set $d_{m_0} := \dim \Phi_{m_0}(X)$ (respectively $d_\Lambda := \dim \Gamma$). The Bertini theorem implies that the general member of the moving part M_Λ of $\pi^*(\Lambda)$ is irreducible whenever $d_\Lambda \geqslant 2$ and, otherwise, $M_\Lambda \equiv a_\Lambda F$, where $a_\Lambda := \deg f_*\mathcal{O}_{X'}(M_\Lambda)$ and F is a general fiber of f. We set

$$\theta_\Lambda := \begin{cases} 1 & \text{if } d_\Lambda \geqslant 2, \\ a_\Lambda & \text{if } d_\Lambda = 1. \end{cases}$$

Recall our definition in [CC10b, Definition 2.4], the *generic irreducible element* Σ of $\pi^*(\Lambda)$ is defined as follows:

$$\Sigma_\Lambda := \begin{cases} \text{the general member of the moving part of } \pi^*(\Lambda) & \text{if } d_\Lambda \geqslant 2, \\ F & \text{if } d_\Lambda = 1. \end{cases}$$

By the above setting, we always have

$$m_0\pi^*(K_X) \sim_{\mathbb{Q}} \theta_\Lambda\Sigma_\Lambda + E'_\Lambda$$

for some effective $\mathbb{Q}$-divisor E'_Λ on X'.

Convention. Whenever we are working on the complete linear system $|m_0K_X|$, we will use parallel notation such as $d_{m_0}, \theta_{m_0}, \ldots$ (or even just $d, \theta, \ldots$, for simplicity).

We discuss the special case with $d_\Lambda = 1$. Clearly the general fiber F is nonsingular projective of dimension $\dim(X) - 1$. Replace X' by its birational model, we may assume that there is a birational contraction morphism $\sigma : F \longrightarrow F_0$ onto a minimal model F_0. We have the following 'canonical restriction inequality'.

Lemma 2.1. *Keep the above settings. Suppose that $d_\Lambda = 1$. The following holds:*

(i) *if $b := g(\Gamma) > 0$, then $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$;*

(ii) *if $b = 0$, then*

$$\pi^*(K_X)|_F \geqslant \frac{\theta_\Lambda}{m_0 + \theta_\Lambda}\sigma^*(K_{F_0}).$$

Proof. Statement (i) follows from Chen [Che10, Lemma 2.5].

Assume $\Gamma \cong \mathbb{P}^1$. Choose a sufficiently large and divisible integer m so that both $|m\pi^*(K_X)|$ and $|mK_{F_0}|$ are base point free. By Kawamata's extension theorem [Kaw99, Theorem A], we have the surjective map

$$H^0(X', m\theta_\Lambda(K_{X'} + F)) \longrightarrow H^0(F, m\theta_\Lambda K_F).$$

Since $|m(\theta_\Lambda + m_0)K_{X'}| \succeq |m\theta_\Lambda(K_{X'} + F)|$, $\mathrm{Mov}|m\theta_\Lambda K_F| = |m\theta_\Lambda\sigma^*(K_{F_0})|$ and $|m(\theta_\Lambda + m_0)\pi^*(K_X)| = |M_{m(\theta_\Lambda+m_0)}|$, we obtain the following inequality:

$$m(\theta_\Lambda + m_0)\pi^*(K_X)|_F = M_{m(\theta_\Lambda+m_0)}|_F \geqslant m\theta_\Lambda\sigma^*(K_{F_0}),$$

which implies statement (ii). □

2.2 Key inequalities on 3-folds

Let X be minimal 3-fold of general type. Assume that $\Lambda \subset |m_0K_X|$ is a linear system of positive dimension. As in § 2.1, we obtain an induced fibration $f : X' \longrightarrow \Gamma$. Pick a generic irreducible element S of $|m_0K_{X'}|$. Let $|G|$ be a given base point free linear system on S. Pick a generic irreducible element C of $|G|$. Since $\pi^*(K_X)|_S$ is nef and big, Kodaira's lemma implies that $\pi^*(K_X)|_S \geqslant \beta C$ for some rational number $\beta > 0$. Then, by [CC10b, (2.1)], one has

$$K_X^3 \geqslant \frac{\theta\beta}{m_0}\xi \tag{1}$$

where $\xi := (\pi^*(K_X) \cdot C)_{X'}$. In addition, by [CC10b, Remark 2.12], one has

$$\xi \geqslant \frac{\deg(K_C)}{1 + m_0/\theta + 1/\beta}. \tag{2}$$

For any positive integer m so that $\alpha_m := (m - 1 - m_0/\theta - 1/\beta)\xi > 1$, by Chen and Zuo [CZ08, Theorem 3.1], one has

$$\xi \geqslant \frac{\deg(K_C) + \lceil\alpha_m\rceil}{m}. \tag{3}$$

We have the following stronger form of inequality (3) when C is 'even'.

Lemma 2.2. *Under the above situation, if C is an even divisor on S (i.e. $\frac{1}{2}C \in Pic(S)$), then, for any $m > 0$ so that $\alpha_m > 0$, one has*

$$\xi \geqslant \frac{\deg(K_C) + 2\lceil\frac{1}{2}\alpha_m\rceil}{m}. \tag{4}$$

Proof. We refer to the proof for Chen and Zuo [CZ08, Theorem 3.1]. The key point is to estimate $\deg(D)$ where $D = \lceil Q\rceil|_C$ and Q is a $\mathbb{Q}$-divisor on S with $(Q \cdot C) = \alpha_m$. Since $\deg(D) \geqslant \alpha_m > 0$ and $\deg(D)$ is even, we naturally have

$$\deg(D) = 2(\lceil Q\rceil \cdot \tfrac{1}{2}C) \geqslant 2\lceil\tfrac{1}{2}\alpha_m\rceil$$

where we note that $(\lceil Q\rceil \cdot \frac{1}{2}C)$ is a positive integer. Clearly the rest of the proof of Chen and Zuo [CZ08, Theorem 3.1] implies inequality (4). □

When $d_\Lambda = 1$, Lemma 2.1(ii) implies the following:

$$\xi = (\pi^*(K_X) \cdot C)_{X'} \geqslant \frac{\theta}{m_0 + \theta}(\sigma^*(K_{F_0}) \cdot C)_F. \tag{5}$$

2.3 Other useful Lemmas

Lemma 2.3 (See [Maş99, Proposition 4] or [Che14, Lemma 2.6]). *Let S be a nonsingular projective surface. Let L be a nef and big $\mathbb{Q}$-divisor on S satisfying the following conditions:*

(1) $L^2 > 8$;

(2) $(L \cdot C_x) \geqslant 4$ *for all irreducible curves C_x passing through any very general point $x \in S$.*

Then the linear system $|K_S + \lceil L\rceil|$ separates two distinct points in very general positions. Consequently, $|K_S + \lceil L\rceil|$ gives a birational map.

Lemma 2.4. *Let $\sigma : S \longrightarrow S_0$ be a birational contraction from a nonsingular projective surface S of general type onto the minimal model S_0. Assume that $(K_{S_0}^2, p_g(S_0)) \neq (1, 2)$ and that C is a moving curve on S. Then $(\sigma^*(K_{S_0}) \cdot C) \geqslant 2$.*

Proof. When $K_{S_0}^2 \geqslant 2$, this is due to Hodge index theorem. When $(K_{S_0}^2, p_g(S_0)) = (1, 0)$, this is due to Miyaoka [Miy76, Lemma 5]. When $(K_{S_0}^2, p_g(S_0)) = (1, 1)$, $(\sigma^*(K_{S_0}) \cdot C) = 1$ implies $K_{S_0} \equiv \sigma_* C$ by the Hodge index theorem. According to Bombieri [Bom73], we know that S_0 is simply connected. Thus, $K_{S_0} \sim \sigma_* C$, which is impossible since $|K_{S_0}|$ is not movable. □

Lemma 2.5. *Let $\sigma : S \longrightarrow S_0$ be the birational contraction onto the minimal model S_0 from a nonsingular projective surface S of general type. Assume that $(K_{S_0}^2, p_g(S_0)) \neq (1, 2)$ and that $\tilde{C}$ is a curve on S passing through very general points. Then $(\sigma^*(K_{S_0}) \cdot \tilde{C}) \geqslant 2$.*

Proof. In fact, by the projection formula, this is equivalent to see $(K_{S_0} \cdot C_0) \geqslant 2$ for any curve $C_0 \subset S_0$ passing through very general points of S_0.

In contrast, let us assume $(K_{S_0} \cdot C_0) \leqslant 1$. Then $g(C_0) \geqslant 2$ implies $C_0^2 \geqslant 1$. The Hodge index theorem says $K_{S_0}^2 = 1$ and $K_{S_0} \equiv C_0$. Recall that S_0 is not a $(1, 2)$ surface. So S_0 must be either a $(1, 0)$ surface or a $(1, 1)$ surface.

If $(K_{S_0}^2, p_g(S_0)) = (1, 0)$, then $q(S_0) = 0$ and the torsion element $\theta := K_{S_0} - C_0$ is of order at most five (see Reid [Rei78]) and $h^0(S_0, C_0) = 1$. Thus, there are at most a finite number of such curves on S_0 since $\#\mathrm{Tor}(S_0) \leqslant 5$, which is absurd by the choice of C_0.

If $(K_{S_0}^2, p_g(S_0)) = (1, 1)$, then $q(S_0) = 0$ and $K_{S_0} \sim C_0$ since $\mathrm{Tor}(S_0) = 0$ by Bombieri [Bom73, Theorem 15] and thus C_0 is the unique canonical curve of S_0, which is absurd as well. □

2.4 The birationality principle

Definition 2.6. Pick two different generic irreducible elements S', S'' (respectively C', C'') in $|M_{m_0}|$ (respectively in $|G|$).

(i) We say that $|mK_{X'}|$ *distinguishes S' and S''* if $\Phi_{|mK_{X'}|}(S') \neq \Phi_{|mK_{X'}|}(S'')$.

(ii) We say that $|mK_{X'}|$ *distinguishes C' and C''* if $\Phi_{|mK_{X'}|}(C') \neq \Phi_{|mK_{X'}|}(C'')$.

We will apply the useful, but technical theorem of Chen and Zuo [CZ08] for the birationality of Φ_m.

Theorem 2.7 (See Chen and Zuo [CZ08, Theorem 3.1] or [CC10b, Theorem 2.11, Part 2]). *Keep the same notation as above. Assume that, for some $m > 0$, $|mK_{X'}|$ distinguishes S' and S'', C' and C'' for generic $S' \neq S''$, $C' \neq C''$. Then Φ_m is birational under one of the following conditions:*

(i) $\alpha_m > 2$*;*

(ii) $\alpha_m > 1$ *and C is not hyper-elliptic.*

3. The lower bound of K^3 in terms of m_0

In the study of three-dimensional explicit birational geometry, a challenging problem is to determine whether a given weighted basket $\mathbb{B}$ is geometric, i.e. equal to $\mathbb{B}_X$ for some 3-fold X or not. By exploiting geometric properties, one might be able to have a better estimation of the lower bound of K_X^3, and hence exclude some non-geometric formal baskets. In fact, in [CC10b, (2.19)–(2.31)], we already proved some effective inequalities for K_X^3. We shall go further along this direction in this section.

Let X be a minimal 3-fold of general type. Assume $P_{m_0}(X) \geqslant 2$. Mostly we will take $\Lambda = |m_0 K_X|$. Keep the settings in §§ 2.1 and 2.2.

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Table A1. Volumes in the case $d_{m_0} = 3$.

$m_0 =$	2	3	4	5	6	7	8
$\xi \geqslant$	4/3	1	3/4	5/8	1/2	6/13	2/5
$K^3 \geqslant$	1/3	1/9	3/64	1/40	1/72	6/637	1/160
$m_0 =$	9	10	11	12	13	14	15
$\xi \geqslant$	4/11	1/3	3/10	5/18	1/4	6/25	2/9
$K^3 \geqslant$	4/891	1/300	3/1210	5/2592	1/696	3/2450	2/2025

Table A2. Volumes in the case $d_{m_0} = 2$.

$m_0 =$	2	3	4	5	6	7	8
$\xi \geqslant$	1/2	2/5	1/3	1/4	2/9	1/5	1/6
$K^3 \geqslant$	1/8	2/45	1/48	1/100	1/162	1/245	1/384
$m_0 =$	9	10	11	12	13	14	15
$\xi \geqslant$	2/13	1/7	1/8	2/17	1/9	1/10	2/21
$K^3 \geqslant$	2/1053	1/700	1/968	1/1224	1/1521	1/1960	2/4725

3.1 The case $d_{m_0} = 3$

If we take $|G|$ to be $|S|_S|$, then $\beta = 1/m_0$. It is known, from [CC10b, (2.19)], that $\deg(K_C) \geqslant 6$, $\xi \geqslant 10/(3m_0+2)$ and $K_X^3 \geqslant \xi/m_0^2$. Take $m = 5m_0+4, \ldots, (2t+1)m_0+2t$, successively. Then, by (3), one has $\xi \geqslant 17/(5m_0+4), 24/(7m_0+6), \ldots, (7t+3)/((2t+1)m_0+2t)$, respectively. Taking the limit, we obtain $\xi \geqslant 7/(2m_0+2)$. Therefore

$$K_X^3 \geqslant \frac{7}{2m_0^2(m_0+1)}. \tag{6}$$

In fact, for each small m_0, the explicit lower bound of K^3 can be slightly improved by the same trick and the results are given in Table A1.

3.2 The case $d_{m_0} = 2$

If we take $|G| = |S|_S|$, then $\beta \geqslant (P_{m_0} - 2)/m_0$. By inequality (3), one has $\xi \geqslant 2/(2m_0+1)$. Take $m = 3m_0+2, 5m_0+4, \ldots, (2t+1)m_0+2t$ successively. One gets from inequality (3) that $\xi \geqslant 4/(3m_0+2), 7/(5m_0+4), \ldots, (3t+1)/((2t+1)m_0+2t)$. Taking the limit, we have $\xi \geqslant 3/(2m_0+2)$. By inequality (1), we have

$$K_X^3 \geqslant \frac{3(P_{m_0}-2)}{2m_0^2(m_0+1)} \geqslant \frac{3}{2m_0^2(m_0+1)}. \tag{7}$$

In fact, we have the estimation in Table A2 for each small m_0, which slightly improves [CC10b, Table A].

Under the same situation, if there exists a number $m_1 > 0$ such that $d_{m_1} = 3$, then, since $(m_1\pi^*(K_X)|_F \cdot C) \geqslant 2$, we have $\xi \geqslant 2/m_1$. Thus, inequality (1) reads

$$K_X^3 \geqslant \frac{2(P_{m_0}-2)}{m_0^2 m_1} \geqslant \frac{2}{m_0^2 m_1}. \tag{8}$$

Table A3. Volumes for the $(1,2)$-fibration case.

$m_0 =$	2	3	4	5	6	7	8
$\xi \geqslant$	1/2	1/3	2/7	1/4	1/5	2/11	1/6
$K^3 \geqslant$	1/12	1/36	1/70	1/120	1/210	1/308	1/432

$m_0 =$	9	10	11	12	13	14	15
$\xi \geqslant$	1/7	2/15	1/8	1/9	2/19	1/10	1/11
$K^3 \geqslant$	1/630	1/825	1/1056	1/1404	1/1729	1/2100	1/2640

3.3 The case $d_{m_0} = 1$, $b = g(\Gamma) > 0$

We have $S = F$ by definition. Pick a very large number $l > 0$. Take $|G| := |l\sigma^*(K_{F_0})|$ which is base point free by the surface theory. By definition, we have $\theta \geqslant P_{m_0} \geqslant 2$. Since $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$ by Lemma 2.1(i), we see $\beta = 1/l$ and thus inequality (1) implies

$$K_X^3 \geqslant \frac{P_{m_0}}{m_0} \cdot \frac{1}{l} \cdot lK_{F_0}^2 \geqslant \frac{P_{m_0}}{m_0}. \tag{9}$$

3.4 The case $d_{m_0} = 1$, $b = 0$

By Lemma 2.1(ii), we have

$$K_X^3 \geqslant \frac{\theta}{m_0}\pi^*(K_X)|_F^2 \geqslant \frac{\theta^3}{m_0(m_0+\theta)^2} \cdot K_{F_0}^2. \tag{10}$$

We will choose suitable linear system $|G|$ on F depending on the numerical type of F. From the surface theory, we know that either $K_{F_0}^2 \geqslant 2$ or $(K_{F_0}^2, p_g(F)) = (1,2)$, $(1,1)$, $(1,0)$.

Subcase 3.4.1. $K_{F_0}^2 \geqslant 2$.

Inequality (10) implies

$$K_X^3 \geqslant \frac{2\theta^3}{m_0(m_0+\theta)^2}. \tag{11}$$

Subcase 3.4.2. $(K_{F_0}^2, p_g(F_0)) = (1,2)$.

Take $|G| := \mathrm{Mov}|K_F|$. Then C, as a generic irreducible element of $|G|$, is a smooth curve of genus 2 (see [BPV84]). By Lemma 2.1(ii), we have $\beta = \theta/(m_0+\theta) \geqslant 1/(m_0+1)$.

Inequality (2) implies $\xi \geqslant \theta/(m_0+\theta)$. Take $m = \lfloor (3m_0+3\theta)/\theta \rfloor + 1 > (3m_0+3\theta)/\theta$. Then, since $\alpha_m \geqslant (m-1-m_0/\theta-1/\beta)\xi > 1$, inequality (3) gives $\xi \geqslant 4/(\lfloor (3m_0+3\theta)/\theta \rfloor + 1) \geqslant 4\theta/(3m_0+4\theta)$. Inductively, take $m = \lfloor ((1+\frac{2}{3}(4^t-1))m_0 + 3\cdot 4^{t-1}\theta)/4^{t-1}\theta \rfloor + 1$, one gets $\xi \geqslant 4^t\theta/((1+\frac{2}{3}(4^t-1))m_0 + 4^t\theta)$ and hence $\xi \geqslant 3\theta/(2m_0+3\theta)$ by taking the limit. Thus we have

$$K_X^3 \geqslant \frac{3\theta^3}{m_0(m_0+\theta)(2m_0+3\theta)} \geqslant \frac{3}{m_0(m_0+1)(2m_0+3)}. \tag{12}$$

A similar calculation leads to better estimation given in Table A3 for smaller m_0.

Subcase 3.4.3. $(K_{F_0}^2, p_g(F_0)) = (1,1)$.

Since $|\sigma^*(K_{F_0})|$ is not moving, we have to take $|G| := |2\sigma^*(K_{F_0})|$ which is base point free by the surface theory. Naturally the generic irreducible element C of $|G|$ is even and $\deg(K_C) = 6$.

Table A4. Volumes for the $(1,1)$-fibration case.

$m_0 =$	2	3	4	5	6	7	8
$\xi \geqslant$	6/7	2/3	1/2	4/9	3/8	1/3	2/7
$K^3 \geqslant$	1/14	1/36	1/80	1/135	1/224	1/336	1/504
$m_0 =$	9	10	11	12	13	14	15
$\xi \geqslant$	4/15	6/25	2/9	1/5	4/21	14/79	1/6
$K^3 \geqslant$	1/675	3/2750	1/1188	1/1560	1/1911	1/2370	1/2880

By Lemma 2.1(ii), we have $\beta = \theta/(2m_0 + 2\theta)$. Take $m = \lfloor (3m_0 + 3\theta)/\theta \rfloor + 1$. Since $\xi > 0$, we have $\alpha_m > 0$. Thus, Lemma 2.2 implies $\xi \geqslant 8\theta/(3m_0 + 4\theta)$. Thus, inequality (1) reads

$$K_X^3 \geqslant \frac{4\theta^3}{m_0(m_0+\theta)(3m_0+4\theta)}. \tag{13}$$

For each small m_0, we have the better estimation given in Table A4.

Subcase 3.4.4. $(K_{F_0}^2, p_g(F_0)) = (1,0)$.

Modulo further birational modification, we may assume that $\mathrm{Mov}|2K_F|$ is base point free. Take $|G| = \mathrm{Mov}|2K_F|$. By Catanese and Pignatelli [CP06], the generic irreducible element C of $|G|$ is a smooth curve of genus at least three. By Lemma 2.1(ii), we have $\beta = \theta/(2m_0 + 2\theta) \geqslant 1/(2m_0 + 2)$. Lemma 2.4 implies $\xi \geqslant \theta/(m_0 + \theta) \cdot (\sigma^*(K_{F_0}) \cdot C) \geqslant 2\theta/(m_0 + \theta)$. Thus, we have

$$K_X^3 \geqslant \frac{\theta^3}{m_0(m_0+\theta)^2}. \tag{14}$$

Of course, for each small m_0, one might obtain a slightly better estimation for ξ and K_X^3.

Variant 3.4.5. If there exists a positive integer m_1 such that $P_{m_1} \geqslant 2$ and that $|m_0 K_{X'}|$ and $|m_1 K_{X'}|$ are not composed with the same pencil. We may take $|G| = |M_{m_1}|_F|$ and then we have $\beta = 1/m_1$. Thus, inequality (1) and Lemma 2.4 imply

$$K_X^3 \geqslant \frac{2\theta_{m_0}^2}{m_0 m_1 (m_0 + \theta_{m_0})}, \tag{15}$$

provided that $(K_{F_0}^2, p_g(F_0)) \neq (1,2)$.

3.5 Some other inequalities

Corollary 3.1. *Let X be a minimal 3-fold of general type. Assume $P_{m_0} = 2$. Keep the same notation as above. Suppose that the general fiber F of the induced fibration from Φ_{m_0} is not a $(1,2)$ surface, and that $P_{m_1} \geqslant 2$ for some integer $m_1 > 0$. Then*

$$K_X^3 \geqslant \min\left\{\frac{(P_{m_1}-1)^3}{m_1(m_1+P_{m_1}-1)^2}, \frac{2}{m_0 m_1 (m_0+1)}\right\}.$$

Proof. If $|m_0 K_{X'}|$, $|m_1 K_{X'}|$ are composed with the same pencil, then both $|m_0 K_{X'}|$ and $|m_1 K_{X'}|$ induce the same fibration $f : X' \longrightarrow \Gamma$. Consider $\tilde{\Lambda} = |m_1 K_{X'}|$. Then, $\theta_{m_1} \geqslant P_{m_1} - 1$. Since F is not a (1,2) surface and by comparing inequalities (9), (11), (13) and (14), we have

$$K_X^3 \geqslant \frac{(P_{m_1}-1)^3}{m_1(m_1+P_{m_1}-1)^2}.$$

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Suppose that $|m_0K_{X'}|$, $|m_1K_{X'}|$ are not composed with the same pencil. We have $\beta = 1/m_1$. Then we have inequality (15) as in Variant 3.4.5. □

Now we are able to study the more restricted case.

Proposition 3.2. *Let X be a minimal 3-fold of general type. Assume that $P_{m_0}(X) \geqslant 4$ and $d_{m_0} = 2$, then*

$$K_X^3 \geqslant \min\left\{\frac{8}{m_0(m_0+2)^2}, \frac{6}{m_0^2(m_0+2)}\right\}.$$

Proof. We need to study the image surface W' of X' through the morphism $\Phi_{|m_0K_{X'}|}$. In fact, we have the Stein factorization

$$\Phi_{m_0} := \Phi_{|m_0K_{X'}|} : X' \xrightarrow{f} \Gamma \xrightarrow{s} W' \subset \mathbb{P}^{P_{m_0}-1}.$$

Denote by H' a very ample divisor on W' such that $M_{m_0} \sim \Phi_{m_0}^*(H')$. Furthermore, one has $M_{m_0}|_S \equiv \tilde{a}_{m_0}C$ for a general member $S \in |M_{m_0}|$ and the integer $\tilde{a}_{m_0} \geqslant \deg(s)\deg(W') \geqslant \deg(W') \geqslant P_{m_0} - 2$, where C is a general fiber of f. Set $|G| := |M_{m_0}|_S|$.

Case 1: $\tilde{a}_{m_0} \geqslant 3$.
We have $\beta \geqslant 3/m_0$. Inequality (2) implies $\xi \geqslant 6/(4m_0+3)$. Take $m = 2m_0+2$. Then inequality (3) gives $\xi \geqslant 2/(m_0+1)$. Take $m = \lfloor(11m_0+9)/6\rfloor + 1$. Since $\alpha_m > ((11m_0+9)/6 - 1 - m_0 - 1/\beta)\,\xi \geqslant 1$, inequality (3) implies $\xi \geqslant 24/(11m_0+15)$. Thus, we have

$$K_X^3 \geqslant \frac{72}{m_0^2(11m_0+15)}. \tag{16}$$

Case 2: $\tilde{a}_{m_0} = 2$.
Automatically we have $P_{m_0} = 4$, which also implies that $\deg(W') = 2$ and $\deg(s) = 1$. Recall that an irreducible surface (in $\mathbb{P}^3$) of degree 2 is one of the following surfaces (see, for instance, Reid [Rei97, p. 30, Example 19]):

(a) W' is the cone $\overline{\mathbb{F}}_2$ obtained by blowing down the unique section with the self-intersection (-2) on the Hirzebruch ruled surface $\mathbb{F}_2$;
(b) $W' \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Case 2(a): $W' = \overline{\mathbb{F}}_2$.
Replacing by its birational model, we may assume that Φ_{m_0} factors through the minimal resolution $\mathbb{F}_2$ of W'. So we have the factorization of $\Phi_{m_0} : X' \xrightarrow{h} \mathbb{F}_2 \xrightarrow{\nu} W'$ where h is a fibration and ν is the minimal resolution of W'. Set $\hat{H} = \nu^*(H')$. We know that ${H'}^2 = 2$ and hence $\hat{H}^2 = 2$. Noting that $\hat{H}$ is nef and big on $\mathbb{F}_2$, we can write

$$\hat{H} \sim \mu G_0 + nT,$$

where μ and n are integers, G_0 denotes the unique section with $G_0^2 = -2$, and T is the general fiber of the ruling on $\mathbb{F}_2$. The property of $\hat{H}$ being nef and big implies that $\mu > 0$ and $n \geqslant 2\mu \geqslant 2$. Now let $pr : \mathbb{F}_2 \longrightarrow \mathbb{P}^1$ be the ruling. Set $\tilde{f} := pr \circ h : X' \longrightarrow \mathbb{P}^1$, which is a fibration with connected fibers. Denote by F a general fiber of $\tilde{f}$. We have

$$M_{m_0} \sim \Phi_{m_0}^*(H') = h^*(\hat{H}) \geqslant 2F.$$

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Let $\Lambda = |2F| \preceq |m_0 K_{X'}|$. Clearly we have $\theta_\Lambda = 2$, $d_\Lambda = 1$ and $b = 0$. By inequalities (11)–(14), we have

$$K_X^3 \geqslant \frac{8}{m_0(m_0+2)^2}. \tag{17}$$

Case 2(b): $W' = \mathbb{P}^1 \times \mathbb{P}^1$.
We have an induced fibration $f : X' \longrightarrow W' = \mathbb{P}^1 \times \mathbb{P}^1$. Since a very ample divisor H' on W' with ${H'}^2 = 2$ is linearly equivalent to $L_1 + L_2 = q_1^*(\text{point}) + q_2^*(\text{point})$ where q_1, q_2 are projections from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ respectively. Set $\tilde{f}_i := q_i \circ f : X' \longrightarrow \mathbb{P}^1$, $i = 1, 2$. Then $\tilde{f}_1$ and $\tilde{f}_2$ are two fibrations onto $\mathbb{P}^1$. Let F_1 and F_2 be general fibers of $\tilde{f}_1$ and $\tilde{f}_2$, respectively. Then $F_1 \cap F_2$ is simply a general fiber C of f. We will estimate ξ in an alternative way. In fact, the following argument is similar to the proof of [CZ08, Theorem 3.1].

Since $\tilde{a}_{m_0} = 2$, we have $S|_S \sim 2C$. On the other hand, we have $S \geqslant F_1 + F_2$. Modulo further birational modifications, we may write $m_0\pi^*(K_X) \equiv F_1 + F_2 + H'_{m_0}$ where H'_{m_0} is an effective $\mathbb{Q}$-divisor with simple normal crossing supports. For any integer $m > m_0 + 1$, we consider the linear system

$$|K_{X'} + \lceil (m - m_0 - 1)\pi^*(K_X)\rceil + F_1 + F_2| \preceq |mK_{X'}|.$$

Since $(m-m_0-1)\pi^*(K_X)+F_2$ is nef and big, Kawamata and Viehweg vanishing [Kaw82, Vie82] gives the surjective map

$$\begin{aligned} &H^0(K_{X'} + \lceil (m - m_0 - 1)\pi^*(K_X)\rceil + F_2 + F_1) \\ &\qquad \longrightarrow H^0(F_1, K_{F_1} + \lceil (m - m_0 - 1)\pi^*(K_X)\rceil|_{F_1} + C). \end{aligned}$$

Using the vanishing theorem again, one obtains the surjective map

$$H^0(F_1, K_{F_1} + \lceil (m - m_0 - 1)\pi^*(K_X)|_{F_1}\rceil + C) \longrightarrow H^0(C, K_C + \hat{D}_m),$$

where $\hat{D}_m := \lceil (m - m_0 - 1)\pi^*(K_X)|_{F_1}\rceil|_C$ with

$$\deg(\hat{D}_m) \geqslant (m - m_0 - 1)\xi.$$

When m is large enough so that $\deg(\hat{D}_m) \geqslant 2$, the above two surjective maps directly implies

$$m\xi \geqslant \deg(K_C) + \deg(\hat{D}_m) \geqslant 2 + \lceil (m - m_0 - 1)\xi\rceil. \tag{18}$$

In particular, we have $\xi \geqslant 2/(m_0 + 1)$.

Take $m = 2m_0 + 3$. Then $(m - m_0 - 1)\xi > 2$ and inequality (18) gives $\xi \geqslant 5/(2m_0 + 3)$.

Assume $m_0 > 1$ and take $m = 2m_0 + 2$. One gets $\xi \geqslant 5/(2m_0 + 2)$. Take $m = \lfloor (7m_0 + 12)/5\rfloor = \lfloor (7m_0 + 7)/5\rfloor + 1 > (7m_0 + 7)/5$, one has $\xi \geqslant 4/m \geqslant 20/(7m_0 + 12)$. Inductively, take $m = \lfloor ((2 + \frac{5}{3}(4^t - 1))m_0 + 2 + \frac{10}{3}(4^t - 1))/(5 \cdot 4^{t-1})\rfloor$ for $t \geqslant 1$, one has $\xi \geqslant (5 \cdot 4^t)/((2 + \frac{5}{3}(4^t - 1))m_0 + 2 + \frac{10}{3}(4^t - 1))$. We have $\xi \geqslant 3/(m_0 + 2)$ by taking the limit and, hence,

$$K_X^3 \geqslant \frac{1}{m_0} \cdot (\pi^*(K_X)|_S)^2 \geqslant \frac{2}{m_0^2} \cdot \xi \geqslant \frac{6}{m_0^2(m_0+2)}. \tag{19}$$

We conclude the statement by comparing (16), (17) and (19). □

Corollary 3.3. *Let X be a minimal 3-fold of general type. The following holds:*

$$K_X^3 \geqslant \begin{cases} \min\left\{\dfrac{8}{m_0(m_0+2)^2}, \dfrac{7}{2m_0^2(m_0+1)}\right\} & \text{when } P_{m_0} \geqslant 4, \\ \dfrac{3}{2m_0^2(m_0+1)} & \text{when } P_{m_0} = 3. \end{cases}$$

Proof. When $P_{m_0} \geqslant 4$, $d_{m_0} = 3, 2, 1$ and the inequality follows from comparing inequality (6), Proposition 3.2, inequalities (9) and (11)–(14) (with $\theta_{m_0} = 3$), respectively.

When $P_{m_0} = 3$, $d_{m_0} = 2, 1$ and the inequality follows immediately by comparing inequality (7) with inequalities (9) and (11)–(14) (with $\theta_{m_0} = 2$). □

4. Threefolds with $\delta(V) \leqslant 12$

The purpose of this section is to prove the following sharper bounds.

Theorem 4.1. *Let X be a minimal projective 3-fold of general type with $2 \leqslant \delta(X) \leqslant 12$. Then $K_X^3 \geqslant v(\delta(X))$, where the function $v(x)$ is defined as follows:*

x	2	3	4	5	6	7
$v(x)$	1/14	1/36	1/90	1/135	1/224	1/336
x	8	9	10	11	12	—
$v(x)$	1/504	1/675	3/2750	1/1188	1/1560	—

We are going to estimate the lower bound of the volume, case by case, for a given δ. The discussion here relies on those formulae in [CC10a, (3.6)–(3.12)].

Proposition 4.2. *If $P_2(X) \geqslant 2$, then $K_X^3 \geqslant \frac{1}{14}$.*

Proof. Set $m_0 = 2$. By Tables A1 and A2, inequalities (9) and (11), Tables A3 and A4 and Corollary 3.3, we have $K_X^3 \geqslant \frac{1}{14}$ unless $P_2 = 2$, $d_2 = 1$, $b = 0$ and F is of type $(1, 0)$.

In the remaining case, we have that $\chi(\mathcal{O}_X) = 1$ by [CC10b, Lemma 2.32]. By [CC10b, Lemma 3.2], one has $P_4 \geqslant 2P_2 \geqslant 4$. If $d_4 \geqslant 2$, then $K_X^3 \geqslant \frac{1}{12}$ by inequality (15) (with $m_0 = 2$, $m_1 = 4$, $\theta_2 = 1$). If $d_4 = 1$, then $|2K_{X'}|$ and $|4K_{X'}|$ are composed with the same pencil. Thus, we have $K_X^3 \geqslant \frac{27}{196} > \frac{1}{8}$ by inequality (14) (with $m_0 = 4$, $\theta_4 = 3$). □

Proposition 4.3. *If $P_3(X) \geqslant 2$, then $K_X^3 \geqslant \frac{1}{36}$.*

Proof. Take $m_0 = 3$ and $\Lambda = |3K_{X'}|$. One has $K_X^3 \geqslant \frac{1}{36}$ by Tables A1 and A2, inequalities (9), (11), Tables A3 and A4 and Corollary 3.3 ($m_0 = 3$) unless we are in Subcase 3.4.4 with $P_3 = 2$. That is, $P_3 = 2, d_3 = 1, b = 0$ and F is of type $(1, 0)$. Again, $\chi(\mathcal{O}_X) = 1$. Thus, for any $m \geqslant 2$, [CC10b, Lemma 3.2] implies $P_{m+2} \geqslant P_m + P_2$.

By Corollary 3.1, if $P_4 \geqslant 3$ (respectively $P_5 \geqslant 3$), then $K_X^3 \geqslant \frac{1}{24}$ (respectively $\frac{1}{30}$). Suppose that both $P_4 \leqslant 2$ and $P_5 \leqslant 2$, then $P_5 = 2$ and $P_2 = 0$. By [CC10a, (3.6)], $n_{1,2}^0 = 5 - 8 + P_4 < 0$, which is a contradiction. Hence, either P_4 or $P_5 \geqslant 3$ in this case and we are done. □

Proposition 4.4. *If $P_4(X) \geqslant 2$, then $K_X^3 \geqslant \frac{1}{90}$.*

Proof. Similarly, we have $K_X^3 \geqslant \frac{1}{80}$ unless $P_4 = 2$, $b = 0$ and F is of $(1, 0)$ type. In fact, in this situation, we have at least $K_X^3 \geqslant \frac{1}{100}$ by inequality (14). We will go a little bit further to investigate this situation.

(0) We may and do assume that $P_2 \leqslant 1$ and $P_3 \leqslant 1$.

(1) If $P_7 \geqslant 3$ (respectively $P_6 \geqslant 3$, $P_5 \geqslant 3$), then $K^3 \geqslant \frac{8}{567} > \frac{1}{80}$ (respectively $\frac{1}{60}, \frac{1}{50}$) by Corollary 3.1 (with $m_0 = 4$, and $m_1 = 7, 6, 5$ respectively). So we may assume $P_5, P_6, P_7 \leqslant 2$. Since $P_6 \geqslant P_4 + P_2$, we see that $P_2 = 0$ and $P_6 = P_4 = 2$.

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(2) If $P_3 = 0$, then $n^0_{1,3} = P_5 - 2 \geqslant 0$ implies $P_5 = 2$. Now $n^5_{1,4} = 3 - \sigma_5 \geqslant 0$ gives $\sigma_5 \leqslant 3$. However, $n^5_{1,3} \geqslant 0$ implies $\sigma_5 \geqslant 4$, a contradiction. We thus assume that $P_3 = 1$ from now on.

(3) We thus can make the following complete table for $B^{(5)}$ depending on P_5, σ_5.

No.	P_5	σ_5	$B^{(5)}$	K^3	$\epsilon + P_7$
1	1	0	$\{2 \times (1,2), (2,5), 5 \times (1,4)\}$	$1/20$	4
2	1	1	$\{3 \times (1,2), (1,3), 4 \times (1,4), (1,r)\}$	$1/r - 1/6$	4
3	2	1	$\{(1,2), 2 \times (2,5), 3 \times (1,4), (1,r)\}$	$1/r - 3/20$	5
4	2	2	$\{2 \times (1,2), (2,5), (1,3), 2 \times (1,4), (1,r_1), (1,r_2)\}$	$1/r_1 + 1/r_2 - 11/30$	5
5	2	3	$\{3 \times (1,2), 2 \times (1,3), (1,4), (1,r_1), (1,r_2), (1,r_3)\}$	$1/r_1 + r_2 + r_3 - 7/12$	5

(4) By definition, one has $\sigma_5 \leqslant \epsilon \leqslant 2\sigma_5$. Note that No. 1 is impossible because $\epsilon = 0$ but $P_7 \leqslant 2$ implies that $\epsilon \geqslant 2$, a contradiction. In No. 3, $P_5 = 2$ implies $P_7 = 2$ and hence $\epsilon = 3 > 2\sigma_5$, a contradiction.

In No. 2, one must have $P_7 = 2$ and $\epsilon = 2 = 2\sigma_5$. Hence, $r \geqslant 6$. Then it follows that $K^3 \leqslant K^3(B^{(5)}) \leqslant 0$, a contradiction. Similarly, in No. 4, $K^3(B^{(5)}) > 0$ only when $r_1 = r_2 = 5$. But then $\epsilon = 2$, a contradiction.

(5) It remains to consider No. 5. Note that $K^3(B^{(5)}) > 0$ only when $r_1 = r_2 = r_3 = 5$ and $K^3(B^{(5)}) = \frac{1}{60}$. There are only finitely many possible packings. Among them, we search for baskets with $K^3 \geqslant \frac{1}{100}$. It turns out there is only one new baskets

$$B_{90} = \{3 \times (1,2), 2 \times (1,3), (2,9), 2 \times (1,5)\}$$

with $K^3(B_{90}) = \frac{1}{90}$. □

Proposition 4.5. *If $P_5 \geqslant 2$, then $K^3_X \geqslant \frac{1}{135}$.*

Proof. Similarly, we have $K^3_X \geqslant \frac{1}{135}$ unless $P_5 = 2$, $b = 0$ and F a $(1,0)$ surface, for which we have $K^3_X \geqslant \frac{1}{180}$. Furthermore, we may assume that $P_m \leqslant 2$ for $m = 6, 7, 8$ by Corollary 3.1. It suffices to consider: $\chi(\mathcal{O}_X) = 1$, $P_2 = 0$, $P_3 = 0, 1$, $P_4 = 0, 1$, $P_5 = P_7 = 2$ and $P_4 \leqslant P_6 \leqslant P_8 \leqslant 2$.

We look at $B^{(5)}$ with $K^3 > 0$ according to (P_3, P_4, P_6) and σ_5. It turns out that there is only one,

$$B^{(5)} = \{2 \times (2,5), 3 \times (1,3), (1,4), (1,6)\}$$

with $K^3(B^{(5)}) = \frac{1}{60}$, given by $(P_3, P_4, P_6) = (1,1,2)$ and $\sigma_5 = 2$. Now $P_8 = 2$ and, hence,

$$B^{(7)} = \{2 \times (2,5), 2 \times (1,3), (2,7), (1,6)\}.$$

However, $K^3(B^{(7)}) = \frac{1}{210} < \frac{1}{180}$, which is impossible. □

Proposition 4.6. *If $P_6 \geqslant 2$, then $K^3_X \geqslant \frac{1}{224}$.*

Proof. Similarly, we have $K^3_X \geqslant \frac{1}{224}$ unless $P_6 = 2$, $b = 0$ and F a $(1,0)$ surface, for which we have $K^3_X \geqslant \frac{1}{294}$. Again, we may assume that $P_m \leqslant 2$ for $m = 7, 8, 9, 10$. Therefore, it remains to consider such a situation that $\chi(\mathcal{O}_X) = 1$, $P_2 = 0$, $P_4 \leqslant 1$, $P_3 \leqslant P_5 \leqslant 1$, $P_7 \leqslant P_9 \leqslant 2$ and $P_8 = P_{10} = 2$. According to the value of (P_3, P_4, P_5) and σ_5, we have the following table.

No.	(P_3, P_4, P_5)	σ_5	$B^{(5)}$	K^3	$\epsilon + P_7$
1	$(0,0,0)$	0	$\{5\times(1,2), 4\times(1,3), (1,4)\}$	$1/12$	2
2	$(0,0,1)$	0	$\{3\times(1,2), 2*(2,5), 3*(1,3)\}$	$1/10$	3
3	$(0,1,0)$	0	$\{6*(1,2), (1,3), 3*(1,4)\}$	$1/12$	3
4	$(0,1,1)$	0	$\{4*(1,2), 2*(2,5), 2*(1,4)\}$	$1/10$	4
5	$(0,1,1)$	1	$\{5*(1,2), 1*(2,5), (1,3), (1,4), (1,r)\}$	$1/r - 7/60$	4
6	$(0,1,1)$	2	$\{6*(1,2), 2*(1,3), (1,r_1), (1,r_2)\}$	$1/r_1 + 1/r_2 - 1/3$	4
7	$(1,0,1)$	0	$\{(2,5), 6*(1,3), (1,4)\}$	$1/20$	2
8	$(1,0,1)$	1	$\{(1,2), 7*(1,3), (1,r)\}$	$1/r - 1/6$	2
9	$(1,1,1)$	0	$\{(1,2), (2,5), 3*(1,3), 3*(1,4)\}$	$1/20$	3
10	$(1,1,1)$	1	$\{2*(1,2), 4*(1,3), 2*(1,4), (1,r)\}$	$1/r - 1/6$	3

(1) It is clear that No. 2, 3, 4 and 9 are not allowed for $\epsilon = 0$ and, hence, $P_7 \geqslant 3$.

(2) In No. 1 and 7, the baskets allow at most one packing at level 7, i.e. $\epsilon_7 \leqslant 1$. However, $P_7 = 2$ and $P_8 = 2$ yield $\epsilon_7 \geqslant 2$, a contradiction.

(3) Consider No. 10. Since $K^3 = 1/r - \frac{1}{6} > 0$, it follows that $r = 5$. So $\epsilon = 1$ and $P_7 = 2$. Then $\epsilon_7 = 2$ and

$$B^{(7)} = \{2\times(1,2), 2\times(1,3), 2\times(2,7), (1,5)\}.$$

This already implies $\epsilon_8 = 0$ and so we get $P_9 = 3$, a contradiction.

(4) Consider No. 8. Since $K^3 > 0$, thus we get

$$B^{(5)} = \{(1,2), 7\times(1,3), (1,5)\}.$$

Since $B^{(5)}$ allows no further packing, hence $K_X^3 = \frac{1}{30}$ in this case.

(5) Consider No. 5. Since $K^3 > 0$, $r = 6, 7, 8$. It is easy to see that the basket with the smallest volume and dominated by $B^{(5)}$ is

$$B_{210} = \{(7,15), (2,7), (1,6)\}$$

with $K^3 = \frac{1}{210}$. Thus, $K_X^3 \geqslant \frac{1}{210}$.

(6) Finally Consider No. 6. Since $K^3 > 0$, $(r_1, r_2) = (5,5), (5,6), (5,7)$. It is easy to see that the basket with the smallest volume and dominated by $B^{(5)}$ is

$$B_{105} = \{6\times(1,2), 2\times(1,3), (1,5), (1,7)\}$$

with $K^3 = \frac{1}{105}$. Thus, $K_X^3 \geqslant \frac{1}{105}$. □

Note that, when $\delta(X) \geqslant 7$, we can utilize our explicit classification in [CC10b, § 3]. We shall omit some details to avoid unnecessary redundancy.

Proposition 4.7. *If $P_7 \geqslant 2$, then $K_X^3 \geqslant \frac{1}{336}$.*

Proof. Similarly, we have $K_X^3 \geqslant \frac{1}{336}$ unless $P_7 = 2$, $b = 0$, F a $(1,0)$ surface and $\chi(\mathcal{O}_X) = 1$. Again, we may assume that $P_m \leqslant 2$ for $m = 8, 9$. Hence, $P_9 = 2$ and $P_2 = 0$.

By $\epsilon_6 = 0$, we have $P_4 + P_5 + P_6 = P_3 + 2 + \epsilon$. Hence $(P_3, P_4, P_5, P_6) = (0,0,1,1), (0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$ which corresponds to cases IV, V, VI and VIII in [CC10b, § 3], respectively. The classification implies that, if $K_X^3 < \frac{1}{336}$, then $B_X \succeq B_{\min}$, where $B_{\min}$ is a minimal positive basket and belongs to one of the following:

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(b1) $B_{6,4} = \{(1,2),(6,13),(1,3),2\times(1,5)\}$ with $K^3(B_{6,4}) = \frac{1}{390}$ and $P_9(B_{6,4}) = 3$;
(b2) $B_{6,6} = \{3\times(1,2),(3,7),(2,5),(1,4),(1,6)\}$ with $K^3(B_{6,6}) = \frac{1}{420}$ and $P_9(B_{6,4}) = 3$;
(b3) $B_{8,3} = \{2\times(2,5),(1,3),(3,11),(1,4)\}$ with $K^3(B_{8,3}) = \frac{1}{660}$.

Clearly, case (b1) cannot happen because $P_9(B_X) \geqslant P_9(B_{\min}) = 3$.

In case (b2), for a similar reason, $B_X \neq B_{6,6}$. Thus, $B_X \succeq B_{60} := \{4\times(1,2), 2\times(2,5), (1,4),(1,6)\}$ and so $K_X^3 \geqslant K^3(B_{60}) = \frac{1}{60}$.

Finally, in case (b3), the proof of [CC10b, Theorem 3.11] implies that $B_X \neq B_{8,3}$ and $B_X \succeq B_{210} = \{2\times(2,5),(1,3),(2,7),2\times(1,4)\}$ with $K_X^3 \geqslant K^3(B_{210}) = \frac{1}{210}$. We have proved the statement. □

It is now immediate to see the following consequences.

Corollary 4.8 (Corollary 1.5). *Let X be a minimal projective 3-fold of general type with $K_X^3 < \frac{1}{336}$. Then $\delta(X) \geqslant 8$.*

Proposition 4.9. *Let X be a minimal projective 3-fold of general type.*

(1) *If $P_8 \geqslant 2$, then $K_X^3 \geqslant \frac{1}{504}$.*
(2) *If $P_9 \geqslant 2$, then $K_X^3 \geqslant \frac{1}{675}$.*
(3) *If $P_{10} \geqslant 2$, then $K_X^3 \geqslant \frac{3}{2750}$.*
(4) *If $P_{11} \geqslant 2$, then $K_X^3 \geqslant \frac{1}{1188}$.*
(5) *If $P_{12} \geqslant 2$, then $K_X^3 \geqslant \frac{1}{1560}$*

Proof. We only prove statement (1). Other statements can be proved similarly.

When $P_8 \geqslant 2$, Tables A1 and A2, inequalities (9) and (11), Tables A3 and A4 imply $K_X^3 \geqslant \frac{1}{504}$ unless we are in Subcase 3.4.4, for which one has $K_X^3 \geqslant \frac{1}{420}$ by [CC10b, Theorem 1.2(2)] since $\chi(\mathcal{O}_X) = 1$. □

Propositions 4.2–4.7 and 4.9 imply Theorem 4.1.

An interesting by-product is the following corollary.

Corollary 4.10 (Corollary 1.7(1)). *Let X be a minimal projective 3-fold of general type with $p_g(X) = 1$. Then $K_X^3 \geqslant \frac{1}{75}$.*

Proof. We distinguish the following cases.

Case 1: $P_4 \geqslant 3$.
By Corollary 3.3, $K_X^3 \geqslant \frac{3}{160}$.

Case 2: $P_4 = 2$.
We have $K_X^3 \geqslant \frac{1}{70}$ by inequalities (9), (11) and Table A3 unless $b = 0$ and F is either a $(1,1)$ or a $(1,0)$ surface, for which we necessarily have $h^2(\mathcal{O}_X) = 0$ and thus $\chi(\mathcal{O}_X) = 0$. Reid's Riemann–Roch formula implies $P_5 > P_4 = 2$. Now Corollary 3.1 (with $m_0 = 4$, $m_1 = 5$) yields $K_X^3 \geqslant \frac{1}{50}$.

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Case 3: $P_4 = 1$.
Since $p_g(X) = 1$, one has $P_m > 0$ for all $m > 1$. By [CC10a, (3.10)], we have

$$P_4 + P_5 + P_6 = 3P_2 + P_3 + P_7 + \epsilon \geqslant 3P_2 + P_3 + P_7.$$

If $P_4 = 1$ (which implies $P_3 = P_2 = 1$), then we have

$$P_5 \geqslant (P_7 - P_6) + 3 \geqslant 3.$$

Then, from [CC10a, (3.6)], $n^0_{1,4} \geqslant 0$ implies $\chi(\mathcal{O}_X) \geqslant 3$. Owing to our previous result [CC08, Corollary 1.2] for irregular 3-folds, we may assume $q(X) = 0$. Thus, we have $h^2(\mathcal{O}_X) = \chi(\mathcal{O}_X) \geqslant 3$. Take a sub-pencil Λ of $|5K_X|$. Then Λ induces a fibration $f : X' \longrightarrow \Gamma$ after Stein factorization. Let F be the general fiber and F_0 be the minimal model of F.

Claim. $K^2_{F_0} \geqslant 2$.

Proof. Clearly we may write

$$f_*\omega_{X'} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(e_2) \oplus \cdots \oplus \mathcal{O}_\Gamma(e_{p_g(F)-1})$$

with $-2 \leqslant e_j \leqslant -1$ for all j, since $p_g(X') = 1$. Note that we have

$$\begin{aligned} h^2(\mathcal{O}_X) &= h^1(f_*\omega_{X'}) + h^0(R^1 f_*\omega_{X'}) \\ &\leqslant (p_g(F) - 1) + h^0(R^1 f_*\omega_{X'}). \end{aligned}$$

If $q(F) > 0$, we have $K^2_{F_0} \geqslant 2$ by the surface theory. If $q(F) = 0$, we have $R^1 f_*\omega_{X'} = 0$ and thus $p_g(F) \geqslant h^2(\mathcal{O}_X) + 1 \geqslant 4$. Hence, we have $K^2_{F_0} \geqslant 4$ by the Noether inequality. □

If $d_5 \geqslant 2$, then we may set $m_1 = 5$ and apply inequality (15), which gives $K^3_X \geqslant \frac{1}{75}$.

If $d_5 = 1$, then $|5K_{X'}|$ and Λ are composed with the same pencil. Thus, we have $\theta_5 \geqslant 2$ and inequality (11) gives $K^3_X \geqslant \frac{16}{245}$. □

5. Threefolds with $\delta(V) \geqslant 13$

Let X be a minimal projective 3-fold of general type with $\delta(X) \geqslant 13$. Now we are in the natural position to classify baskets $\mathbb{B}(X)$ with $\delta(X) \geqslant 13$. In fact, we have $\mathbb{B}^{12} \succeq \mathbb{B}(X) \succeq \mathbb{B}_{\min}$ for certain minimal positive basket $\mathbb{B}_{\min}$ listed in [CC10b, Table C], where $\mathbb{B}^{12}$ is also listed there. However, as pointed out in [CC10b, Proposition 4.5], our earlier classification in [CC10b, Table C] is not clean since some minimal baskets in Table C are actually known to be 'non-geometric'.

Recall that, by definition, a geometric weighted basket is a basket of a projective threefold of general type. Hence, the following properties hold:

(A) $P_m P_n \leqslant P_{m+n}$ if $P_m = 1$ and $n > 0$;
(B) $P_m \geqslant 0$ for all $m > 0$;
(C) $K^3 \geqslant f(m_0)$ for some explicit function $f(x)$ given in §§ 3 and 4 provided that $P_{m_0} \geqslant 2$.

Indeed, if $\mathbb{B}^{12}$ violates one of A, B, C, then so does $\mathbb{B}(X)$. Therefore $\mathbb{B}(X)$ is non-geometric. If $\mathbb{B}_{\min}$ is non-geometric (e.g. cases No. 3a, 5b, 10a, ..., etc.), then we need to check all baskets between $\mathbb{B}^{12}$ and $\mathbb{B}_{\min}$. The following Table H consists of non-geometric baskets with $\delta \geqslant 13$. We keep the same notation as in Table C.

Table H.

No.	$(P_{12},\dots,P_{24})$	$(n_{1,2},n_{4,9},\dots,n_{1,5})$ or $B_{\min}$	K^3	Offending
3a	$(1,0,0,1,0,0,2,0,3,1,1,1,3)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{17}{30030}$	$P_8P_8>P_{16}$
5b	$(1,0,1,2,0,0,3,0,2,1,2,2,3)$	$\{(5,13),(4,15),*\}$	$\frac{1}{1170}$	$P_8P_8>P_{16}$
8	$(1,0,2,1,0,1,3,1,4,3,2,2,5)$	$(7,1,0,1,0,2,0,0,6,0,2,0,0,0,1)$	$\frac{1}{770}$	$P_6P_{10}>P_{16}$
9	$(1,0,2,-1,1,0,2,0,1,2,1,0,2)$	$(9,0,0,2,0,0,1,1,4,0,1,0,0,1,0)$	$\frac{1}{5544}$	$P_{15}=-1$
10a	$(1,0,2,1,2,-1,2,0,2,2,1,2,4)$	$\{(4,9),(3,7),*\}\succ\{(7,16),*\}$	$\frac{1}{1680}$	$P_{17}=-1$
11a	$(1,0,2,0,2,0,2,2,2,1,1,1,3)$	$\{(3,8),(4,11),*\}\succ\{(7,19),*\}$	$\frac{1}{2660}$	$P_8P_{14}>P_{22}$
13	$(1,0,3,-1,1,1,3,1,3,3,3,1,4)$	$(12,0,0,2,0,2,0,2,4,0,2,0,0,1,0)$	$\frac{4}{3465}$	$P_{15}=-1$
15a	$(1,0,3,0,1,0,2,0,3,1,1,1,4)$	$\{(4,11),(1,3),*\}\succ\{(5,14),*\}$	$\frac{1}{2520}$	$P_8P_{14}>P_{22}$
15b	$(1,0,2,0,1,0,3,0,3,2,1,1,4)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{23}{36036}$	$P_8P_{14}>P_{22}$
15c	$(1,0,3,1,2,0,3,1,3,2,2,2,5)$	$\{(7,16),(7,19),*\}$	$\frac{31}{31920}$	$P_8P_{14}>P_{22}$
16c	$(1,0,2,1,1,-1,3,-1,2,2,1,1,3)$	$\{\{(5,13),(7,16)*\}$	$\frac{3}{16016}$	$P_{17}=-1$
18a	$(1,0,3,0,1,0,2,1,2,2,2,1,3)$	$\{(4,11),(1,3),*\}\succ\{(5,14),*\}$	$\frac{1}{3080}$	$P_6P_{11}>P_{17}$
19	$(1,0,2,0,1,1,3,0,2,2,2,1,3)$	$(8,0,1,1,0,1,0,1,5,0,1,0,0,1,0)$	$\frac{2}{3465}$	$P_9P_{14}>P_{23}$
20a	$(1,0,1,1,1,0,3,-1,2,1,0,1,3)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{1}{16380}$	$P_{19}=-1$
21a	$(1,1,1,1,2,0,2,1,2,1,2,2,3)$	$\{(1,3),(3,10),*\}\succ\{(4,13),*\}$	$\frac{1}{4680}$	$P_8P_9>P_{17}$
22	$(1,0,1,1,1,0,2,1,3,1,1,1,3)$	$(7,1,0,1,0,1,1,0,5,1,0,0,1,0,1)$	$\frac{1}{9240}$	$P_8P_9>P_{17}$
23a	$(1,0,2,1,2,0,2,1,3,1,2,2,3)$	$\{(4,9),(3,7),*\}\succ\{(7,16),*\}$	$\frac{1}{2640}$	$P_8P_9>P_{17}$
24	$(1,0,2,0,0,1,3,0,3,2,2,0,3)$	$(10,1,0,1,0,3,0,1,6,0,2,0,0,1,0)$	$\frac{1}{3465}$	$P_8P_8>P_{16}$
26a	$(1,0,3,1,1,1,3,0,4,1,2,2,5)$	$\{(4,11),(1,3),*\}\succ\{(5,14),*\}$	$\frac{1}{1260}$	$P_9P_{10}>P_{19}$
27.1	$(1,0,2,2,1,1,5,0,4,3,3,3,6)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{71}{45045}$	$P_9P_{10}>P_{19}$
27.2	$(1,0,2,2,1,1,5,-1,3,2,2,2,4)$	$\{(2,5),(5,13),*\}\succ\{(7,18),*\}$	$\frac{1}{1386}$	$P_{19}=-1$
27a	$(1,0,2,2,1,1,5,-1,3,2,2,2,3)$	$\{(2,5),(7,18),*\}\succ\{(9,23),*\}$	$\frac{1}{1386}$	$P_{19}=-1$
27b	$(1,0,2,2,1,1,5,-1,3,2,2,2,5)$	$\{(5,13),(5,18),*\}$	$\frac{1}{1170}$	$P_{19}=-1$
29a	$(1,1,3,1,2,2,2,1,3,1,2,2,3)$	$\{(5,14),(1,3),*\}\succ\{(6,17),*\}$	$\frac{1}{5335}$	$P_9P_{14}>P_{23}$
32b	$(1,0,3,1,1,1,3,1,3,2,3,2,4)$	$\{(4,11),(1,3),*\}\succ\{(5,14),*\}$	$\frac{1}{1386}$	$P_9P_{14}>P_{23}$
33a	$(1,1,2,0,2,1,1,1,2,2,1,2,3)$	$\{(3,10),(2,7),*\}\succ\{(5,17),*\}$	$\frac{1}{2856}$	$P_6P_{16}>P_{22}$
34b	$(1,1,2,0,1,1,3,0,3,3,1,2,4)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{1}{1170}$	$P_6P_{13}>P_{19}$
39a	$(1,1,2,1,3,0,2,1,3,2,2,3,4)$	$\{(4,9),(3,7),*\}\succ\{(7,16),*\}$	$\frac{1}{1680}$	$P_6P_{16}>P_{22}$
39b	$(1,1,2,1,3,1,2,1,3,2,2,3,5)$	$\{(3,10),(2,7),*\}\succ\{(5,17),*\}$	$\frac{4}{5355}$	$P_6P_{16}>P_{22}$
40.1	$(1,1,2,1,2,1,4,0,4,3,2,3,6)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{41}{32760}$	$P_6P_{13}>P_{19}$
40a	$(1,1,2,1,2,1,4,-1,3,2,1,2,4)$	$\{(4,10),(3,8),*\}\succ\{(7,18),*\}$	$\frac{1}{2520}$	$P_6P_{13}>P_{19}$
40b	$(1,1,2,1,2,1,4,0,4,3,1,2,5)$	$\{(2,5),(6,16),*\}\succ\{(8,21),*\}$	$\frac{1}{1260}$	$P_6P_{13}>P_{19}$
43a	$(1,1,3,0,2,1,2,1,3,2,2,2,4)$	$\{(4,11),(1,3),*\}\succ\{(5,14),*\}$	$\frac{1}{2520}$	$P_7P_8>P_{15}$
43b	$(1,1,2,0,2,1,3,1,3,3,2,2,4)$	$\{(2,5),(3,8),*\}\succ\{(5,13),*\}$	$\frac{23}{36036}$	$P_7P_8>P_{15}$

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Table H. Continued.

No.	$(P_{12}, \dots, P_{24})$	$(n_{1,2}, n_{4,9}, \dots, n_{1,5})$ or $B_{\min}$	K^3	Offending
44a	$(1,1,2,1,2,1,4,1,3,4,2,2,4)$	$\{(2,5),(6,16),*\} \succ \{(8,21),*\}$	$\frac{1}{1386}$	$P_7P_{18} > P_{25} = 3$
44b	$(1,1,2,1,2,0,3,0,2,3,2,2,3)$	$\{(7,16),(5,13),*\}$	$\frac{3}{16016}$	$P_7P_{10} > P_{17}$
46a	$(1,1,1,1,2,1,3,0,3,1,1,2,3)$	$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{1}{16380}$	$P_9P_{10} > P_{19}$
50a	$(1,1,3,1,2,2,3,1,4,2,3,3,5)$	$\{(4,11),(1,3),*\} \succ \{(5,14),*\}$	$\frac{1}{1260}$	$P_7P_{14} > P_{21}$
51a	$(1,1,2,2,2,2,5,0,3,3,3,3,4)$	$\{(4,10),(3,8),*\} \succ \{(7,18),*\}$	$\frac{1}{1386}$	$P_6P_{13} > P_{19}$
51b	$(1,1,2,2,2,2,5,0,3,3,3,3,5)$	$\{(5,13),(5,18),*\}$	$\frac{1}{1170}$	$P_6P_{13} > P_{19}$
52a	$(1,1,2,1,1,0,2,1,2,2,1,2,3)$	$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{1}{2184}$	$P_5P_{12} > P_{17}$
56a	$(1,1,2,2,1,1,2,1,3,2,2,3,3)$	$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$	$P_5P_{14} > P_{19}$
57	$(1,0,2,2,0,1,3,1,3,2,2,2,3)$	$(3,0,1,2,0,5,0,0,4,0,0,1,0,0,0)$	$\frac{1}{1386}$	$P_7P_9 > P_{16}$
58a	$(1,1,2,2,2,0,2,1,3,2,2,3,4)$	$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{1680}$	$P_5P_{12} > P_{17}$
59a	$(1,1,2,1,2,1,2,3,2,2,2,2,3)$	$\{(3,8),(4,11),*\} \succ \{(7,19),*\}$	$\frac{1}{2660}$	Item C
60a	$(1,1,1,2,1,1,3,0,3,1,1,2,3)$	$\{(2,5),(3,8),*\} \succ \{(5,13),*\}$	$\frac{1}{16380}$	$P_9P_{10} > P_{19}$
61	$(1,1,1,2,1,1,2,2,3,2,2,2,3)$	$(0,1,0,1,0,3,1,0,2,0,0,0,1,0,0)$	$\frac{1}{9240}$	Item C
62a	$(1,1,2,2,2,1,2,2,3,2,3,3,3)$	$\{(4,9),(3,7),*\} \succ \{(7,16),*\}$	$\frac{1}{2640}$	Item C
63	$(1,1,3,1,2,1,3,2,3,3,2,2,4)$	$(5,0,1,2,0,1,1,1,3,0,1,0,0,0,1)$	$\frac{1}{5544}$	Item C

By eliminating non-geometric baskets, we obtain a shorter list of baskets, listed in Tables F0, F1 and F2 in Appendix A. We summarize some observations from the tables.

Theorem 5.1 (Theorem 1.4). *Let X be a minimal projective 3-fold of general type with the weighted basket $\mathbb{B}(X) := \{B_X, P_2, \chi(\mathcal{O}_X)\}$. If $\delta(X) \geqslant 13$, then $P_2 = 0$ and $\mathbb{B}(X)$ belongs to one of the types listed in Tables F0–F2 in Appendix A. Furthermore, the following hold:*

(1) *$\delta(X) = 18$ if and only if $\mathbb{B}(X) = \{B_{2a}, 0, 2\}$ (see Table F0 for B_{2a}) with $K_X^3 = \frac{1}{1170}$;*
(2) *$\delta(X) \neq 16, 17$;*
(3) *$\delta(X) = 15$ if and only if $\mathbb{B}(X)$ is among one of the cases in Table F1; one has $K_X^3 \geqslant \frac{1}{1386}$;*
(4) *$\delta(X) = 14$ if and only if $\mathbb{B}(X)$ is among one of the cases in Table F2; one has $K_X^3 \geqslant \frac{1}{1680}$;*
(5) *$\delta(X) = 13$ if and only if $\mathbb{B}(X) = \{B_{41}, 0, 2\}$ (see Table F0 for B_{41}) with $K_X^3 = \frac{1}{252}$.*

Theorems 4.1 and 5.1 and [Che07, Theorem 1.4] imply the following corollary.

Corollary 5.2 (Theorem 1.6(2)). *Let X be a minimal projective 3-fold of general type. Then $K_X^3 \geqslant \frac{1}{1680}$, and equality holds if and only if $\chi(\mathcal{O}_X) = 2$, $P_2 = 0$ and $B_X = B_{7a}$ or $B_X = B_{36a}$ (cf. Table F2).*

Theorem 5.1, together with the explicit calculation, also implies the following result.

Corollary 5.3. *Let X be a minimal projective 3-fold of general type. Then:*

(1) *if $\delta(X) = 13$, $P_m > 0$ for all $m \geqslant 10$;*
(2) *if $\delta(X) = 14, 15, 18$, $P_m > 0$ for all $m \geqslant 20$.*

6. Birationality

Theorem 6.1. *Let X be a minimal projective 3-fold of general type. If $\delta(X)=18$, then Φ_m is birational for all $m\geqslant 61$.*

Proof. Set $m_0=18$. By Theorem 5.1, we know that $B_X=B_{2a}$, $P_2=0$, $\chi(\mathcal{O}_X)=2$, $P_{19}=0$, $P_{24}=3$ and $K_X^3=\frac{1}{1170}$. By [CC08, Corollary 1.2], we see $q(X)=0$. Thus, $|18K_X|$ induces a fibration $f: X'\longrightarrow \Gamma\cong\mathbb{P}^1$. We have $h^2(\mathcal{O}_{X'})=h^2(\mathcal{O}_X)=1$. Pick a general fiber F. Since $P_{19}(X)=P_{19}(\mathbb{B}_{2a})=0$, we have $H^0(X',K_{X'}+F)=0$.

Claim 6.1.1. $p_g(F)=1$.

Proof. Since $\chi(\mathcal{O}_{X'})>1$, we have $p_g(F)>0$ by [CC10b, Lemma 2.32]. On the other hand, we have the long exact sequence

$$H^0(X',K_{X'}+F)\longrightarrow H^0(F,K_F)\longrightarrow H^1(X',K_{X'})\longrightarrow H^1(X',K_{X'}+F)$$

which implies $h^0(K_F)\leqslant h^1(X',K_{X'})=h^2(\mathcal{O}_{X'})=1$. Thus, we get $p_g(F)=1$. □

We have $P_m>0$ for all $m\geqslant 20$ by Corollary 5.3(2). Consider the linear systems

$$|K_{X'}+\lceil n\pi^*(K_X)\rceil+F|\preceq|(n+19)K_{X'}|.$$

Clearly $|(n+19)K_{X'}|$ distinguish different general fibers F as long as $n\geqslant 19$. By Kawamata and Viehweg vanishing,

$$\begin{aligned}|K_{X'}+\lceil n\pi^*(K_X)\rceil+F||_F&=|K_F+\lceil n\pi^*(K_X)\rceil|_F|\\&\succeq|K_F+\lceil L_n\rceil|\end{aligned}$$

where we set $L_n:=n\pi^*(K_X)|_F$.

Claim 6.1.2. *$L_n^2>8$ whenever $n\geqslant 42$.*

Proof. Since $p_g(F)=1$, we are in Subcase 3.4.1 or Subcase 3.4.3.

Let us consider Subcase 3.4.1 (i.e. $K_{F_0}^2\geqslant 2$) first. We have

$$(\pi^*(K_X)|_F)^2\geqslant\frac{1}{19^2}K_{F_0}^2\geqslant\frac{2}{19^2}$$

by Lemma 2.1(ii). Thus, $L_n^2>8$ whenever $n>38$.

If $K_{F_0}^2=1$, we shall estimate L_n^2 in an alternative way. Suppose that $|24K_{X'}|$ and $|18K_{X'}|$ are not composed with the same pencil. Take $|G|:=|M_{24}|_F|$. Pick a generic irreducible element C of $|G|$. Then we have $\xi=(\pi^*(K_X)|_F\cdot C)\geqslant\frac{2}{19}$ by Lemma 2.4. Thus, $(\pi^*(K_X)|_F)^2\geqslant\frac{1}{24}\xi\geqslant\frac{1}{12\cdot 19}$. Since $r(X)=2340$ and $r(X)(\pi^*(K_X)|_F)^2$ is an integer, we see $(\pi^*(K_X)|_F)^2\geqslant\frac{11}{2340}$. So we have $L_n^2>8$ whenever $n\geqslant 42$.

Assume that $|24K_{X'}|$ and $|18K_{X'}|$ are composed with the same pencil. Since $P_{24}=3$, we may set $m_0=24$ and $\Lambda=|24K_{X'}|$. We have $\theta=2$. The argument in Subcase 3.4.3 implies that

$$(\pi^*(K_X)|_F)^2\geqslant\frac{4\theta^2}{(\tilde{m}_0+\theta)(3m_0+4\theta)}=\frac{1}{130}.$$

We have $L_n^2>8$ whenever $n\geqslant 33$. □

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For very general curves $\tilde{C}$ on F, one has

$$(L_n \cdot \tilde{C}) \geqslant \frac{n}{19}(\sigma^*(K_{F_0}) \cdot \tilde{C}) \geqslant \frac{2n}{19}$$

by Lemma 2.5. Therefore, $(L_n \cdot \tilde{C}) \geqslant 4$ for $n \geqslant 38$. Lemma 2.3 implies that $|K_F + \lceil L_n \rceil|$ gives a birational map for $n \geqslant 42$. Thus, Φ_m is birational for all $m \geqslant 61$. □

Theorem 6.2. *Let X be a minimal projective 3-fold of general type. If $\delta(X) \leqslant 15$, then Φ_m is birational for all $m \geqslant 56$.*

Proof. Set $m_0 = \delta(X)$. By considering a sub-pencil Λ of $|m_0 K_X|$, we may always assume that we have an induced fibration $f : X' \longrightarrow \Gamma$ onto a curve Γ. By Chen and Hacon [CH07], we may assume $q(X) = 0$. Thus, $\Gamma \cong \mathbb{P}^1$. By [CC10b, Corollary 3.13] and [CC10b, Lemma 2.32], we know that $\delta(X) \leqslant 10$ as long as F is a $(1, 0)$ surface. Therefore, it suffices to consider the following three cases:

(1) $\delta(X) \leqslant 15$ and F is a $(1, 2)$ surface;
(2) $\delta(X) \leqslant 15$ and F is neither a $(1, 2)$ surface nor a $(1, 0)$ surface;
(3) $\delta(X) \leqslant 10$ and F is a $(1, 0)$ surface.

Case 1. Without losing of generality, let us assume $\delta(X) = 15$. Take $|G|$ to be the moving part of $|K_F|$. Then, by Table A3, we have $\xi \geqslant \frac{1}{11}$. We have $m_0 = 15$ and $\beta \mapsto \frac{1}{16}$. So $\alpha_m > 2$ whenever $m \geqslant 55$. By Corollary 5.3, $|mK_{X'}|$ separates different general fibers F as long as $m \geqslant 35$. On the other hand, Kawamata and Viehweg vanishing and Lemma 2.1 imply the following, whenever $m \geqslant 49$:

$$\begin{aligned} |mK_{X'}||_F &\succeq |K_{X'} + \lceil (m-16)\pi^*(K_X) \rceil + F||_F \\ &\succeq |K_F + \lceil (m-16)\pi^*(K_X)|_F \rceil \\ &\succeq |(K_F + \lceil Q_m \rceil + C) + C| \end{aligned}$$

where Q_m is a nef and big $\mathbb{Q}$-divisor. Thus, by [CC10b, Lemma 2.17], Φ_m distinguishes different generic curves C for $m \geqslant 49$. Finally Theorem 2.7 implies that Φ_m is birational for all $m \geqslant 55$.

Case 2. Still assume $\delta(X) = 15$. Parallel to the respective argument in the proof of Theorem 6.1, one knows that $|mK_{X'}|$ distinguishes different general fibers F for $m \geqslant 35$. By the surface theory, we see that F is either a surface with $K_{F_0}^2 \geqslant 2$ or a $(1, 1)$ surface. We want to study the linear system $|K_F + \lceil L_n \rceil|$. In fact, by the estimation in Subcase 3.4.1 and Table A4, we have $L_n^2 \geqslant n^2/(32 \cdot 6) > 8$ whenever $n \geqslant 40$. Similarly we have $(L_n \cdot \tilde{C}) \geqslant 4$ for all $n \geqslant 32$ and for all curves $\tilde{C}$ on F passing through very general points. By Lemma 2.3, we see that $|K_F + \lceil L_n \rceil|$ gives a birational map for all $n \geqslant 40$. Similar to what discussed in the proof of Theorem 6.1, we have proved that Φ_m is birational for all $m \geqslant n + 16 \geqslant 56$.

Case 3. When $\delta(X) \leqslant 10$, we have much better birationality result even though F is a $(1, 0)$ surface. In fact, parallel argument shows that Φ_m is birational for all $m \geqslant 39$. The proof is more or less similar to the above proofs. We leave it as an exercise to interested readers. □

Theorems 5.1, 6.1, and 6.2 imply Theorem 1.6(2).

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7. Threefolds with $\delta(V) = 2$

This section is devoted to classifying minimal projective 3-folds of general type with $\delta(X) = 2$, that is, $p_g(X) \leqslant 1$ and $P_2(X) \geqslant 2$.

Assume that $P_2 \geqslant 2$. We first recall the following known results:

(a) if $d_2 = 3$, then Φ_m is birational for all $m \geqslant 7$ by [CC10b, Theorem 2.20];

(b) if $d_2 = 2$, Φ_m is birational for all $m \geqslant 10$ by [CC10b, Theorem 2.22];

(c) if $q(X) > 0$, then Φ_m is birational for all $m \geqslant 7$ by Chen and Hacon [CH07] and for $m = 6$ by Chen *et al.* [CCJ13].

The purpose of this section is to prove that Φ_m is birational for $m \geqslant 11$ and classify 3-folds such that Φ_{10} is not birational. Therefore, we may and do assume that $q(X) = 0$, $d_2 = 1$ and $b = g(\Gamma) = 0$. Let F be the general fiber of the induced fibration $f : X' \to \mathbb{P}^1$ from Φ_2.

7.1 Birationality of Φ_m for $m \geqslant 11$

Lemma 7.1. *The linear system $|mK_{X'}|$ distinguishes different general fibers of f for all $m \geqslant 9$.*

Proof. When $p_g(F) > 0$, by [CC10b, Proposition 2.15(i)], one has $P_k > 0$ for $k \geqslant 7$. Thus, for all $m \geqslant 9$, $mK_{X'} \geqslant F$, hence $|mK_{X'}|$ distinguishes different general fibers of f.

When $p_g(F) = 0$, one has $\chi(\mathcal{O}_X) \leqslant 1$ (cf. [CC10b, Lemma 2.32]). By [CC10b, Lemma 3.2], one has $P_5 \geqslant P_2 > 0$. Then clearly $P_k > 0$ for all $k \geqslant 5$. Thus, for all $m \geqslant 7$, $mK_{X'} \geqslant F$ and, hence, $|mK_{X'}|$ distinguishes different general fibers of f. □

Proposition 7.2. *Assume $P_2(X) \geqslant 2$, $q(X) = 0$, $d_2 = 1$ and F is not a $(1, 2)$ surface. Then Φ_m is birational for all $m \geqslant 10$.*

Proof. Set $L_n := n\pi^*(K_X)|_F$ which is a nef and big $\mathbb{Q}$-divisor on F. Kawamata and Viehweg vanishing gives the following surjective map:

$$H^0(X', K_{X'} + \lceil n\pi^*(K_X)\rceil + F) \longrightarrow H^0(F, K_F + \lceil n\pi^*(K_X)\rceil|_F).$$

Together with Lemma 7.1, it is sufficient to prove that $|K_F + \lceil L_n\rceil|$ gives a birational map for $n \geqslant 7$ because

$$|(n+3)K_{X'}| \ \succeq\ |K_{X'} + \lceil n\pi^*(K_X)\rceil + F|.$$

Claim 7.2.1. If $K_{F_0}^2 \geqslant 2$ or F_0 is of type $(1, 0)$, then $|K_F + \lceil L_n\rceil|$ is birational for $n \geqslant 7$.

First of all, for any curve $\tilde{C} \subset F$ passing through very general points of F, we estimate $(L_n \cdot \tilde{C})$ for $n \geqslant 7$. Clearly we have $g(\tilde{C}) \geqslant 2$. Set $m_0 = 2$ and $\Lambda = |2K_{X'}|$. By Lemmas 2.1 and 2.5, we have

$$(L_n \cdot \tilde{C}) \geqslant 7(\pi^*(K_X)|_F \cdot \tilde{C}) \geqslant \tfrac{7}{3}(\sigma^*(K_{F_0}) \cdot \tilde{C}) > 4.$$

If $K_{F_0}^2 \geqslant 2$, then we have

$$L_n^2 \geqslant 49(\pi^*(K_X)|_F)^2 \geqslant 49(\tfrac{1}{3}\sigma^*(K_{F_0}))^2 \geqslant \tfrac{98}{9} > 8.$$

If F_0 is a $(1, 0)$ surface, we have $P_4 \geqslant 2P_2 \geqslant 4$ since $\chi(\mathcal{O}_X) \leqslant 1$. When $d_4 \geqslant 2$, we set $m_0 = 2$, $\Lambda = |2K_{X'}|$ and $|G| = |M_4|_F|$. Then $\beta = \frac{1}{4}$, $\xi \geqslant \frac{1}{3}(\sigma^*(K_{F_0}) \cdot C) \geqslant \frac{2}{3}$ and so $L_n^2 \geqslant \frac{49}{6} > 8$.

When $d_4 = 1$, we set $m_0 = 4$ and $\Lambda = |4K_{X'}|$. Clearly $|2K_{X'}|$ and $|4K_{X'}|$ induce the same fibration f. Take $|G| = |2\sigma^*(K_{F_0})|$. Since $\theta \geqslant 3$, we have $\beta \geqslant \frac{3}{14}$ by Lemma 2.1. Thus, $\xi \geqslant \frac{6}{7}$ and so $L_n^2 \geqslant 49 \cdot \frac{3}{14} \cdot \frac{6}{7} > 8$. By Lemma 2.3, the claim follows.

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Claim 7.2.2. If F_0 is a $(1,1)$ surface, then $|K_F+\lceil L_n\rceil|$ is birational for $n\geqslant 7$.

Following the similar argument as above, it is easy to see that $L_n^2\geqslant \frac{64}{7}>8$ and $(L_n\cdot \tilde{C})\geqslant 4$ for all $n\geqslant 8$. We consider the linear system $|K_F+\lceil 7\pi^*(K_X)|_F\rceil|$ in an alternative way. Note that $|2\sigma^*(K_{F_0})|$ is base point free. Pick a generic irreducible element $C\in|2\sigma^*(K_{F_0})|$. Since $\mathcal{O}_\Gamma(1)\hookrightarrow f_*\omega_{X'}$, we have $f_*\omega^2_{X'/\Gamma}\hookrightarrow f_*\omega^{10}_{X'}$. The semi-positivity implies that $f_*\omega^2_{X'/\Gamma}$ is generated by global sections, which directly implies $10K_{X'}|_F\geqslant C$. Thus, Φ_{10} distinguishes different C. By Lemma 2.1, we have $6\pi^*(K_X)|_F\equiv C+H_6$ for an effective $\mathbb{Q}$-divisor H_6 on F. Thus, the vanishing theorem implies

$$|K_F+\lceil 7\pi^*(K_X)|_F-H_6\rceil||_C=|K_C+D|$$

with $\deg(D)\geqslant 2(\lceil 7\pi^*(K_X)|_F-C-H_6\rceil\cdot\sigma^*(K_{F_0}))\geqslant 2$. Since C is non-hyperelliptic, $|K_C+D|$ gives a birational map. Thus $|K_F+\lceil 7\pi^*(K_X)|_F\rceil|$ is birational. □

Proposition 7.3. *Assume $P_2(X)\geqslant 2$, $q(X)=0$, $d_2=1$ and F a $(1,2)$ surface. Then Φ_m is birational for all $m\geqslant 11$.*

Proof. Take $|G|$ to be the moving part of $|\sigma^*(K_{F_0})|$. Modulo birational modifications, we may assume that $|G|$ is base point free. Pick a generic irreducible element C of $|G|$. It is also known that $g=2$.

Claim 7.3.1. *The linear system $|mK_{X'}|$ distinguishes different general members of $|G|$ for $m\geqslant 9$.*

Proof. Clearly $|G|$ is composed with a rational pencil since $q(F)=0$. We shall prove $|mK_{X'}|_F\succeq |G|$ and thus the statement follows. In fact, by Lemma 2.1, we have

$$3\pi^*(K_X)\equiv\sigma^*(K_{F_0})+H_3$$

for an effective $\mathbb{Q}$-divisor H_3 on F. Thus, for $m\geqslant 10$,

$$Q_m:=(m-3)\pi^*(K_X)_{|F}-2H_3-2\sigma^*(K_{F_0})\equiv(m-9)\pi^*(K_X)|_F$$

is nef and big. It follows that $K_F+\lceil Q_m\rceil+\sigma^*(K_{F_0})>0$ by [CC10b, Lemma 2.14]. We thus have the following:

$$\begin{aligned}|mK_{X'}|_{|F}&\succeq|K_{X'}+F+\lceil(m-3)\pi^*(K_X)\rceil|_{|F}\\&=|K_F+\lceil(m-3)\pi^*(K_X)\rceil_{|F}|\\&\succeq|K_F+\lceil(m-3)\pi^*(K_X)_{|F}-2H_3\rceil|\\&=|(K_F+\lceil Q_m\rceil+\sigma^*(K_{F_0}))+\sigma^*(K_{F_0})|\\&\succeq|\sigma^*(K_{F_0})|\ \succeq\ |G|\end{aligned}$$

where the first equality follows from the Kawamata and Viehweg vanishing [Kaw82, Vie82]. Therefore, $|mK_{X'}|$ distinguishes general members of $|G|$ for $m\geqslant 10$. Moreover, for $m=9$,

$$\begin{aligned}|9K_{X'}|_{|F}\succeq|5K_{X'}|_{|F}\ &\succeq\ |K_{X'}+\lceil 2\pi^*(K_X)\rceil+F|_{|F}\\&=|K_F+\lceil 2\pi^*(K_X)\rceil|_F|\ \succeq\ |G|\end{aligned}$$

where the equality is again due to Kawamata and Viehweg vanishing. Hence, $|9K_{X'}|$ distinguishes general members of $|G|$ as well, which asserts the claim. □

From Table A3, one has $\xi\geqslant\frac{1}{2}$. Take $m\geqslant 11$, then $\alpha_m\geqslant\frac{5}{2}>2$. This means that $|mK_{X'}|_{|C}$ distinguishes points on C. Thus, by Theorem 2.7 and Claim 7.3.1, Φ_m is birational for all $m\geqslant 11$. □

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Now Theorem 1.8.1 follows from Propositions 7.2 and 7.3. That is, if $P_2 \geqslant 2$, then Φ_m is birational for $m \geqslant 11$.

If either $\xi > \frac{1}{2}$ or $\beta > \frac{1}{3}$, then $\alpha_{10} > 2$. Hence the following consequence is immediate.

Corollary 7.4. *Let X be a minimal projective 3-fold of general type. Assume $P_2(X) \geqslant 2$, $q(X) = 0$, $d_2 = 1$ and F_0 a $(1, 2)$ surface. If either $\xi > \frac{1}{2}$ or $\beta > \frac{1}{3}$ or $P_2 > 2$, then Φ_{10} is birational.*

Propositions 7.2, 7.3 and Corollary 7.4 also imply the following result.

Corollary 7.5. *Let X be a minimal projective 3-fold of general type. Assume $P_2 \geqslant 2$ and Φ_{10} is not birational. Then $P_2 = 2$, $q(X) = 0$ and $|2K_{X'}|$ is composed with a rational pencil of $(1, 2)$ surfaces.*

7.2 Classification

In the rest of this section, we classify minimal 3-folds X of general type which satisfy the following assumptions:

$$P_2(X) = 2 \textit{ and } \Phi_{10} \textit{ is not birational.} \tag{$\sharp$}$$

Note that Corollary 7.5 implies that $|2K_X|$ induces a fibration $f : X' \longrightarrow \mathbb{P}^1$ with the general fiber F a $(1, 2)$ surface.

Lemma 7.6. *If X satisfies ($\sharp$), then $0 \leqslant \chi(\mathcal{O}_X) \leqslant 3$.*

Proof. Note that the general fiber F of f is a $(1, 2)$ surface. Since $q(F) = 0$, we have $q(X) = 0$, $h^2(\mathcal{O}_X) = h^1(\mathbb{P}^1, f_*\omega_{X'})$ and $p_g(X) = h^0(f_*\omega_{X'})$. Since $P_2(X) = 2$ implies $p_g(X) \leqslant 1$, we see $\chi(\mathcal{O}_X) \geqslant 0$. By Fujita's semi-positivity [Fuj78], we have $\chi(\mathcal{O}_X) \leqslant 3$. □

Theorem 7.7. *Let X be a minimal projective 3-fold of general type. Assume $P_2 = 2$, $q(X) = 0$ and F a $(1, 2)$ surface. Then Φ_{10} is birational under one of the following conditions:*

(1) $P_3 \geqslant 4$;
(2) $P_4 \geqslant 6$;
(3) $P_5 \geqslant 8$;
(4) $P_6 \geqslant 14$.

Proof. We set $m_0 = 2$. Pick a general fiber F of $f : X' \longrightarrow \Gamma$ and a generic irreducible element C of $|G| := \mathrm{Mov}|\sigma^*(K_{F_0})|$ on F. For $m_1 = 3, 4, 5$ and 6, we have $P_{m_1} \geqslant 4$. Modulo further birational modifications to π, we may assume that the moving part $|M_{m_1}|$ of $|m_1 K_{X'}|$ is base point free. We consider the following natural maps:

$$H^0(X', S_{m_1}) \xrightarrow{\mu_{m_1}} H^0(F, S_{m_1}|_F) \xrightarrow{\nu_{m_1}} H^0(C, S_{m_1}|_C)$$

where $S_{m_1} \in |M_{m_1}|$ denotes the general member.

Let $\mathrm{Mov}|S_{m_1}|_F|$ be the moving part of $|S_{m_1}|_F|$ and let T_{m_1} be a general element in $\mathrm{Mov}|S_{m_1}|_F|$ when $h^0(F, S_{m_1}|_F) > 1$. Clearly

$$(S_{m_1} \cdot C)_{X'} \geqslant (T_{m_1} \cdot C)_F \geqslant 0.$$

Since F and C are general, both μ_{m_1} and ν_{m_1} are non-zero maps. In particular, $h^0(F, S_{m_1}|_F) > 0$ and $h^0(C, S_{m_1}|_C) > 0$.

Let $F_{(r)}$ be a general element in $\mathrm{Mov}|S_{m_1} - rF|$ if $h^0(S_{m_1} - rF) \geqslant 2$. Let $C_{(r)}$ be a general element in $\mathrm{Mov}|T_{m_1} - rC|$ if $h^0(T_{m_1} - rC) \geqslant 2$. Replace X' by its birational modification, we may and do assume that $\mathrm{Mov}|S_{m_1} - rF|$ is free.

Clearly, for $0 < r \leqslant h^0(X', S_{m_1})/h^0(F, S_{m_1}|_F)$, we have

$$h^0(X', S_{m_1} - rF) \geqslant h^0(X', S_{m_1}) - r \cdot h^0(F, S_{m_1}|_F). \tag{20}$$

Claim 7.7.1. *If $(T_{m_1} \cdot C) \leqslant 1$, then $(T_{m_1} \cdot C) = 0$.*

Proof. In fact, if $|T_{m_1}| \neq \emptyset$ and $|T_{m_1}|$ is not composed of the same pencil as that of $|C|$, then $\Phi_{|T_{m_1}|}(C)$ is a curve and so $h^0(C, T_{m_1}|_C) \geqslant 2$. Note that $g(C) = 2$. The Riemann–Roch theorem and the Clifford theorem imply that $(T_{m_1} \cdot C) = \deg(T_{m_1}|_C) \geqslant 2$, a contradiction. Hence, either $|T_{m_1}|$ is composed of the same pencil as that of $|C|$ on F or $|T_{m_1}| = \emptyset$. Claim 7.7.1 now follows. □

Claim 7.7.2. *Keep the same notation as above. Then Φ_{10} is birational under one of the following conditions:*

(i) *$(T_{m_1} \cdot C) > m_1/2$;*
(ii) *$T_{m_1} \cdot C = 0$ and $h^0(F, T_{m_1}) > 1 + m_1/3$;*
(iii) *$T_{m_1} \geqslant tC$ for some rational number $t > m_1/3$;*
(iv) *either $|T_{m_1}| = \emptyset$ and $P_{m_1} > 1 + m_1/2$ or $|T_{m_1}| \neq \emptyset$ and $\lfloor (P_{m_1} - 1)/h^0(F, T_{m_1}) \rfloor > m_1/2$;*
(v) *$F_{(r)}$ (respectively $C_{(r)}$) is algebraically equivalent to F (respectively C) and $(r+1)/m_1 > \frac{1}{2}$ (respectively $(r+1)/m_1 > \frac{1}{3}$).*

Proof. If $(T_{m_1} \cdot C) > m_1/2$, then $\xi \geqslant (1/m_1)(S_{m_1} \cdot C) \geqslant (1/m_1)(T_{m_1} \cdot C) > \frac{1}{2}$. Then Corollary 7.4 implies that Φ_{10} is birational, which proves condition (i).

Now we prove condition (iv). We claim that we have

$$m_1 \pi^*(K_X) \geqslant S_{m_1} \geqslant rF$$

for an integer $r > m_1/2$. In fact, when $|T_{m_1}| = \emptyset$, $|S_{m_1}|$ is composed of the same pencil as that of $|F|$ and we may take $r := P_{m_1} - 1$. When $|T_{m_1}| \neq \emptyset$, we may take $r = \lfloor (P_{m_1} - 1)/h^0(F, T_{m_1}) \rfloor$ and then $S_{m_1} \geqslant rF$ since $\dim \mathrm{im}(\mu_{m_1}) \leqslant h^0(F, T_{m_1})$. Then Lemma 2.1 implies $\beta \geqslant r/(m_1 + r) > \frac{1}{3}$. So Φ_{10} is birational by Corollary 7.4, which asserts condition (iv).

Since $m_1 \pi^*(K_X)|_F \geqslant T_{m_1} \geqslant tC$, we have $\beta > \frac{1}{3}$ and Φ_{10} is birational by Corollary 7.4, which proves condition (iii).

If $(T_{m_1} \cdot C) = 0$ and $h^0(F, T_{m_1}) > 1 + m_1/3$, then $|T_{m_1}|$ is composed of the same pencil as that of $|C|$ and $T_{m_1} \geqslant tC$ where $t \geqslant h^0(T_{m_1}) - 1$. Hence, Φ_{10} is birational by condition (iii), which proves condition (ii).

Finally, if $F_{(r)}$ is algebraically equivalent to F, then $S_{m_1} \geqslant F_{(r)} + F \sim (r+1)F$. Hence, $\beta \geqslant (r+1)/(m_1 + r + 1) > \frac{1}{3}$. Thus, Φ_{10} is birational by Corollary 7.4. If $C_{(r)}$ is algebraically equivalent to C, then we have $\beta \geqslant (r+1)/m_1 > \frac{1}{3}$ as well. Hence, Φ_{10} is birational, which verifies condition (v). □

We return to the proof of Theorem 7.7.

Part I. $P_3 \geqslant 4$. Set $m_1 = 3$. By Claims 7.7.2(i) and (ii) and 7.7.1, we may assume $(T_3 \cdot C) = 0$ and $h^0(F, T_3) \leqslant 2$. Also by Claim 7.7.2(iv), we may assume $|T_3| \neq \emptyset$ and $h^0(F, T_3) = 2$.

By inequality (20), one gets $h^0(S_3 - F) \geqslant 2$. Clearly we have that $S_3 \geqslant F + F_{(1)}$ and that, by assumption, $F_{(1)}$ is nef. Since $r = 1$ and $(r+1)/m_1 = \frac{2}{3} > \frac{1}{2}$, we may assume that $F_{(1)}$ is not algebraically equivalent to F by Claim 7.7.2(v).

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Now clearly we have $h^0(F, F_{(1)}|_F) \geqslant 2$. Note that we have

$$|10K_{X'}| \succeq |K_{X'} + \lceil 6\pi^*(K_X)\rceil + F_{(1)} + F|.$$

Kawamata and Viehweg vanishing gives the surjective map

$$H^0(X', K_{X'} + \lceil 6\pi^*(K_X)\rceil + F_{(1)} + F) \longrightarrow H^0(F, K_F + \lceil 6\pi^*(K_X)\rceil|_F + F_{(1)}|_F).$$

It is sufficient to verify the birationality of the rational map defined by $|K_F + \lceil 6\pi^*(K_X)|_F\rceil + \Gamma_{(1)}|$ where $\Gamma_{(1)}$ is a generic irreducible element in $\mathrm{Mov}|F_{(1)}|_F|$.

We claim that $(\pi^*(K_X) \cdot \Gamma_{(1)}) \geqslant \frac{1}{2}$. In fact, if $\Gamma_{(1)}$ is algebraically equivalent to C, then $(\pi^*(K_X) \cdot \Gamma_{(1)}) = \xi \geqslant \frac{1}{2}$ by Table A3. On the other hand, if $\Gamma_{(1)}$ is not algebraically equivalent to C, then we should have $(\Gamma_{(1)} \cdot C) \geqslant 2$. By Lemma 2.1, $(\pi^*(K_X)|_F \cdot \Gamma_{(1)}) \geqslant \frac{1}{3}(C \cdot \Gamma_{(1)}) \geqslant \frac{2}{3}$.

Clearly $|K_F + \lceil 6\pi^*(K_X)|_F\rceil + \Gamma_{(1)}|$ distinguishes different generic $\Gamma_{(1)}$ since $K_F + \lceil 6\pi^*(K_X)|_F\rceil > 0$. Now by the vanishing theorem again we have the following surjective map:

$$H^0(F, K_F + \lceil 6\pi^*(K_X)|_F\rceil + \Gamma_{(1)}) \longrightarrow H^0(\Gamma_{(1)}, K_{\Gamma_{(1)}} + D)$$

where $D := \lceil 6\pi^*(K_X)|_F\rceil|_{\Gamma_{(1)}}$ with $\deg(D) \geqslant 6(\pi^*(K_X) \cdot \Gamma_{(1)}) > 2$. So Φ_{10} is birational by the ordinary birationality principle.

Part II. $P_4 \geqslant 6$. We set $m_1 = 4$. By Claim 7.7.2(i) and (4), we may assume $(T_4 \cdot C) \leqslant 2$ and $h^0(F, T_4) \geqslant 2$. Claim 7.7.1 implies either $(T_4 \cdot C) = 0$ or $(T_4 \cdot C) = 2$.

(II-1). If $h^0(F, T_4) = 2$, we have $h^0(X', S_4 - 2F) \geqslant 2$ by inequality (20). We consider $F_{(2)}$ and may assume that $F_{(2)}$ is not algebraically equivalent to F by Claim 7.7.2(v). Now $h^0(F, F_{(2)}|_F) \geqslant 2$ and pick a generic irreducible element $\Gamma_{(2)}$ of $\mathrm{Mov}|F_{(2)}|_F|$. By Kawamata and Viehweg vanishing, we have

$$\begin{aligned}|10K_{X'}||_F &\succeq |K_{X'} + \lceil 5\pi^*(K_X)\rceil + F_{(2)} + 2F||_F\\ &= |K_F + \lceil 5\pi^*(K_X)\rceil|_F + F_{(2)}|_F|\\ &\succeq |K_F + \lceil 5\pi^*(K_X)|_F\rceil + \Gamma_{(2)}|.\end{aligned}$$

When C is algebraically equivalent to $\Gamma_{(2)}$ (in particular, $C \sim \Gamma_{(2)}$ due to the fact that $q(F) = 0$), since

$$\deg(5\pi^*(K_X)|_C) = 5\xi \geqslant \tfrac{5}{2}$$

and

$$|K_F + \lceil 5\pi^*(K_X)|_F\rceil + \Gamma_{(2)}||_C = |K_C + \lceil 5\pi^*(K_X)|_F\rceil|_C|$$

with $\deg(\lceil 5\pi^*(K_X)|_F\rceil|_C) > 2$, we see that $\Phi_{10}|_C$ is birational by Lemma 7.1 and Claim 7.3.1.

When C is not algebraically equivalent to $\Gamma_{(2)}$, we have $(\Gamma_{(2)} \cdot C) \geqslant 2$ and

$$K_F + \lceil 5\pi^*(K_X)|_F\rceil + \Gamma_{(2)} \geqslant K_F + \lceil Q_1 + C\rceil + \Gamma_{(2)}$$

for certain nef and big $\mathbb{Q}$-divisor Q_1 on F by Lemma 2.1. The vanishing theorem also shows that

$$|K_F + \lceil Q_1\rceil + \Gamma_{(2)} + C||_C = |K_C + (Q_1 + \Gamma_{(2)})|_C|$$

gives a birational map since $\deg((Q_1 + \Gamma_{(2)})|_C) > 2$. Thus, we have shown that Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

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(II-2). If $(T_4 \cdot C) = 0$ and $h^0(F, T_4) \geqslant 3$, Φ_{10} is birational by Claim 7.7.2(ii).

(II-3). If $(T_4 \cdot C) = 2$ and $h^0(F, T_4) \geqslant 3$, then $|T_4|$ is not composed of the same pencil as that of $|C|$ and $h^0(C, T_4|_C) \geqslant 2$. By the Riemann–Roch and the Clifford theorem, we see $\deg(T_4|_C) = h^0(C, T_4|_C) = 2$. Thus, $\dim \mathrm{im}(\nu_4) = 2$.

(II-3-1). If $h^0(F, T_4) \geqslant 4$, we have $h^0(F, T_4 - C) \geqslant 2$. Denote by $C_{(1)}$ a generic irreducible element of $\mathrm{Mov}|T_4 - C|$. Then we have $T_4 \geqslant C + C_{(1)}$ and we may assume that C is not algebraically equivalent to $C_{(1)}$ by Claim 7.7.2(v), which implies $(C_{(1)} \cdot C) \geqslant 2$. By the Kawamata and Viehweg vanishing and properties of the roundup operator, we have

$$\begin{aligned} |10K_{X'}||_F &\succeq |K_{X'} + \lceil 3\pi^*(K_X)\rceil + S_4 + F||_F \\ &= |K_F + \lceil 3\pi^*(K_X)\rceil|_F + S_4|_F| \\ &\succeq |K_F + \lceil 3\pi^*(K_X)|_F\rceil + C_{(1)} + C| \end{aligned}$$

and

$$|K_F + \lceil 3\pi^*(K_X)|_F\rceil + C_{(1)} + C||_C = |K_C + D|,$$

where $D := (\lceil 3\pi^*(K_X)|_F\rceil + C_{(1)})|_C$ with $\deg(D) > (C_{(1)} \cdot C) \geqslant 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(II-3-2). If $h^0(F, T_4) = 3$, we have $h^0(S_4 - F) \geqslant 3$. Again, we pick a general member $F_{(1)} \in \mathrm{Mov}|S_4 - F|$. Consider the natural map

$$H^0(X', F_{(1)}) \xrightarrow{\mu_4'} H^0(F, F_{(1)}|_F) \subset H^0(F, S_4|_F).$$

When $\dim \mathrm{im}(\mu_4') = 3$, we see $\dim \nu_4(\mathrm{im}(\mu_4')) = \dim \nu_4(\mathrm{im}(\mu_4)) = 2$; when $\dim \mathrm{im}(\mu_4') = 2$, we consider the situation $\dim \nu_4(\mathrm{im}(\mu_4')) \leqslant 1$ at first. In both cases, $h^0(F, F_{(1)}|_F - C) > 0$ and thus $F_{(1)}|_F - C \geqslant 0$. By the vanishing theorem once more, we have

$$\begin{aligned} |10K_{X'}||_F &\succeq |K_{X'} + \lceil 5\pi^*(K_X)\rceil + F_{(1)} + F||_F \\ &= |K_F + \lceil 5\pi^*(K_X)\rceil|_F + F_{(1)}|_F| \\ &\succeq |K_F + \lceil 5\pi^*(K_X)|_F\rceil + C|. \end{aligned}$$

Applying the vanishing theorem again, we see

$$|K_F + \lceil 5\pi^*(K_X)|_F\rceil + C||_C = |K_C + D|,$$

where $D := (\lceil 5\pi^*(K_X)|_F\rceil)|_C$ with $\deg(D) \geqslant 5\xi > 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

When $\dim \mathrm{im}(\mu_4') = \dim \nu_4(\mathrm{im}(\mu_4')) = 2$, then $|F_{(1)}|_F|$ is not composed with the same pencil as that of $|C|$. In particular, $(F_{(1)} \cdot C) \geqslant 2$. By Lemma 2.1, we have

$$K_F + \lceil 5\pi^*(K_X)|_F\rceil + F_{(1)}|_F \geqslant K_F + \lceil Q_2 + C\rceil + F_{(1)}|_F$$

for certain nef and big $\mathbb{Q}$-divisor Q_2. Since the vanishing theorem gives

$$|K_F + \lceil Q_2\rceil + F_{(1)}|_F + C||_C = |K_C + D'|$$

with $\deg(D') > (F_{(1)} \cdot C) \geqslant 2$, we see Φ_{10} is birational too by Lemma 7.1 and Claim 7.3.1.

Consider the last case $\dim \mathrm{im}(\mu_4') = 1$. We see that $|F_{(1)}|$ is composed of the same pencil as that of $|F|$ and $F_{(1)} \geqslant 2F$. Thus $S_4 \geqslant 3F$ and, since $3/m_1 > \frac{1}{2}$, Φ_{10} is birational by Claim 7.7.2(v).

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Part III. $P_5 \geqslant 8$. We set $m_1 = 5$. By Claims 7.7.1 and 7.7.2(i), (ii) and (iv), we may assume $(T_5 \cdot C) = 2$ and $h^0(F, T_5) \geqslant 3$. Clearly $|T_5|$ is not composed of the same pencil as that of $|C|$ and so that $h^0(C, T_5|_C) \geqslant 2$. By the Riemann–Roch and the Clifford theorem, we see $\deg(T_5|_C) = h^0(C, T_5|_C) = 2$. Thus, $\dim \operatorname{im}(\nu_5) = 2$.

(III-1). If $h^0(F, T_5) \geqslant 4$, we have $h^0(F, T_5 - C) \geqslant 2$. Let $C_{(1)}$ be a generic irreducible element in $\operatorname{Mov}|T_5 - C|$. Thus, we have $T_5 \geqslant C + C_{(1)}$ and we may assume that $C_{(1)}$ is not algebraically equivalent to C by Claim 7.7.2(v). Hence, $(C_{(1)} \cdot C) \geqslant 2$. By the Kawamata and Viehweg vanishing and properties of the roundup operator, we have the following:

$$\begin{aligned} |10K_{X'}||_F &\succeq |K_{X'} + \lceil 2\pi^*(K_X)\rceil + S_5 + F||_F \\ &= |K_F + \lceil 2\pi^*(K_X)\rceil|_F + S_5|_F| \\ &\succeq |K_F + \lceil 2\pi^*(K_X)|_F\rceil + C_{(1)} + C| \end{aligned}$$

and $|K_F + \lceil 2\pi^*(K_X)|_F\rceil + C_{(1)} + C||_C = |K_C + D|$, with

$$\deg(D) > (C_{(1)} \cdot C) \geqslant 2.$$

Thus, Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(III-2). If $h^0(F, T_5) = 3$, we have $h^0(S_5 - F) \geqslant 5$. Let $F_{(1)} \in \operatorname{Mov}|S_5 - F|$ be a general member. We consider the natural map

$$H^0(X', F_{(1)}) \xrightarrow{\mu_5'} H^0(F, F_{(1)}|_F) \subset H^0(F, S_5|_F).$$

Clearly we have $\dim \operatorname{im}(\mu_5') \leqslant h^0(F, T_5) = 3$.

When $\dim \operatorname{im}(\mu_5') = 3$, we see $\dim \nu_5(\operatorname{im}(\mu_5')) = \dim \nu_5(\operatorname{im}(\mu_5)) = 2$. Thus, $|F_{(1)}|_F|$ is not composed of the same pencil as that of $|C|$. Pick a generic irreducible element $\Gamma_{(1)}$ in the moving part of $|F_{(1)}|_F|$. Then $(\Gamma_{(1)} \cdot C) \geqslant 2$. By the vanishing theorem, we have

$$\begin{aligned} |10K_{X'}||_F &\succeq |K_{X'} + \lceil 4\pi^*(K_X)\rceil + F_{(1)} + F||_F \\ &= |K_F + \lceil 4\pi^*(K_X)\rceil|_F + F_{(1)}|_F| \\ &\succeq |K_F + \lceil 4\pi^*(K_X)|_F\rceil + \Gamma_{(1)}|. \end{aligned}$$

Applying Lemma 2.1, we have

$$|K_F + \lceil 4\pi^*(K_X)|_F\rceil + \Gamma_{(1)}| \succeq |K_F + \lceil Q_3 + C\rceil + \Gamma_{(1)}|$$

where Q_3 is certain nef and big $\mathbb{Q}$-divisor on F. Applying the vanishing once more, we have

$$|K_F + \lceil Q_3\rceil + \Gamma_{(1)} + C||_C = |K_C + D|$$

with $\deg(D) > (\Gamma_{(1)} \cdot C) \geqslant 2$. Thus, Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

When $\dim \operatorname{im}(\mu_5') \leqslant 2$, we have $h^0(X', F_{(1)} - 2F) \geqslant 1$ and hence $S_5 - 3F \geqslant 0$. Therefore, Φ_{10} is birational by Claim 7.7.2(v).

Part IV. $P_6 \geqslant 14$. We set $m_1 = 6$. By Claims 7.7.1 and 7.7.2(i), (ii) and (iv), we may assume $2 \leqslant (T_6 \cdot C) \leqslant 3$ and $h^0(F, T_6) \geqslant 4$. Clearly $|T_6|$ is not composed of the same pencil as that of $|C|$. Thus, by the Riemann–Roch theorem and the Clifford theorem, $\dim \operatorname{im}(\nu_6) = h^0(C, T_6|_C) = 2$.

(IV-1). If $h^0(F,T_6) \geqslant 5$, then we see $h^0(F,T_6 - C) \geqslant 3$. We pick a general member $C_{(1)}$ in $\mathrm{Mov}|T_6 - C|$. By Claim 7.7.2(v), we may assume that $|C_{(1)}|$ is not composed of the same pencil as that of $|C|$. We shall analyze the natural map $\nu_6' : H^0(F, C_{(1)}) \mapsto H^0(C, C_{(1)}|_C)$. Clearly $2 \leqslant \dim \mathrm{im}(\nu_6') \leqslant h^0(C, T_6|_C) = 2$.

Since $C_{(1)}$ is not algebraically equivalent to C, one has $(C_{(1)} \cdot C) \geqslant 2$. By the vanishing theorem, we have

$$\begin{aligned}|10K_{X'}||_F &\succeq |K_{X'} + \lceil \pi^*(K_X) \rceil + S_6 + F||_F \\ &\succeq |K_F + \lceil \pi^*(K_X)|_F \rceil + C_{(1)} + C|\end{aligned}$$

and $|K_F + \lceil \pi^*(K_X)|_F\rceil + C_{(1)} + C||_C = |K_C + D|$ with $\deg(D) > (C_{(1)} \cdot C) = 2$. Thus, Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(IV-2). If $h^0(F,T_6) = 4$, we have $h^0(S_6 - F) \geqslant 10$. We pick a general member $F_{(1)} \in \mathrm{Mov}|S_6 - F|$ and consider the natural map

$$H^0(X', F_{(1)}) \xrightarrow{\mu_6'} H^0(F, F_{(1)}|_F) \subset H^0(F, S_6|_F).$$

Clearly we have $\dim \mathrm{im}(\mu_6') \leqslant h^0(F, T_6) = 4$.

When $\dim \mathrm{im}(\mu_6') \leqslant 3$, we have $F_{(1)} - 3F \geqslant 0$ and then $S_6 \geqslant 4F$. By Claim 7.7.2(v), Φ_{10} is birational.

When $\dim \mathrm{im}(\mu_6') = 4$, we see $\dim \nu_6(\mathrm{im}(\mu_6')) = \dim \nu_6(\mathrm{im}(\mu_6)) = 2$. Thus, $h^0(F, F_{(1)}|_F - C) = 2$. Furthermore $|F_{(1)}|_F|$ is not composed of the same pencil as that of $|C|$. Noting that a divisor of degree one can not move on C, we see $(F_{(1)} \cdot C) \geqslant 2$. Denote by $\Gamma_{(1)}$ a general irreducible element of $\mathrm{Mov}|F_{(1)}|_F - C|$. Noting that $S_6 \geqslant F_{(1)} + F$ and applying the vanishing theorem, we have

$$\begin{aligned}|10K_{X'}| &\succeq |K_{X'} + \lceil 3\pi^*(K_X) \rceil + F_{(1)} + F| \\ &\succeq |K_F + \lceil 3\pi^*(K_X)|_F \rceil + F_{(1)}|_F|.\end{aligned}$$

If $\Gamma_{(1)}$ is not algebraically equivalent to C, we have $(\Gamma_{(1)} \cdot C) \geqslant 2$. The vanishing theorem gives

$$|K_F + \lceil 3\pi^*(K_X)|_F \rceil + \Gamma_{(1)} + C||_C = |K_C + D|$$

with $\deg(D) > (\Gamma_{(1)} \cdot C) \geqslant 2$. Thus, Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1. If $\Gamma_{(1)}$ is algebraically equivalent to C, we have $F_{(1)}|_F \geqslant 2C$ and write

$$F_{(1)}|_F = 2C + H_6$$

where H_6 is an effective divisor on F. Since $3\pi^*(K_X)|_F + F_{(1)}|_F - C - \frac{1}{2}H_6$ is nef and big, the Kawamata and Viehweg vanishing theorem implies the following surjective map

$$H^0(F, K_F + \lceil 3\pi^*(K_X)|_F + F_{(1)}|_F - \tfrac{1}{2}H_6 \rceil) \longrightarrow H^0(C, D')$$

where $D' := \lceil 3\pi^*(K_X)|_F + F_{(1)}|_F - \frac{1}{2}H_6 - C\rceil|_C$ with $\deg(D') \geqslant 3\xi + \frac{1}{2}(F_{(1)} \cdot C) > 2$. Thus, we see that Φ_{10} is birational again by Lemma 7.1 and Claim 7.3.1. So we conclude the theorem. □

COROLLARY 7.8 (Theorem 1.8(2)). *Let X be a minimal projective 3-fold of general type with $\delta(X) = 2$. If Φ_{10} is not birational, then the weighted basket $\mathbb{B}(X) = (B_X, P_2, \chi(\mathcal{O}_X))$ are dominated by an initial basket listed in Tables II1, II2 and II3 in Appendix A.*

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Proof. By Lemma 7.6 and Theorem 7.7, we see $0 \leqslant \chi(\mathcal{O}_X) \leqslant 3$, $P_2(X) = 2$, $P_3(X) \leqslant 3$, $P_4(X) \leqslant 5$, $P_5(X) \leqslant 7$ and $P_6(X) \leqslant 13$. According to [CC10a, § 3], the total number of numerical types of $\mathbb{B}(X)$ is finite. We give a list of $\mathbb{B}^0(X)$ in Tables II1, II2 and II3. □

8. Projective 4-folds of general type with positive geometric genus

In order to study 4-folds of general type, we need to prove a slightly general statement on 3-folds.

Theorem 8.1. *Let $\nu : \tilde{X} \longrightarrow X$ be a birational morphism from a nonsingular projective 3-fold $\tilde{X}$ of general type onto a minimal model X with $p_g(X) > 0$. Let Q_λ be any $\mathbb{Q}$-divisor on $\tilde{X}$ satisfying $Q_\lambda \equiv \lambda\nu^*(K_X)$ for some rational number $\lambda > 16$. Then $|K_{\tilde{X}} + \lceil Q_\lambda \rceil|$ gives a birational map onto its image. In particular, Φ_m is birational for all $m \geqslant 18$.*

Proof. From the proof of Corollary 4.10, we only need to consider the following two cases.

Case 1: $P_4 \geqslant 2$.

Case 2: $P_4 = 1$ and $P_5 \geqslant 3$.
Set $m_0 = 4$ (respectively 5) in case 1 (respectively case 2). Take a sub-pencil $\Lambda \subset |m_0K_X|$. We use the same setup as in § 2.1. We may and do assume that π factors through ν, i.e. there is a birational morphism $\mu : X' \longrightarrow \tilde{X}$ so that $\pi = \nu \circ \mu$ and that $\mu^*(\{Q_\lambda\}) \cup \{\text{exc. divisors of } \mu\}$ has simple normal crossing supports.

Since

$$\mu_*\mathcal{O}_{X'}(K_{X'} + \lceil \mu^*(Q_\lambda) \rceil) \subseteq \mu_*\mathcal{O}_{X'}(K_{X'} + \mu^*\lceil Q_\lambda \rceil) \subseteq \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + \lceil Q_\lambda \rceil),$$

it is sufficient to prove the birationality of $\Phi_{|K_{X'}+\lceil \mu^*(Q_\lambda)\rceil|}$. We write $Q'_\lambda := \mu^*(Q_\lambda) \equiv \lambda\pi^*(K_X)$.

We have an induced fibration $f : X' \longrightarrow \Gamma$ onto a smooth projective curve. Let F be a general fiber of f. Recall that we have $m_0\pi^*(K_X) \sim_{\mathbb{Q}} \theta F + E'_\Lambda$ for a positive integer θ and an effective $\mathbb{Q}$-divisor E'_Λ on X'.

Without loss of generality, we may assume $p_g(X) = 1$ (the case with $p_g(X) > 1$ is much easier). Clearly one has $p_g(F) > 0$.

Claim 8.1.1. One has $h^0(X', K_{X'} + \lceil Q'_\lambda \rceil) > 0$ for $\lambda > 2m_0 + 1$.

By Lemma 2.1,

$$\pi^*(K_X)|_F \equiv \frac{1}{m_0+1}\sigma^*(K_{F_0}) + H_{m_0}$$

for a certain effective $\mathbb{Q}$-divisor H_{m_0} on F. Since $Q'_\lambda - F - (1/\theta)E'_\Lambda \equiv (\lambda - m_0/\theta)\pi^*(K_X)$ is nef and big, Kawamata and Viehweg vanishing implies the surjective map

$$H^0\left(X', K_{X'} + \left\lceil Q'_\lambda - \frac{1}{\theta}E'_\Lambda \right\rceil\right) \longrightarrow H^0\left(F, K_F + \left\lceil Q'_\lambda - \frac{1}{\theta}E'_\Lambda \right\rceil\Big|_F\right). \tag{21}$$

Let

$$\begin{aligned} Q_{\lambda,F} &:= \left(Q'_\lambda - \frac{1}{\theta}E'_\Lambda\right)\Big|_F - (m_0+1)H_{m_0} - \sigma^*(K_{F_0}) \\ &\equiv \left(\lambda - \frac{m_0}{\theta} - m_0 - 1\right)\pi^*(K_X)\Big|_F, \end{aligned}$$

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which is nef and big. Since $p_g(F)>0$, we have

$$\begin{aligned}h^0\Big(F, K_F+\Big\lceil Q'_\lambda-\frac{1}{\theta}E'_\Lambda\Big\rceil\Big|_F\Big)\\ \geqslant h^0\Big(F, K_F+\Big\lceil\Big(Q'_\lambda-\frac{1}{\theta}E'_\Lambda\Big)\Big|_F-(m_0+1)H_{m_0}\Big)\Big\rceil\\ = h^0(F, K_F+\lceil Q_{\lambda,F}\rceil+\sigma^*(K_{F_0}))\geqslant 2\end{aligned}$$

by [CC10b, Lemma 2.14]. This verifies the claim.

Claim 8.1.2. *The linear system $|K_{X'}+\lceil Q'_\lambda\rceil|$ distinguishes different general fibers of f for any $\lambda>3m_0+1$.*

Proof. When $g(\Gamma)=0$, we consider $Q'_\zeta:=Q'_\lambda-F-(1/\theta)E'_\Lambda\equiv\zeta\pi^*(K_X)$ with $\zeta=\lambda-m_0/\theta$. It is clear that $K_{X'}+\lceil Q'_\lambda\rceil\geqslant(K_{X'}+\lceil Q'_\zeta\rceil)+F$ and hence $|K_{X'}+\lceil Q'_\lambda\rceil|$ distinguishes different general fibers by Claim 8.1.1 since $\zeta>2m_0+1$.

When $g(\Gamma)>0$, we have $\theta\geqslant 2$. Pick two different general fibers F_1 and F_2 of f. The vanishing theorem gives the surjective map

$$\begin{aligned}H^0\Big(X', K_{X'}+\Big\lceil Q'_\lambda-\frac{2}{\theta}E'_\Lambda\Big\rceil\Big)\\ \longrightarrow\bigoplus_{i=1}^{2}H^0\Big(F_i,\Big(K_{X'}+\Big\lceil Q'_\lambda-F_1-F_2-\frac{2}{\theta}E'_\Lambda\Big\rceil+F_1+F_2\Big)\Big|_{F_i}\Big)\end{aligned}$$

where we note that $(K_{X'}+\lceil Q'_\lambda-F_1-F_2-(2/\theta)E'_\Lambda\rceil)|_{F_i}\geqslant 0$ due to Claim 8.1.1 and the fact $(F_1+F_2)|_{F_i}=0$. Hence, $|K_{X'}+\lceil Q'_\lambda\rceil|$ distinguishes F_1 and F_2. □

Now we discuss two cases independently.

Case 1: $P_4\geqslant 2$.
If F is a $(1,2)$ surface, we take $|G|:=\mathrm{Mov}|\sigma^*(K_{F_0})|$ and a general member $C\in|G|$. By the surjection map in (21) and Claim 8.1.2, it is sufficient to study the linear system $|K_F+\lceil(Q'_\lambda-(1/\theta)E'_\Lambda)|_F\rceil|$. For any t, let

$$L_{\lambda,t}:=\Big(Q'_\lambda-\frac{1}{\theta}E'_\Lambda\Big)\Big|_F-t\sigma^*(K_{F_0})-5tH_4\equiv\Big(\lambda-\frac{4}{\theta}-5t\Big)\pi^*(K_X)\Big|_F,$$

which is nef and big as long as $\lambda-(4/\theta)-5t>0$. Note also that $(Q'_\lambda-(1/\theta)E'_\Lambda)|_F\geqslant L_{\lambda,t}+t\sigma^*(K_{F_0})$. For simplicity, $L_{\lambda,0}$ is denoted by L_λ. In fact, for $\lambda>14$ and by [CC10b, Lemma 2.14], one has

$$K_F+\Big\lceil Q'_\lambda-\frac{1}{\theta}E'_\Lambda\Big\rceil\Big|_F\geqslant(K_F+\lceil L_{\lambda,2}\rceil+\sigma^*(K_{F_0}))+C\geqslant C.$$

Thus, $|K_F+\lceil(Q'_\lambda-(1/\theta)E'_\Lambda)|_F\rceil|$ separates different general curves C when $\lambda>14$. Restricting to the curve C, one sees by the vanishing theorem that

$$\Big|K_F+\Big\lceil\Big(Q'_\lambda-\frac{1}{\theta}E'_\Lambda\Big)\Big|_F\Big\rceil\Big|\Big|_C\geqslant|K_F+\lceil L_{\lambda,1}\rceil+C||_C=|K_C+\lceil L_{\lambda,1}\rceil|_C|.$$

Since $\deg(\lceil L_{\lambda,1}\rceil|_C)\geqslant(\lambda-(4/\theta)-5)\xi>2$ for $\xi\geqslant\frac{2}{7}$ (cf. Table A3 with $m_0=4$). Thus, $\Phi_{|K_{X'}+\lceil Q'_\lambda\rceil|}$ separates points on the general curve C and, hence, is birational when $\lambda>16$.

Assume that F is not a $(1, 2)$ surface. We would like to study $|K_F + \lceil L_\lambda \rceil|$ where $L_\lambda := (Q'_\lambda - (1/\theta)E'_\Lambda)|_F$, making use of the relation (21). If $K^2_{F_0} \geqslant 2$, inequalities (9) and (11) imply

$$L^2_\lambda \geqslant \frac{2(\lambda - 4)^2}{25} > 8$$

whenever $\lambda > 14$. If F is a $(1, 1)$ surface, then we have $q(X) = g(\Gamma) \geqslant 0$ and $h^2(\mathcal{O}_X) = 0$ as seen in the proof of case 2 of Corollary 4.10. Hence, we have $\chi(\mathcal{O}_X) \leqslant 0$ and Reid's Riemann–Roch formula gives $P_5 > P_4 \geqslant 2$. In particular, we have $P_5 \geqslant 3$. We omit the discussion for the situation when $|5K_{X'}|$ and $|4K_{X'}|$ are composed with the same pencil since that is a comparatively much better case. So may assume that $|5K_{X'}||_F$ is moving on F. If we take $|G_1| := \mathrm{Mov}|\lceil 5\pi^*(K_X)\rceil|_F|$, we have $\beta_{G_1} = \frac{1}{5}$. Then, by Lemmas 2.1 and 2.4, we have

$$L^2_\lambda \geqslant \frac{(\lambda - 4)^2}{25}(\sigma^*(K_{F_0}) \cdot G_1) \geqslant \frac{2(\lambda - 4)^2}{25} > 8$$

whenever $\lambda > 14$. Finally, for both cases, $(L_\lambda \cdot \tilde{C}) \geqslant (2(\lambda - 4))/5 \geqslant 4$ for $\lambda \geqslant 14$ and for any very general curve $\tilde{C}$ on F. Therefore, by Lemma 2.3, $|K_F + \lceil L_\lambda \rceil|$ gives a birational map when $\lambda \geqslant 14$.

Hence, when $P_4 \geqslant 2$, $\Phi_{|K_{X'} + \lceil Q'_\lambda \rceil|}$ is birational for $\lambda > 16$.

Case 2: $P_4 = 1$ and $P_5 \geqslant 3$.
We set $m_0 = 5$. If $d_5 = 1$, we set $\Lambda = |5K_X|$. Then we are in a much better situation than that of $P_3 = 2$ since we have $\theta \geqslant 2$ (and noting that $\theta/m_0 = \frac{2}{5} > \frac{1}{3}$). We omit the details and leave this as an exercise to interested readers.

If $d_5 \geqslant 2$, we take a sub-pencil $\Lambda \subset |5K_X|$ and Λ induces a fibration $f : X' \longrightarrow \Gamma$ onto a smooth complete curve Γ. As we have seen in case 3 of Corollary 4.10, the general fiber F satisfies $K^2_{F_0} \geqslant 2$. For the similar reason, we can take $m_1 = 5$ and $|G| := \mathrm{Mov}|m_1 K_{X'}|_F|$. Pick a generic irreducible element C in $|G|$. Lemma 2.1 implies $\xi = (\pi^*(K_X) \cdot C) \geqslant \frac{1}{6}(\sigma^*(K_{F_0}) \cdot C) \geqslant \frac{1}{3}$. We may write $5\pi^*(K_X)|_F \equiv C + N_5$ for an effective $\mathbb{Q}$-divisor N_5 on F. For two different generic irreducible curves C_1 and C_2 in $|G|$, we set

$$L_{\lambda,2} := \left(Q'_\lambda - \frac{1}{\theta}E'_\Lambda\right)\bigg|_F - C_1 - C_2 - 2N_5,$$

and

$$L_{\lambda,1} := \left(Q'_\lambda - \frac{1}{\theta}E'_\Lambda\right)\bigg|_F - C - N_5,$$

respectively. It is clear that they are both nef and big whenever $\lambda > 15$.

Thanks to the vanishing theorem, we have the surjective map

$$\begin{aligned} H^0(F, K_F + \lceil L_\lambda - 2N_5 \rceil) \longrightarrow & H^0(C_1, K_{C_1} + \lceil L_{\lambda,2} \rceil|_{C_1} + C_2|_{C_1}) \\ & \oplus H^0(C_2, K_{C_2} + \lceil L_{\lambda,2} \rceil|_{C_2} + C_1|_{C_2}) \end{aligned}$$

if $\lambda > 15$. It is clear that

$$H^0(C_i, K_{C_i} + \lceil L_{\lambda,2} \rceil_{|C_i} + C_{2-i}|_{C_i}) \neq 0$$

since $L_{\lambda,2}$ is nef and big. Hence, $|K_F + \lceil (Q'_\lambda - (1/\theta)E'_\Lambda)|_F - 2N_5 \rceil| = |K_F + \lceil L_\lambda - 2N_5 \rceil|$ separates different general curves C in $|G|$. This also implies that $|K_F + \lceil (Q'_\lambda - (1/\theta)E'_\Lambda) \rceil|$

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can distinguish C_1 and C_2. Now applying the vanishing theorem once more, we get the surjective map

$$H^0(F, K_F + \lceil L_\lambda - N_5 \rceil) \longrightarrow H^0(C, K_C + \lceil L_{\lambda,1} \rceil_{|C})$$

with

$$\deg(\lceil L_{\lambda,1} \rceil_{|C}) \geqslant \left(\lambda - \frac{5}{\theta} - 5\right)\xi > 2$$

whenever $\lambda > 16$ for $\xi \geqslant \frac{1}{3}$. Thus, by Theorem 2.7, $|K_{X'} + \lceil Q'_\lambda \rceil|$ gives a birational map for $\lambda > 16$. So we conclude the statement of the theorem. $\square$

Theorem 8.2 (Theorem 1.11). *Let V be a nonsingular projective 4-fold of general type. Then:*

(1) *when $p_g(V) \geqslant 2$, $\Phi_{m,V}$ is birational for all $m \geqslant 35$;*
(2) *when $p_g(V) \geqslant 19$, $\Phi_{m,V}$ is birational for all $m \geqslant 18$.*

Proof. Let Z be the minimal model of V. We set $m_0 = 1$, $\Lambda = |K_Z|$ and use the setup in § 2.1. Thus, we have an induced fibration $f : Z' \longrightarrow \Gamma$.

First we consider the case $\dim \Gamma = 1$. Recall that we have $M_\Lambda \equiv \theta F$ for a general fiber F of f, where $\theta \geqslant p_g(Z) - 1$. It is clear that, when $m \geqslant 3$, $|mK_{Z'}|$ distinguishes different general fibers of f. Pick a general fiber $F = X'$, which is a nonsingular projective 3-fold of general type with $p_g(X') > 0$. Replace by its birational model, we may assume that there is a birational morphism $\nu : X' \longrightarrow X$ onto a minimal model. By Lemma 2.1, we have

$$\pi^*(K_Z)|_{X'} \equiv \frac{\theta}{\theta+1}\nu^*(K_X) + J_1$$

for an effective $\mathbb{Q}$-divisor J_1 on X'. When m is large, since $(m-1)\pi^*(K_Z) - X' - (1/\theta)E'_\Lambda$ is nef and big, Kawamata and Viehweg vanishing implies

$$\begin{aligned}\left|K_{Z'} + \left\lceil (m-1)\pi^*(K_Z) - \frac{1}{\theta}E'_\Lambda \right\rceil\right|\Bigg|_{X'} &= \left|K_{X'} + \left\lceil (m-1)\pi^*(K_Z) - \frac{1}{\theta}E'_\Lambda \right\rceil_{X'}\right| \\ &\succeq |K_{X'} + \lceil R_m \rceil|\end{aligned}$$

where $R_m := ((m-1)\pi^*(K_Z) - X' - (1/\theta)E'_\Lambda)|_{X'}$. In fact, we have

$$\begin{aligned}R_m &\equiv \left(m - 1 - \frac{1}{\theta}\right)\pi^*(K_Z)\Big|_{X'} \\ &\equiv \left(\frac{m\theta}{\theta+1} - 1\right)\nu^*(K_X) + \left(m - 1 - \frac{1}{\theta}\right)J_1.\end{aligned}$$

Since $m\theta/(\theta+1) - 1 > 16$ whenever either $m \geqslant 18$ and $p_g(Z) \geqslant 19$ or $m \geqslant 35$ and $p_g(Z) \geqslant 2$, Theorem 8.1 implies that $|K_{X'} + \lceil R_m - (m-1-1/p)J_1 \rceil|$ gives a birational map. Thus, statements of the theorem follow in this case.

Next we consider the case $\dim \Gamma \geqslant 2$. By definition, $\theta = 1$. Clearly it is sufficient to consider $\Phi_{|mK_{Z'}|}|_{X'}$ for a general member $X' \in |M_\Lambda|$. We consider a general X' and, similarly, we may assume that there is a birational morphism $\nu : X' \longrightarrow X$ onto a minimal model X. Then Kawamata's extension theorem [Kaw99, Theorem A] still implies

$$\pi^*(K_Z)|_{X'} \geqslant \tfrac{1}{2}\nu^*(K_X). \tag{22}$$

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We consider the linear system $|M_\Lambda|_{X'}|$, which may be assumed to be base point free modulo further birational modifications. Pick a generic irreducible element S of this linear system. We clearly have

$$\pi^*(K_Z)|_{X'} \geqslant M_\Lambda|_{X'} \geqslant S.$$

Modulo $\mathbb{Q}$-linear equivalence, one has

$$2S \leqslant (\pi^*(K_Z) + X')|_{X'} \leqslant K_{X'}.$$

Thus, Kawamata's extension theorem gives

$$\nu^*(K_X)|_S \geqslant \tfrac{2}{3}\sigma^*(K_{S_0}) \tag{23}$$

where $\sigma : S \longrightarrow S_0$ is the contraction onto the minimal model S_0 of S. Both (22) and (23) imply

$$\pi^*(K_Z)|_S \geqslant \tfrac{1}{3}\sigma^*(K_{S_0}).$$

Write $\pi^*(K_Z)|_{X'} \equiv S + H_\Lambda$ where H_Λ is an effective $\mathbb{Q}$-divisor on X'. Since $R_m - S - H_\Lambda \equiv (m-3)\pi^*(K_Z)|_{X'}$ is nef and big, the vanishing theorem implies

$$\begin{aligned} |K_{X'} + \lceil R_m - H_\Lambda \rceil|_{|S} &= |K_S + \lceil R_m - S - H_\Lambda \rceil_{|S}| \\ &\succeq |K_S + \lceil R_{m,S} \rceil| \end{aligned}$$

where $R_{m,S} := (R_m - S - H_\Lambda)|_S$. Note that

$$\begin{aligned} R_{m,S} &\equiv (m-3)\pi^*(K_Z)|_S \\ &\equiv \frac{m-3}{3}\sigma^*(K_{S_0}) + E_{m,S} \end{aligned}$$

where $E_{m,S}$ is an effective $\mathbb{Q}$-divisor on S. Now it is clear by Lemma 2.3 that $|K_S + \lceil R_{m,S} - E_{m,S} \rceil|$ gives a birational map whenever $m \geqslant 15$. Again Kawamata and Viehweg vanishing shows that $|K_{X'} + \lceil R_m \rceil|$ distinguishes different elements S. Thus, we have shown that $\Phi_{m,Z}$ is birational for all $m \geqslant 15$ in this case. We are done. □

Brown and Reid kindly informed us of the following interesting canonical 4-folds.

Example 8.3. The general hypersurfaces $W_{36} \subset \mathbb{P}(1,1,3,5,7,18)$ and $Y_{36} \subset \mathbb{P}(1,1,4,5,6,18)$ have canonical singularities, $p_g = 2$. It is clear that the 17-canonical maps of these two 4-folds are not birational.

Problem 8.4. It is a very interesting problem to find more examples of 4-folds of general type so that Φ_m is not birational for large m.

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Appendix A. Tables

Table F0.

Types	B_X	χ	K_X^3	$\delta(X)$
2a	$\{4\times(1,2),(4,9),(2,5),(5,13),3\times(1,3),2\times(1,4)\}$	2	1/1170	18
41	$\{5\times(1,2),(4,9),2\times(3,8),(1,3),2\times(2,7)\}$	2	1/252	13

Table F1.

Types	B_X	χ	K_X^3	$\delta(X)$
2	$\{4\times(1,2),(4,9),2\times(2,5),(3,8),3\times(1,3),2\times(1,4)\}$	2	1/360	15
3	$\{6\times(1,2),(5,11),4\times(2,5),(3,8),4\times(1,3),(2,7),2\times(1,4)\}$	3	23/9240	15
5.1	$\{7\times(1,2),(4,9),3\times(2,5),(5,13),4\times(1,3),(3,11),(1,4)\}$	3	61/25740	15
5.2	$\{7\times(1,2),(4,9),2\times(2,5),(7,18),4\times(1,3),(3,11),(1,4)\}$	3	1/660	15
5.3	$\{7\times(1,2),(4,9),(2,5),(9,23),4\times(1,3),(3,11),(1,4)\}$	3	47/45540	15
5a	$\{7\times(1,2),(4,9),(11,28),4\times(1,3),(3,11),(1,4)\}$	3	1/1386	15
5b	$\{7\times(1,2),(4,9),3\times(2,5),(5,13),4\times(1,3),(4,15)\}$	3	1/1170	15
53a	$\{3\times(1,2),(4,9),2\times(2,5),(5,13),3\times(1,3),(1,5)\}$	2	1/1170	15

Table F2.

Types	B_X	χ	K_X^3	$\delta(X)$
1	$\{5\times(1,2),(3,7),3\times(2,5),3\times(1,3),(3,11)\}$	2	3/770	14
4	$\{7\times(1,2),(4,9),4\times(2,5),(4,11),3\times(1,3),(2,7),2\times(1,4)\}$	3	13/3465	14
4.5	$\{7\times(1,2),(4,9),4\times(2,5),(5,14),2\times(1,3),(2,7),2\times(1,4)\}$	3	1/630	14
5	$\{7\times(1,2),(4,9),4\times(2,5),(3,8),4\times(1,3),(3,11),(1,4)\}$	3	17/3960	14
5.4	$\{7\times(1,2),(4,9),4\times(2,5),(3,8),4\times(1,3),(4,15)\}$	3	1/360	14
6	$\{9\times(1,2),2\times(3,7),(2,5),(4,11),4\times(1,3),2\times(2,7),(1,5)\}$	3	1/462	14
7	$\{5\times(1,2),(4,9),(3,7),5\times(1,3),(2,7),(1,5)\}$	2	1/630	14
7a	$\{5\times(1,2),(7,16),5\times(1,3),(2,7),(1,5)\}$	2	1/1680	14
10	$\{8\times(1,2),(4,9),(3,7),2\times(3,8),5\times(1,3),(2,7),(1,4),(1,5)\}$	3	1/630	14
11	$\{9\times(1,2),2\times(3,7),(3,8),(4,11),3\times(1,3),(3,10),(1,4),(1,5)\}$	3	3/3080	14
12	$\{9\times(1,2),(4,9),(2,5),2\times(3,8),4\times(1,3),2\times(2,7),(1,5)\}$	3	1/252	14
12.1	$\{9\times(1,2),(4,9),(5,13),(3,8),4\times(1,3),2\times(2,7),(1,5)\}$	3	67/32760	14
12a	$\{9\times(1,2),(4,9),(8,21),4\times(1,3),2\times(2,7),(1,5)\}$	3	1/630	14
14	$\{10\times(1,2),(3,7),2\times(2,5),2\times(3,8),6\times(1,3),2\times(2,7),$ $(1,4),(1,5)\}$	4	1/770	14
15	$\{11\times(1,2),(4,9),(3,7),2\times(2,5),(3,8),(4,11),5\times(1,3),2\times(2,7),$ $(1,4),(1,5)\}$	4	71/27720	14

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TABLE F2. Continued.

Types	B_X	χ	K_X^3	$\delta(X)$
15.1	$\{11\times(1,2),(4,9),(3,7),2\times(2,5),(7,19),5\times(1,3),2\times(2,7),(1,4),(1,5)\}$	4	47/23940	14
15.2	$\{11\times(1,2),(7,16),2\times(2,5),(3,8),(4,11),5\times(1,3),2\times(2,7),(1,4),(1,5)\}$	4	29/18480	14
16	$\{11\times(1,2),(4,9),(3,7),2\times(2,5),2\times(3,8),6\times(1,3),(2,7),(3,11),(1,5)\}$	4	43/13860	14
16.1	$\{11\times(1,2),(4,9),(3,7),(2,5),(5,13),(3,8),6\times(1,3),(2,7),(3,11),(1,5)\}$	4	85/72072	14
16.2	$\{11\times(1,2),(7,16),2\times(2,5),2\times(3,8),6\times(1,3),(2,7),(3,11),(1,5)\}$	4	13/6160	14
16.4	$\{11\times(1,2),(7,16),2\times(2,5),2\times(3,8),6\times(1,3),(5,18),(1,5)\}$	4	1/720	14
16.5	$\{11\times(1,2),(4,9),(3,7),2\times(2,5),2\times(3,8),6\times(1,3),(5,18),(1,5)\}$	4	1/420	14
17	$\{9\times(1,2),2\times(3,7),2\times(4,11),3\times(1,3),(2,7),(1,4),(1,5)\}$	3	3/1540	14
18	$\{9\times(1,2),2\times(3,7),(3,8),(4,11),4\times(1,3),(3,11),(1,5)\}$	3	23/9240	14
18b	$\{9\times(1,2),2\times(3,7),(7,19),4\times(1,3),(3,11),(1,5)\}$	3	83/43890	14
20	$\{7\times(1,2),2\times(4,9),(2,5),(3,8),6\times(1,3),(2,7),(1,4),(1,5)\}$	3	1/504	14
21	$\{6\times(1,2),(4,9),(3,8),3\times(1,3),(3,10),(1,5)\}$	2	1/360	14
23	$\{8\times(1,2),(4,9),(3,7),(2,5),(4,11),4\times(1,3),(3,10),(1,4),(1,5)\}$	3	19/13860	14
25	$\{9\times(1,2),(5,11),(4,9),3\times(2,5),(3,8),7\times(1,3),2\times(2,7),(1,4),(1,5)\}$	4	47/27720	14
25a	$\{9\times(1,2),(9,20),3\times(2,5),(3,8),7\times(1,3),2\times(2,7),(1,4),(1,5)\}$	4	1/840	14
26	$\{10\times(1,2),2\times(4,9),3\times(2,5),(4,11),6\times(1,3),2\times(2,7),(1,4),(1,5)\}$	4	41/13860	14
27	$\{10\times(1,2),2\times(4,9),3\times(2,5),(3,8),7\times(1,3),(2,7),(3,11),(1,5)\}$	4	97/27720	14
27.3	$\{10\times(1,2),2\times(4,9),3\times(2,5),(3,8),7\times(1,3),(5,18),(1,5)\}$	4	1/360	14
28	$\{5\times(1,2),(5,11),(3,8),4\times(1,3),(2,7),(1,5)\}$	2	23/9240	14
29	$\{6\times(1,2),(4,9),(4,11),3\times(1,3),(2,7),(1,5)\}$	2	13/3465	14
29.1	$\{6\times(1,2),(4,9),(5,14),2\times(1,3),(2,7),(1,5)\}$	2	1/630	14
30	$\{7\times(1,2),(5,11),(3,7),(2,5),(4,11),5\times(1,3),(2,7),(1,4),(1,5)\}$	3	1/924	14
31	$\{7\times(1,2),(5,11),(3,7),(2,5),(3,8),6\times(1,3),(3,11),(1,5)\}$	3	1/616	14
32	$\{8\times(1,2),(4,9),(3,7),(2,5),(4,11),5\times(1,3),(3,11),(1,5)\}$	3	2/693	14
32a	$\{8\times(1,2),(7,16),(2,5),(4,11),5\times(1,3),(3,11),(1,5)\}$	3	1/528	14

Table F2. Continued.

Types	B_X	χ	K_X^3	$\delta(X)$
33	$5\times(1,2),2\times(3,7),(3,8),(1,3),(3,10),(2,7)\}$	2	1/840	14
34	$\{7\times(1,2),(4,9),(3,7),2\times(2,5),(3,8),3\times(1,3),3\times(2,7)\}$	3	1/360	14
34a	$\{7\times(1,2),(7,16),2\times(2,5),(3,8),3\times(1,3),3\times(2,7)\}$	3	1/560	14
35	$\{5\times(1,2),2\times(3,7),(4,11),(1,3),2\times(2,7)\}$	2	1/462	14
36	$\{4\times(1,2),(4,9),(3,7),(2,5),2\times(1,3),(3,10),(2,7)\}$	2	1/630	14
36a	$\{4\times(1,2),(7,16),(2,5),2\times(1,3),(3,10),(2,7)\}$	2	1/1680	14
36b	$\{4\times(1,2),(4,9),(3,7),(2,5),2\times(1,3),(5,17)\}$	2	4/5355	14
37	$6\times(1,2),2\times(4,9),3\times(2,5),4\times(1,3),3\times(2,7)\}$	3	1/315	14
38	$\{3\times(1,2),(5,11),(3,7),(2,5),3\times(1,3),2\times(2,7)\}$	2	1/770	14
39	$\{7\times(1,2),(4,9),(3,7),(2,5),2\times(3,8),2\times(1,3),(3,10),(2,7),(1,4)\}$	3	1/630	14
40	$\{9\times(1,2),2\times(4,9),3\times(2,5),2\times(3,8),4\times(1,3),3\times(2,7),(1,4)\}$	4	1/315	14
42	$\{6\times(1,2),(5,11),(3,7),(2,5),2\times(3,8),3\times(1,3),2\times(2,7),(1,4)\}$	3	1/770	14
43	$\{7\times(1,2),(4,9),(3,7),(2,5),(3,8),(4,11),2\times(1,3),2\times(2,7),(1,4)\}$	3	71/27720	14
43.1	$\{7\times(1,2),(7,16),(2,5),(3,8),(4,11),2\times(1,3),2\times(2,7),(1,4)\}$	3	29/18480	14
43c	$\{7\times(1,2),(7,16),(2,5),(7,19),2\times(1,3),2\times(2,7),(1,4)\}$	3	31/31920	14
43.2	$\{7\times(1,2),(4,9),(3,7),(2,5),(7,19),2\times(1,3),2\times(2,7),(1,4)\}$	3	47/23940	14
44	$\{7\times(1,2),(4,9),(3,7),(2,5),2\times(3,8),3\times(1,3),(2,7),(3,11)\}$	3	43/13860	14
44.1	$\{7\times(1,2),(4,9),(3,7),(5,13),(3,8),3\times(1,3),(2,7),(3,11)\}$	3	85/72072	14
44.2	$\{7\times(1,2),(4,9),(3,7),(2,5),2\times(3,8),3\times(1,3),(5,18)\}$	3	1/420	14
44.3	$\{7\times(1,2),(7,16),(2,5),2\times(3,8),3\times(1,3),(2,7),(3,11)\}$	3	13/6160	14
44c	$\{7\times(1,2),(7,16),(2,5),2\times(3,8),3\times(1,3),(5,18)\}$	3	1/720	14
45	$\{3\times(1,2),2\times(4,9),(3,8),3\times(1,3),(2,7),(1,4)\}$	2	1/504	14
46	$\{6\times(1,2),2\times(4,9),2\times(2,5),(3,8),3\times(1,3),(3,10),(2,7),(1,4)\}$	3	1/504	14
46b	$\{6\times(1,2),2\times(4,9),2\times(2,5),(3,8),3\times(1,3),(5,17),(1,4)\}$	3	7/6120	14
48	$\{4\times(1,2),(4,9),(3,7),(4,11),(1,3),(3,10),(1,4)\}$	2	19/13860	14
49	$\{5\times(1,2),(5,11),(4,9),2\times(2,5),(3,8),4\times(1,3),2\times(2,7),(1,4)\}$	3	47/27720	14
49a	$\{(5\times(1,2),(9,20),2\times(2,5),(3,8),4\times(1,3),2\times(2,7),(1,4)\}$	3	1/840	14
50	$\{6\times(1,2),2\times(2,9),2\times(2,5),(4,11),3\times(1,3),2\times(2,7),(1,4)\}$	3	41/13860	14
51	$\{6\times(1,2),2\times(4,9),2\times(2,5),(3,8),4\times(1,3),(2,7),(3,11)\}$	3	97/27720	14
51.1	$\{6\times(1,2),2\times(4,9),(2,5),(5,13),4\times(1,3),(2,7),(3,11)\}$	3	71/45045	14
52	$\{4\times(1,2),(3,7),2\times(2,5),2\times(3,8),2\times(1,3),(1,5)\}$	2	1/420	14
53	$3\times(1,2),(4,9),3\times(2,5),(3,8),3\times(1,3),(1,5)\}$	2	1/360	14
54	$\{2\times(1,2),2\times(3,7),3\times(2,5),(3,8),(1,3),(2,7)\}$	2	1/840	14
56	$\{(1,2),(4,9),(3,7),4\times(2,5),2\times(1,3),(2,7)\}$	2	1/630	14
58	$\{4\times(1,2),(4,9),(3,7),4\times(2,5),2\times(3,8),2\times(1,3),(2,7),(1,4)\}$	3	1/630	14
59	$\{2\times(1,2),2\times(3,7),2\times(2,5),(3,8),(4,11),(1,4)\}$	2	3/3080	14
60	$3\times(1,2),2\times(4,9),5(2,5),(3,8),3\times(1,3),(2,7),(1,4)\}$	3	1/504	14
62	$\{(1,2),(4,9),(3,7),3\times(2,5),(4,11),(1,3),(1,4)\}$	2	19/13860	14

Table II1.

No.	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
1	$\{5 * (1,2), 2 * (1,3)\}$	$1/6$	0	$(3,5,7,11)$
2	$\{5 * (1,2), (1,3), (1,4)\}$	$1/12$	0	$(3,5,6,9)$
3	$\{18 * (1,2), (1,3), \}$	$1/3$	1	$(1,5,6,13)$
4	$\{(18-4t) * (1,2), 3t * (1,3), (1,4)\}, t = 0,1,2$	$1/4$	1	$(1+t, 5, 5+t, 11+t)$
5	$\{(18-4t) * (1,2), 3t * (1,3), (1,5)\}, 5 \leqslant r \leqslant 12; t = 0,1,2$	$1/r$	1	$(1+t, 5, 5+t, 10+t)$
6	$\{(17-4t) * (1,2), (2+3t) * (1,3)\}, t = 0,1,2$	$1/6$	1	$(1+t, 4, 4+t, 9+t)$
7	$\{(14-4t) * (1,2), (2+3t) * (1,3), 2 * (1,4)\}, t = 0,1$	$1/6$	1	$(2+t, 5, 5+t, 10+t)$
8	$\{(14-4t) * (1,2), (2+3t) * (1,3), (1,4), (1,5)\}, t = 0,1$	$7/60$	1	$(2+t, 5, 5+t, 9+t)$
9	$\{(14-4t) * (1,2), (2+3t) * (1,3), (1,4), (1,6)\}, t = 0,1$	$1/12$	1	$(2+t, 5, 5+t, 9+t)$
10	$\{(14-4t) * (1,2), (1+3t) * (1,3), 3 * (1,4)\}, t = 0,1$	$1/12$	1	$(2+t, 5, 4+t, 8+t)$
11	$\{(17-4t) * (1,2), (1+3t) * (1,3), (1,4)\}, t = 0,1,2$	$1/12$	1	$(1+t, 4, 3+t, 7+t)$

Table II2.

No.	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
1	$\{27 * (1,2), 2 * (1,3), (1,r)\}$	$\frac{1}{6} + \frac{1}{r}$	2	$(0,5,5,13)$
2	$\{(27-4t) * (1,2), (1+3t) * (1,3),$ $2 * (1,4)\}, t = 0,1$	$1/3$	2	$(t, 5, 4+t, 12+t)$
3	$\{(27-4t) * (1,2), (1+3t) * (1,3),$ $(1,4), (1,r)\}, 5 \leqslant r; t = 0,1,2$	$\frac{1}{12} + \frac{1}{r}$	2	$(t, 5, 4+t, 11+t)$
4	$\{(27-4t) * (1,2), (1+3t) * (1,3),$ $(1,r_1), (1,r_2)\}, (r_1, r_2) \in I_4; t = 0,1,2,3$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{6}$	2	$(t, 5, 4+t, 10+t)$
5	$\{(26-4t) * (1,2), (4+3t) * (1,3)\}, t = 0,1$	$1/3$	2	$(t, 4, 4+t, 12+t)$
6	$\{(27-4t) * (1,2), 3t * (1,3), 3 * (1,4)\},$ $t = 0,1,2,3$	$1/4$	2	$(t, 5, 3+t, 10+t)$
7	$\{(27-4t) * (1,2), 3t * (1,3), 2 * (1,4),$ $(1,r)\}, 5 \leqslant r \leqslant 12; t = 0,1,2,3$	$1/r$	2	$(t, 5, 3+t, 9+t)$
8	$\{(27-4t) * (1,2), 3t * (1,3), (1,4), (1,r_1),$ $(1,r_2)\}, (r_1, r_2) \in I_3; t = 0,1,2,3$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{4}$	2	$(t, 5, 3+t, 8+t)$
9	$\{(27-4t) * (1,2), 3t * (1,3), 3 * (1,5)\},$ $t = 0,1,2,3$	$1/10$	2	$(t, 5, 3+t, 7+t)$
10	$\{(26-4t) * (1,2), (3+3t) * (1,3), (1,4)\},$ $t = 0,1,2,3$	$1/4$	2	$(0, 4, 3+t, 10+t)$
11	$\{(26-4t) * (1,2), (3+3t) * (1,3), (1,r)\},$ $5 \leqslant r \leqslant 12; t = 0,1,2,3$	$1/r$	2	$(0, 4, 3+t, 9+t)$
12	$\{(25-4t) * (1,2), (5+3t) * (1,3)\}, t = 0,1,2,3$	$1/6$	2	$(t, 3, 2+t, 8+t)$

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Table II2. Continued.

No.	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
13	$\{(26-4t)*(1,2),(2+3t)*(1,3),2*(1,4)\}$, $t=0,1,2,3$	1/6	2	$(t,4,2+t,8+t)$
14	$\{(26-4t)*(1,2),(2+3t)*(1,3),(1,4),(1,5)\}$, $t=0,1,2,3$	7/60	2	$(t,4,2+t,7+t)$
15	$\{(26-4t)*(1,2),(2+3t)*(1,3),(1,4),(1,6)\}, t=0,1,2,3$	1/12	2	$(t,4,2+t,7+t)$
16	$\{(25-4t)*(1,2),(4+3t)*(1,3),(1,4)\}$, $t=0,1,2,3$	1/12	2	$(t,3,1+t,6+t)$
17	$\{(26-4t)*(1,2),(1+3t)*(1,3),3*(1,4)\}$, $t=0,1,2,3$	1/12	2	$(t,4,1+t,6+t)$

where

$$\begin{aligned} I_4 &= \{(r_1,r_2)|1/r_1+1/r_2 \geqslant 1/4, r_i \geqslant 5\} \\ &= \{(5,5),\ldots,(5,20),(6,6),\ldots,(6,12),(7,7),(7,8),(7,9),(8,8)\} \\ I_3 &= \{(r_1,r_2)|1/r_1+1/r_2 \geqslant 1/3, r_i \geqslant 5\} \\ &= \{(5,5),(5,6),(5,7),(6,6)\}. \end{aligned}$$

Table II3.

	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
1	$\{32*(1,2),5*(1,3),2*(1,4),(1,r)\}, 5 \leqslant r$	$\frac{1}{6}+\frac{1}{r}$	3	$(0,5,4,13)$
2	$\{(32-4t)*(1,2),(5+3t)*(1,3),(1,4),(1,r_1),(1,r_2)\},(r_1,r_2)\in I_6, t\leqslant 1$	$\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{12}$	3	$(t,5,4+t,12+t)$
3	$\{(32-4t)*(1,2),(5+3t)*(1,3),(1,r_1),(1,r_2),(1,r_3)\},(r_1,r_2,r_3)\in J, t\leqslant 2$	$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{3}$	3	$(t,5,4+t,11+t)$
4	$\{(31-4t)*(1,2),(7+3t)*(1,3),2*(1,4)\}, t\leqslant 1$	1/3	3	$(t,4,3+t,12+t)$
5	$\{(31-4t)*(1,2),(7+3t)*(1,3),(1,4),(1,r)\}, 5\leqslant r; t\leqslant 2$	$\frac{1}{12}+\frac{1}{r}$	3	$(t,4,3+t,11+t)$
6	$\{(31-4t)*(1,2),(7+3t)*(1,3),(1,r_1),(1,r_2)\},(r_1,r_2)\in I_4; t\leqslant 3$	$\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{6}$	3	$(t,4,3+t,10+t)$
7	$\{(30-4t)*(1,2),(10+3t)*(1,3)\}, t=0,1$	1/3	3	$(t,3,3+t,12+t)$
8	$\{(31-4t)*(1,2),(6+3t)*(1,3),3*(1,4)\}, t=0,1,2,3$	1/4	3	$(t,4,2+t,10+t)$

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Table II3. Continued.

	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
9	$\{(31-4t)*(1,2),(6+3t)*(1,3),$ $2*(1,4),(1,r)\}, 5 \leqslant r \leqslant 12; t=0,1,2,3$	$1/r$	3	$(t,4,2+t,9+t)$
10	$\{(31-4t)*(1,2),(6+3t)*(1,3),$ $(1,4),(1,r_1),(1,r_2)\}, (r_1,r_2) \in I_3; t \leqslant 3$	$\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{4}$	3	$(t,4,2+t,8+t)$
11	$\{(31-4t)*(1,2),(6+3t)*(1,3),$ $3*(1,5)\}, t=0,1,2,3$	$1/10$	3	$(t,4,2+t,7+t)$
12	$\{(30-4t)*(1,2),(9+3t)*(1,3),$ $(1,4)\}, t=0,1,2,3$	$1/4$	3	$(0,3,2+t,10+t)$
13	$\{(30-4t)*(1,2),(9+3t)*(1,3),$ $(1,r)\}, 5 \leqslant r \leqslant 12; t=0,1,2,3$	$1/r$	3	$(0,3,2+t,9+t)$
14	$\{(30-4t)*(1,2),(8+3t)*(1,3),$ $2*(1,4)\}, t=0,1,2,3$	$1/6$	3	$(t,3,1+t,8+t)$
15	$\{(30-4t)*(1,2),(8+3t)*(1,3),$ $(1,4),(1,5)\}, t=0,1,2,3$	$7/60$	3	$(t,3,1+t,7+t)$
16	$\{(30-4t)*(1,2),(8+3t)*(1,3),$ $(1,4),(1,6)\}, t=0,1,2,3$	$1/12$	3	$(t,3,1+t,7+t)$
17	$\{(30-4t)*(1,2),(7+3t)*(1,3),$ $3*(1,4)\}, t=0,1,2,3$	$1/12$	3	$(t,3,t,6+t)$

where

$$I_4 = \{(r_1,r_2) | 1/r_1 + 1/r_2 \geqslant 1/4, r_i \geqslant 5\}$$
$$= \{(5,5),\ldots,(5,20),(6,6),\ldots,(6,12),(7,7),(7,8),(7,9),(8,8)\}$$
$$I_3 = \{(r_1,r_2) | 1/r_1 + 1/r_2 \geqslant 1/3, r_i \geqslant 5\}$$
$$= \{(5,5),(5,6),(5,7),(6,6)\}.$$
$$I_6 = \{(r_1,r_2) | 1/r_1 + 1/r_2 \geqslant 1/6, r_i \geqslant 5\}$$
$$= \{(5,s_5),(6,s_6),(7,s_7),(8,s_8),(9,s_9),(10,s_{10}),(11,11),(11,12),(11,13),(12,12)\},$$
$$5 \leqslant s_1, 6 \leqslant s_2, 7 \leqslant s_7 \leqslant 42, 8 \leqslant s_8 \leqslant 24, 9 \leqslant s_9 \leqslant 18, 10 \leqslant s_{10} \leqslant 15.$$
$$J = \{(r_1,r_2,r_3) | 1/r_1 + 1/r_2 + 1/r_3 \geqslant 5/12, r_i \geqslant 5\}$$
$$= \{(5,5,s_1),(5,6,s_2),(5,7,s_3),(5,8,8),(5,8,9),(5,8,10),(5,9,9),(6,6,s_4),(6,7,7),(6,7,8),$$
$$(6,7,9),(6,8,8),(7,7,7)\}, 5 \leqslant s_1 \leqslant 60, 6 \leqslant s_2 \leqslant 20, 7 \leqslant s_3 \leqslant 13, 6 \leqslant s_4 \leqslant 12.$$

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