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Uniform Convergence of Trigonometric Series with General Monotone Coefficients

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Abstract. We study criteria for the uniform convergence of trigonometric series with general monotone coefficients. We also obtain necessary and sufficient conditions for a given rate of convergence of partial Fourier sums of such series.

1 Introduction

In this paper, we consider the trigonometric series

(1.1) $$\sum_{n=1}^{\infty} a_n \sin nx,$$

(1.2) $$\sum_{n=1}^{\infty} a_n \cos nx$$

with general monotone coefficients $\{a_n\}_{n=1}^{\infty}$.

1.1 Uniform Convergence and General Monotone Sequences

In [3], T. W. Chaundy and A. E. Jolliffe proved the following theorem on the uniform convergence of the sine trigonometric series.

Theorem A ([3]) *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative nonincreasing sequence. Then series (1.1) converges uniformly on $[0, 2\pi]$ if and only if $na_n \to 0$ as $n \to \infty$.*

For the cosine series we highlight the following obvious fact.

Theorem B *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative sequence. Then series (1.2) converges uniformly on $[0, 2\pi]$ if and only if $\sum_{n=1}^{\infty} a_n$ converges.*

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The main goal of this paper is to generalize both Theorems A and B and provide necessary and sufficient conditions for the trigonometric series

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.3}$$

to be uniformly convergent. We will assume that the coefficients satisfy the general monotone condition. On a heuristic level, this indicates that the local variations of a sequence can be majorized by the (local) sums of absolute values of coefficients.

Very recently, several generalizations of Theorems A and B have been proved where different extensions of monotonicity condition were considered (see, *e.g.,* [4, 10, 12, 17, 19] and the references therein). Many generalizations involve the consideration of general monotone sequences. Let us recall the definition of the $GM(\boldsymbol{\beta})$ sequences (see [17]).

Definition Let $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ and $\boldsymbol{\beta} = \{\beta_n\}_{n=1}^{\infty}$ be two sequences of complex and nonnegative numbers, respectively. We will say that $\mathbf{a}$ is a general monotone sequence with majorant $\boldsymbol{\beta}$ if there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\sum_{k=n}^{2n} |\Delta a_k| \leqslant C\beta_n. \tag{1.4}$$

In this case, we will write $\mathbf{a} \in GM(\boldsymbol{\beta})$.

Note that the widest class of general monotone sequences is when $\beta_n = \sum_{k=n}^{2n+1}|a_k|$, since in this case any sequence belongs to this class with $C = 2$. Let us give some examples of majorants $\boldsymbol{\beta}$ that are useful in the study of trigonometric series:

(a) $\beta_n^1 = |a_n|$;
(b) $\beta_n^2 = \frac{1}{n}\sum_{s=\frac{n}{\gamma}}^{\gamma n}|a_s|, \quad \gamma > 1$;
(c) $\beta_n^3 = \frac{1}{n}\max_{k\geqslant\frac{n}{\gamma}}\sum_{s=k}^{2k}|a_s|, \quad \gamma > 1$.

It is known that $GM(\boldsymbol{\beta}^1) \subsetneq GM(\boldsymbol{\beta}^2) \subsetneq GM(\boldsymbol{\beta}^3)$; see [4, 18]. Moreover, if $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^i)$, $i = 1, 2, 3$, and $a_n \geqslant 0$, then series (1.1) converges uniformly on $[0, 2\pi]$ if and only if $na_n \to 0$ as $n \to \infty$; see [4, 17–19]. Recently, the authors of [8] proved an analogue of Theorem A for $\{a_n\} \in GM(\boldsymbol{\beta}^2)$ without the assumption that a_n is a nonnegative sequence.

In this paper, we study uniform convergence of sine and cosine trigonometric series with general monotone coefficients, which are not necessarily nonnegative. In particular, following the idea from [8], we prove that for a rather general class of majorants $\boldsymbol{\beta}$, the series (1.1) converges uniformly on $[0, 2\pi]$ if and only if $\sum_{k=n}^{2n}|a_k| \to 0$ as $n \to \infty$. In particular, this result holds for $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^3)$. The condition $\sum_{k=n}^{2n}|a_k| \to 0$ as $n \to \infty$ turns out to be equivalent to the condition $na_n \to 0$ as $n \to \infty$.

We will need the following general result.

Theorem C ([5, Theorem 2.1, part (C)]) *Let $\boldsymbol{\beta} = \{\beta_n\}_{n=1}^{\infty}$ be a majorant such that $n\beta_n \to 0$ as $n \to \infty$. Then series (1.1) converges uniformly on $[0, 2\pi]$.*

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1.2 Several Important Classes of General Monotone Sequences

In this paper, we consider the $GM(\boldsymbol{\beta})$ sequences with majorants $\boldsymbol{\beta}$ having the form described below. Let S be a set of numerical sequences. Denote by $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ any element of S.

We will say that a sequence of functionals on S, that is, $F_n: S \to \mathbb{R}_+$, $n \in \mathbb{N}$, is *admissible* if

(i) $F_n(\mathbf{x}) \to 0$ as $n \to \infty$ for any $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ vanishing at infinity,
(ii) $\{F_n(\mathbf{x})\}_{n=1}^{\infty}$ is bounded whenever $\mathbf{x} = \{x_k\}_{k=1}^{\infty}$ is bounded.

Examples of such $F = \{F_n\}_{n=1}^{\infty}$ are

(a) $F_n^1(\mathbf{x}) = |x_n|^{\alpha}$, $\alpha > 0$;
(b) $F_n^2(\mathbf{x}) = \sum_{k=\frac{n}{\gamma}}^{\gamma n} \frac{|x_k|}{k}$, $\gamma > 1$;
(c) $F_n^3(\mathbf{x}) = \max_{k \geqslant \frac{n}{\gamma}} |x_k|$, $\gamma > 1$;
(d) $F_n^4(\mathbf{x}) = \frac{1}{n} \sum_{k=1}^{n} |x_k|$;
(e) $F_n^5(\mathbf{x}) = \sum_{k=1}^{\infty} a_{nk} |x_k|$, where $\{a_{nk}\}_{n,k=1}^{\infty}$ is a regular matrix (see [20, Ch. III, §1]);
(f) the composition $F = G \circ H$, $F_n(\mathbf{x}) := G_n(H_k(\mathbf{x}))$, of admissible sequences $\{H_n\}_{n=1}^{\infty}$, $\{G_n\}_{n=1}^{\infty}$ is also admissible.

A typical example of a non-admissible $\{F_n\}_{n=1}^{\infty}$ is $F_n(\mathbf{x}) = \sum_{k=n}^{n+\lambda_n} \frac{|x_k|}{k}$, where a positive sequence $\{\lambda_n\}$ is such that $\lambda_n/n \to \infty$. Note also that conditions (i) and (ii) in the definition of admissible functionals are independent; take for example $F_n(\mathbf{x}) = |x_n|^{\alpha} + c$ with $\alpha, c > 0$ and $F_n(\mathbf{x}) = \sum_{k=n}^{n^2} k^{x_k - 2}$.

For a given sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, denote by $\widetilde{\mathbf{a}}$ the following sequence:

$$\widetilde{a}_n := \sum_{k=n}^{2n} |a_k|.$$

We study a class of general monotone sequences $GM(\boldsymbol{\beta})$ with

$$\beta_n = \frac{1}{n} F_n(\widetilde{\mathbf{a}}).$$

In particular, the class $GM(\boldsymbol{\beta}^3)$ coincides with the class $GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n} F_n^3(\widetilde{\mathbf{a}})$. Moreover, $GM(\boldsymbol{\beta}^2)$ coincides with $GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n} F_n^2(\widetilde{\mathbf{a}})$. Indeed, considering the sum $\sum_{k=M}^{N} \frac{\widetilde{a}_k}{k}$, where $N > 2M$, we note that

$$\sum_{k=M}^{N} \frac{1}{k} \sum_{s=k}^{2k} |a_s| = \sum_{s=M}^{2M} |a_s| \sum_{k=M}^{s} \frac{1}{k} + \sum_{s=2M+1}^{N} |a_s| \sum_{k=\frac{s}{2}}^{s} \frac{1}{k} + \sum_{s=N+1}^{2N} |a_s| \sum_{k=\frac{s}{2}}^{N} \frac{1}{k},$$

and therefore,

$$C_1 \sum_{s=2M}^{N} |a_s| \leqslant \sum_{k=M}^{N} \frac{\widetilde{a}_k}{k} \leqslant C_2 \sum_{s=M}^{2N} |a_s|.$$

It will be a key observation in our further study that any $GM(\boldsymbol{\beta})$ sequence preserves the monotonicity properties. This is given by the following result.

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Lemma 1.1 ([13, Lemma 3.1]) *Let* $\mathbf{a} \in GM(\boldsymbol{\beta})$*; then for all* $n \in \mathbb{N}$ *we have*

$$|a_k| \leqslant C\beta_n + |a_m| \quad \textit{for all} \quad k, m = n, \dots, 2n;$$

$$(1.5)\qquad |a_k| \leqslant C\beta_n + \frac{1}{n}\sum_{j=n+1}^{2n} |a_j| \quad \textit{for all} \quad k = n, \dots, 2n;$$

$$|a_n| \leqslant \frac{C}{n}\Big(\sum_{k=[\frac{n}{2}]}^{n-1} \beta_k + \sum_{j=n}^{2n-1} |a_j|\Big).$$

1.3 Main Results

Theorem 1.2 *Let* $\{F_n\}_{n=1}^{\infty}$ *be admissible. Also let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ *and* $\widetilde{\mathbf{a}}$ *is a bounded sequence. Then the following conditions are equivalent:*

(i) *the series* (1.1) *converges uniformly on* $[0, 2\pi]$;

(ii) $\lim_{n\to\infty} na_n = 0$;

(iii) $\lim_{n\to\infty} \widetilde{a}_n = 0$.

Remark 1.3 (i) It is clear that the condition of boundedness of $\widetilde{\mathbf{a}}$ is needed only to show the implication (i)⇒(ii).

(ii) Generally speaking, the statement of Theorem 1.2 is not true without assuming that the sequence $\{\widetilde{a}_n\}_{n=1}^{\infty}$ is bounded. The corresponding counterexample is constructed in Theorem 4.1. More precisely, there exists a uniformly converging sine series with coefficients satisfying $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta}^3)$ such that $na_n \nrightarrow 0$ and $\widetilde{a}_n \nrightarrow 0$ as $n \to \infty$.

(iii) It is easy to see that dealing with admissible $\{F_n\}_{n=1}^{\infty}$ allows us to expect that $F_n(\widetilde{\mathbf{a}})$ is bounded for a bounded sequence $\widetilde{\mathbf{a}}$. In light of the previous remark, this property is essential in the proof. In general, Theorem 1.2 is not valid for non-admissible sequences. In particular, the corresponding example can be given using lacunary series. Take the non-admissible functionals $F_n(\mathbf{x}) = n|x_n|$ and the lacunary sequence

$$a_k = \begin{cases} m^{-2} & k = 2^m, \\ 0 & k \neq 2^m. \end{cases}$$

Then $\lim_{n\to\infty} \widetilde{a}_n = 0$, the series $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly, but $\{ka_k\}$ is not bounded.

Another example can be given for non-admissible functionals $F_n(\mathbf{x}) = \sum_{k=n}^{n+\lambda_n} \frac{|x_k|}{k}$ with $\lambda_n/n \to \infty$ using the Rudin–Shapiro construction; see Remark 4.2(ii).

(iv) Regarding the fact that $GM(\boldsymbol{\beta}^2) \subsetneq GM(\boldsymbol{\beta}^3)$, we note that there exists a sequence $\mathbf{a} \in GM(\boldsymbol{\beta}^3) \setminus GM(\boldsymbol{\beta}^2)$ such that $\widetilde{\mathbf{a}}$ is bounded (see Section 3). This shows that Theorem 1.2 extends the results from [8].

A counterpart for the cosine series reads as follows.

Theorem 1.4 *Let* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ *with admissible* $\{F_n\}_{n=1}^{\infty}$ *and bounded* $\widetilde{\mathbf{a}}$*. Then series* (1.2) *converges uniformly on* $[0, 2\pi]$ *if and only if the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

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Remark 1.5 The condition of boundedness of $\widetilde{\mathbf{a}}$ in Theorem 1.4 is needed only to prove the "only if" part.

Corollary 1.6 *Let $\{F_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ be admissible. Let $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$ with $\beta_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$ and $\{b_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$ with $\beta_n = \frac{1}{n}G_n(\widetilde{\mathbf{b}})$. Suppose that $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ are bounded sequences. Then for the series (1.3) the following conditions are equivalent:*

(i) *series (1.3) is the Fourier series of a continuous function;*
(ii) *series (1.3) converges uniformly on $[0, 2\pi]$;*
(iii) *$\sum_{n=1}^{\infty} a_n$ converges and $nb_n \to 0$ as $n \to \infty$.*

1.4 Approximation by Partial Sums of Fourier Series

Here, we study the convergence rate of $\|h - S_n(h)\|_{C[0,2\pi]}$, where $S_n(h)$ is the n-th partial sum of the Fourier series of h. In [11] (see also [20, Ch. II, §10]), Lebesgue proved that for a function h from the Lipschitz space Lip α, where

$$\operatorname{Lip} \alpha = \Big\{ f \in C[0, 2\pi] : \omega(f, \delta)_C = O(\delta^{\alpha}) \Big\},$$

one has

$$\|h - S_n(h)\|_{C[0,2\pi]} = O\Big(\frac{\ln n}{n^{\alpha}} \Big). \tag{1.6}$$

Here $\omega(f, \delta)_C$ is the modulus of continuity of f, *i.e.*,

$$\omega(f, \delta)_C = \sup_{|h| \leqslant \delta} \|\Delta_h f(\cdot)\|_C \quad \text{and} \quad \Delta_h f(x) = f(x + h) - f(x).$$

Salem and Zygmund [15] showed that the logarithm cannot be suppressed even if, in addition to the hypothesis $h \in \operatorname{Lip} \alpha$, we suppose that h is of bounded variation. However, they demonstrated that if a function $h \in \operatorname{Lip} \alpha$ is of monotonic type, then the logarithm can be omitted in (1.6).

Theorem D ([15]) *Let h be a continuous function of monotonic type; that is, there exists a real constant K such that the function $h(x) + Kx$ is either non-decreasing or non-increasing on $(-\infty, \infty)$. Let $h \in \operatorname{Lip} \alpha$, where $0 < \alpha < 1$. Then*

$$\|h - S_n(h)\|_{C[0,2\pi]} = O\Big(\frac{1}{n^{\alpha}} \Big). \tag{1.7}$$

We will show (see Corollaries 1.10–1.11) that estimate (1.7) also holds for functions from Lip α having a Fourier series with coefficients from the $GM(\boldsymbol{\beta}^2)$ class. Denote by $g(x)$ and $f(x)$ the sums of series (1.1) and (1.2), respectively.

Theorem 1.7 *Let $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n} \sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$. Then, for $0 < \alpha \leqslant 1$,*

$$\|f - S_n(f)\|_{C[0,2\pi]} = o\Big(\frac{1}{n^{\alpha}} \Big) \iff a_n = o\Big(\frac{1}{n^{\alpha+1}} \Big).$$

$$\|g - S_n(g)\|_{C[0,2\pi]} = o\Big(\frac{1}{n^{\alpha}} \Big) \iff a_n = o\Big(\frac{1}{n^{\alpha+1}} \Big).$$

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Theorem 1.8 *Let* $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}\sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$*. Then, for* $0 < \alpha \leqslant 1$*,*

$$\|f - S_n(f)\|_{C[0,2\pi]} = O\Big(\frac{1}{n^\alpha}\Big) \iff a_n = O\Big(\frac{1}{n^{\alpha+1}}\Big),$$
$$\|g - S_n(g)\|_{C[0,2\pi]} = O\Big(\frac{1}{n^\alpha}\Big) \iff a_n = O\Big(\frac{1}{n^{\alpha+1}}\Big).$$

Remark 1.9 (i) Note that the condition $\|f - S_n(f)\|_{C[0,2\pi]} = O(\frac{1}{n^\alpha})$ implies that the sum f is a continuous function and $\{a_n\}_{n=1}^\infty$ is the sequence of Fourier coefficients of f.

(ii) For $\alpha = 0$, Theorem 1.7 also holds in the case of the sine series, which gives an alternative proof of the main result in [8].

Moreover, Theorem 1.8 along with [6, Theorem 2.2 and Corollary 3.4] imply the following results.

Corollary 1.10 *Let* $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}\sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$*. Also let* (1.2) *be the Fourier series of a continuous function* f*. Then for* $0 < \alpha \leqslant 1$ *the following conditions are equivalent:*

(i) $f \in \operatorname{Lip}\alpha$,
(ii) $\|f - S_n(f)\|_C = O(\frac{1}{n^\alpha})$,
(iii) $E_n(f)_C = O(\frac{1}{n^\alpha})$.

Here, $E_n(f)_C$ is the best approximation of a function f by trigonometric polynomials of degree n in $C[0, 2\pi]$.

Corollary 1.11 *Let* $\{a_n\}_{n=1}^\infty \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}\sum_{k=\frac{n}{\gamma}}^{\gamma n} |a_k|$*. Let also* (1.1) *be the Fourier series of a continuous function* g*. Then for* $0 < \alpha < 1$ *the following conditions are equivalent:*

(i) $g \in \operatorname{Lip}\alpha$,
(ii) $\|g - S_n(g)\|_C = O(\frac{1}{n^\alpha})$,
(iii) $E_n(g)_C = O(\frac{1}{n^\alpha})$.

Moreover, for $\alpha = 1$*, conditions* (i), (iii), *and*

(iv) $a_n = O(\frac{1}{n^2})$

are pairwise equivalent, but the condition $g \in \operatorname{Lip} 1$ *is not equivalent to any of them.*

Remark 1.12 Regarding the case $\alpha = 1$ in Corollaries 1.10 and 1.11, we first note that the direct and inverse theorems of trigonometric approximation; namely,

$$E_n(\psi)_C \leqslant C\omega\Big(\psi, \frac{1}{n}\Big)_C \leqslant \frac{C}{n}\sum_{\nu=1}^{n+1} E_{\nu-1}(\psi)_C,$$

immediately imply that $\psi \in \operatorname{Lip}\alpha$ if and only if $E_n(\psi)_C = O(\frac{1}{n^\alpha})$ for $0 < \alpha < 1$. We see that dealing with series with general monotone coefficients allows one to prove a similar result in the limiting case $\alpha = 1$ when $\psi = f$. A similar result does not hold for sine series ($\psi = g$), because of the following reason. For series with monotone coefficients,

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a necessary and sufficient condition for $g \in \mathrm{Lip}\,1$ is already given by $\sum_k ka_k < \infty$. This fact was first observed by Boas [2], and in turn is related to the behavior of the derivative of g at the origin. In particular, the function $g(x) = \sum_k \frac{\sin kx}{k^2}$ is such that $E_n(g)_C \leqslant \|g - S_n(g)\|_C = O(\frac{1}{n})$, but $g \notin \mathrm{Lip}\,1$. See [18] for the related results regarding series with non-negative GM coefficients.

1.5 Organization of the Paper

In Section 2, we prove our main results, Theorems 1.2–1.8 and Corollary 1.6. Section 3 provides some examples of sequences $\mathbf{a} \in GM(\boldsymbol{\beta}^3) \setminus GM(\boldsymbol{\beta}^2)$ with different behaviour of $\widetilde{\mathbf{a}}$ and different convergence properties of the series $\sum_{n=1}^{\infty} a_n \sin nx$. In Section 4, we give an example of a uniformly converging series (1.1) with unbounded $\widetilde{\mathbf{a}}$. We conclude with final remarks in Section 5.

Throughout this paper, we denote by C positive constants that may be different on various occasions. In addition, $F \asymp G$ means that $\frac{1}{C} F \leqslant G \leqslant CF$.

2 Proofs of Main Results

Remark 2.1 Without loss of generality, we can assume in Theorems 1.2 and 1.4 that the inequality

$$\widetilde{a}_n \leqslant F_n(\widetilde{\mathbf{a}}) \tag{2.1}$$

is valid for all sequences $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ and for all $n \in \mathbb{N}$. Indeed, if this is not the case, then we can consider the majorant

$$G_n(\widetilde{\mathbf{a}}) = \max\{\widetilde{a}_n, F_n(\widetilde{\mathbf{a}})\},$$

which satisfies (2.1). It is clear that conditions (i)–(ii) hold for the sequence $\{G_n\}_{n=1}^{\infty}$. Moreover, instead of the class $GM(\boldsymbol{\beta})$ with $\beta_n = (F_n(\widetilde{\mathbf{a}}))/n$ we can consider the class $GM(\boldsymbol{\beta}^*) \supseteq GM(\boldsymbol{\beta})$, where $\beta_n^* = (G_n(\widetilde{\mathbf{a}}))/n$. Throughout this paper, we will assume that $\{F_n\}_{n=1}^{\infty}$ satisfies (2.1).

Lemma 2.2 *Let $\mathbf{a} \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{F_n(\widetilde{\mathbf{a}})}{n}$. Then for all $n \in \mathbb{N}$,*

$$|a_k| \leqslant C\frac{F_n(\widetilde{\mathbf{a}})}{n} \quad \text{for all} \quad k = n, \dots, 2n. \tag{2.2}$$

Proof The proof follows from (2.1) and inequality (1.5). ∎

2.1 Proof of Theorem 1.2

We will prove this theorem as follows: (ii)⇒(iii)⇒(i)⇒(ii).
The implication (ii)⇒(iii) is clear.
(iii)⇒(i). Let $\widetilde{a}_n \to 0$ as $n \to \infty$. Then from property (i) of F_n, we get

$$F_n(\widetilde{\mathbf{a}}) \to 0 \quad \text{as} \quad n \to \infty.$$

Finally, we use Theorem C with $\beta_n = F_n(\widetilde{\mathbf{a}})/n$.
(i)⇒(ii). From Lemma 2.2 and property (ii) on $\{F_n\}_{n=1}^{\infty}$ it follows that it is sufficient to prove $\lim_{n\to\infty} \widetilde{a}_n = 0$. Let $\varepsilon > 0$; then by Cauchy's criterion, we can choose $N \in \mathbb{N}$

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such that for all $N \leqslant k \leqslant l$,

$$\Big\| \sum_{j=k}^{l} a_j \sin jx \Big\|_{C[0,2\pi]} < \varepsilon.$$

Let $n > N$ and $\widetilde{a}_n \neq 0$. By (2.1), note that $F_n(\widetilde{\mathbf{a}}) \neq 0$. We put

$$A_n := \Big\{ k : |a_k| \geqslant \frac{\widetilde{a}_n}{4n}, n \leqslant k \leqslant 2n \Big\}. \tag{2.3}$$

Note that A_n is not the empty set. Let us obtain a lower estimate for the cardinality of A_n denoted by $|A_n|$. By (2.2), we have $|a_k| \leqslant \frac{C}{n} F_n(\widetilde{\mathbf{a}})$ for $n \leqslant k \leqslant 2n$, and therefore,

$$\begin{aligned}
\widetilde{a}_n = \sum_{s=n}^{2n} |a_s| &= \sum_{s\in[n,2n]\setminus A_n} |a_s| + \sum_{s\in A_n} |a_s| \\
&\leqslant \sum_{s\in[n,2n]\setminus A_n} \frac{\widetilde{a}_n}{4n} + \sum_{s\in A_n} \frac{C}{n} F_n(\widetilde{\mathbf{a}}) \\
&\leqslant \frac{2n\widetilde{a}_n}{4n} + |A_n| \frac{C}{n} F_n(\widetilde{\mathbf{a}}) = \frac{\widetilde{a}_n}{2} + |A_n| \frac{C}{n} F_n(\widetilde{\mathbf{a}}).
\end{aligned}$$

Hence,

$$|A_n| \geqslant \frac{n}{2C} \frac{\widetilde{a}_n}{F_n(\widetilde{\mathbf{a}})}. \tag{2.4}$$

Following [8], we construct disjoint subsets $S_1, \dots, S_{\kappa_n}$ of $[n, 2n]$. Put $m_1 = \min A_n$, and select ν_1 according to the following procedure:

(a) If there exists a $j_0 \geqslant 1$ such that for $j = 0, 1, \dots, j_0$, $n \leqslant m_1 + j \leqslant 2n$ the numbers a_{m_1+j} have the same sign and for $j = 0, 1, \dots, j_0 - 1$, $|a_{m_1+j}| \geqslant \frac{\widetilde{a}_n}{8n}$ and $|a_{m_1+j_0}| < \frac{\widetilde{a}_n}{8n}$, then we set $\nu_1 = j_0$.

(b) If such j_0 does not exist, then let $\nu_1 = l_0$ such that $m_1 + l_0 \in [n, 2n]$ and $a_{m_1+l_0}$ is the first element to become zero or of the opposite sign to a_{m_1}.

(c) If neither (a) nor (b) happens, then simply let $\nu_1 = l_0$ for which $m_1 + l_0$ is the first number greater than $2n$.

Define a set

$$S_1 = \{m_1, m_1 + 1, \dots, m_1 + \nu_1 - 1\}.$$

If the set $A_n \setminus S_1$ is not empty, we put $m_2 = \min(A_n \setminus S_1)$. Using the same procedure as above, we select ν_2 and define

$$S_2 = \{m_2, m_2 + 1, \dots, m_2 + \nu_2 - 1\}.$$

We continue this procedure until we reach an S_{κ_n} for which

$$A_n \setminus (S_1 \cup \cdots \cup S_{\kappa_n}) = \varnothing.$$

Now we obtain the upper estimate for κ_n. If $\kappa_n > 1$, we note first that for all $1 \leqslant j < \kappa_n$, we have

$$\sum_{k\in S_j} |\Delta a_k| \geqslant |a_{m_j} - a_{m_j+\nu_j}| \geqslant \frac{\widetilde{a}_n}{8n}.$$

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From the definition of $GM(\boldsymbol{\beta})$, $\beta_n = \frac{F_n(\widetilde{\mathbf{a}})}{n}$ we get

$$\sum_{s=n}^{2n} |\Delta a_s| \leqslant \frac{C}{n} F_n(\widetilde{\mathbf{a}}).$$

Hence,

$$\frac{C}{n} F_n(\widetilde{\mathbf{a}}) \geqslant \sum_{s=n}^{2n} |\Delta a_s| \geqslant \sum_{j=1}^{\kappa_n - 1} \sum_{k \in S_j} |\Delta a_k| \geqslant \sum_{j=1}^{\kappa_n - 1} \frac{\widetilde{a}_n}{8n} = (\kappa_n - 1) \frac{\widetilde{a}_n}{8n}.$$

Therefore,

$$\kappa_n \leqslant \frac{8CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n} + 1 \leqslant \frac{9CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n}. \tag{2.5}$$

If $\kappa_n = 1$, then (2.5) also holds. Let $x = \frac{\pi}{4n}$ and $n \leqslant k \leqslant 2n$. Then

$$\sin kx \geqslant \frac{2}{\pi} \frac{\pi k}{4n} \geqslant \frac{1}{2}.$$

Since all a_k, $k \in S_j$ have the same sign, we get

$$\frac{1}{2} \sum_{k \in S_j} |a_k| \leqslant \Big| \sum_{k \in S_j} a_k \sin \frac{\pi k}{4n} \Big| < \varepsilon \tag{2.6}$$

for all $n > N$. Hence,

$$\sum_{k \in A_n} |a_k| \leqslant \sum_{j=1}^{\kappa_n} \sum_{k \in S_j} |a_k| < \varepsilon \frac{18CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n}.$$

From the definition (2.3) of the set A_n and estimate (2.4), we get

$$\frac{1}{8C} \frac{\widetilde{a}_n^2}{F_n(\widetilde{\mathbf{a}})} \leqslant \varepsilon \frac{18CF_n(\widetilde{\mathbf{a}})}{\widetilde{a}_n}.$$

Hence,

$$\frac{\widetilde{a}_n^3}{F_n(\widetilde{\mathbf{a}})^2} \to 0 \qquad \text{as} \quad n \to \infty.$$

Since $\{F_n(\widetilde{\mathbf{a}})\}_{n=1}^{\infty}$ is bounded, we obtain that $\widetilde{a}_n \to 0$ as $n \to \infty$. Hence, $F_n(\widetilde{\mathbf{a}}) \to 0$ as $n \to \infty$. ∎

2.2 Proof of Theorem 1.4

Here we will need the following result; see [5, Theorem 2.1, part (B)].

Theorem E *Let* $\mathbf{a} \in GM(\boldsymbol{\beta})$. *If* $n\beta_n = o(1)$ *as* $n \to \infty$, *then series* (1.2) *converges uniformly on* $[0, 2\pi]$ *if and only if the series* $\sum_n a_n$ *converges.*

To make the paper self-contained, we sketch the proof of this result. The "only if" part is obvious. To prove the "if" part, we first note that

$$n \sum_{\nu=n}^{\infty} |\Delta a_\nu| = n \sum_{s=0}^{\infty} \sum_{\nu=2^s n}^{2^{s+1}n-1} |\Delta a_\nu| \leqslant C \max_{\nu \geqslant n} \big(\nu \beta_\nu \big) =: C\varepsilon_n \to 0.$$

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Therefore, $\{a_n\}$ is of bounded variation and (1.2) converges on $(0, \pi]$ to $f(x)$. Setting $S_n(f, x) = \sum_{j=1}^{n} a_j \cos jx$ and $D_n^*(x) = \sum_{j=1}^{n} \cos jx$, we have

$$f(x) - S_{n-1}(f, x) = \sum_{k=n}^{\infty} \triangle a_k D_k^*(x) - a_n D_{n-1}^*(x) =: I_1 + I_2.$$

Then by (2.7),

$$|I_2| = |a_n D_{n-1}^*(x)| \leqslant n|a_n| \leqslant n \sum_{\nu=n}^{\infty} |\Delta a_\nu| \leqslant C\varepsilon_n.$$

To estimate I_1, for fixed $x \in (0, \pi]$, we define $l \in \mathbf{N}$ such that $x \in (\frac{\pi}{l+1}, \frac{\pi}{l}]$. If $l \leqslant n$, we obtain

$$|I_1| \leqslant \frac{C}{x} \sum_{k=n}^{\infty} |\Delta a_k| \leqslant Cl \frac{\varepsilon_n}{n} \leqslant C\varepsilon_n. \tag{2.8}$$

If $l > n$, then we write $I_1 = \sum_{k=n}^{l-1} + \sum_{k=l}^{\infty}$. Similarly to (2.8), we derive that

$$\Big| \sum_{k=l}^{\infty} \triangle a_k D_k^*(x) \Big| \leqslant Cl \sum_{k=l}^{\infty} |\Delta a_k| \leqslant C\varepsilon_l \leqslant C\varepsilon_n.$$

Further,

$$\Big| \sum_{k=n}^{l-1} \triangle a_k D_k^*(x) \Big| \leqslant \Big| \sum_{k=n}^{l-1} k \triangle a_k \Big| + \sum_{k=n}^{l-1} |\triangle a_k| \Big| D_k^*(x) - k \Big| =: K + L.$$

Since

$$\sum_{k=n}^{l-1} k \triangle a_k = \sum_{k=n}^{l-1} a_k + (n-1)a_n - (l-1)a_l,$$

we have $K \leqslant 2\nu_n + C\varepsilon_n$, where ν_n is a non-increasing null-sequence satisfying $|\sum_{j=n}^{\infty} a_j| \leqslant \nu_n$. To estimate L, we use $|D_k^*(x) - k| \leqslant k^2 x$ to get

$$\begin{aligned} L &\leqslant x \sum_{k=n}^{l-1} k^2 |\triangle a_k| \leqslant \frac{C}{l} \Big(\sum_{m=n}^{l} m \sum_{j=m}^{l} |\triangle a_j| + n^2 \sum_{j=n}^{l} |\triangle a_j| \Big) \\ &\leqslant \frac{C}{l} \Big(\sum_{m=n}^{l} \varepsilon_m + n\varepsilon_n \Big) \leqslant C\varepsilon_n. \end{aligned}$$

Collecting the obtained estimates, we have $|f(x) - S_n(f, x)| \leqslant C(\varepsilon_n + \nu_n)$, which implies the uniform convergence of (1.2).

Proof of Theorem 1.4 The "only if" part is clear.

To show the "if" part, as in the proof of Theorem 1.2, it is enough to show that

$$\lim_{n\to\infty} \widetilde{a}_n = 0. \tag{2.9}$$

For $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that for all $l \geqslant k \geqslant N$,

$$\Big| \sum_{j=k}^{l} a_j \Big| < \varepsilon.$$

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Relation (2.9) is proved in the same way and with the same notation as in Theorem 1.2, using the inequality

$$\frac{1}{2}\sum_{k\in S_j}|a_k| = \frac{1}{2}\Big|\sum_{k\in S_j} a_k\Big| < \varepsilon, \quad S_j \subset [n, 2n], \quad n > N,$$

instead of inequality (2.6). Then since $\{F_n\}_{n=1}^{\infty}$ is admissible, we obtain that $F_n(\widetilde{\mathbf{a}}) \to 0$ as $n \to \infty$. Thus, $\mathbf{a} \in GM(\boldsymbol{\beta})$ with $n\beta_n = o(1)$ and Theorem E concludes the proof. ∎

2.3 Proof of Corollary 1.6

We divide the proof into two parts: (i)⇔(ii) and (ii)⇔(iii).

(i)⇒(ii). Let series (1.3) be the Fourier series of a continuous function $h(x)$. Note that for a sequence $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$ with $\boldsymbol{\beta}_n = \frac{1}{n}F_n(\widetilde{\mathbf{a}})$, the boundedness of the sequence $\{na_n\}_{n=1}^{\infty}$ is equivalent to the boundedness of the sequence $\{\widetilde{a}_n\}_{n=1}^{\infty}$. From the boundedness of $\{na_n\}_{n=1}^{\infty}$, it follows that $a_n \geqslant -\frac{C}{n}$ for all $n \geqslant 1$ and some $C > 0$. The last inequality with the Paley–Fekete theorem in [7, Theorem C] implies the uniform convergence of series (1.3).

(ii)⇒(i). This part is clear.

(ii)⇒(iii). Let series (1.3) converge uniformly. Denote by $h(x)$ the sum of series (1.3). Note that the series $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} b_n \sin nx$ are the Fourier series of the continuous functions

$$\frac{h(x)+h(-x)}{2} \quad \text{and} \quad \frac{h(x)-h(-x)}{2},$$

respectively. Since both series converge uniformly, Theorems 1.2 and 1.4 imply (iii).

(iii)⇒(ii). This part follows from Theorems 1.2 and 1.4. ∎

2.4 Proof of Theorems 1.7 and 1.8

Here we follow the proof of [6]. Without loss of generality, in the definition of $GM(\boldsymbol{\beta})$ with $\beta_n = \frac{1}{n}\sum_{k=n/\gamma}^{\gamma n}|a_k|$, we can assume that $\gamma = 2^{\nu}$, where ν is an integer number. Using notation from [6] we denote for any $n > 2\nu$,

$$A_n := \max_{2^n \leqslant k \leqslant 2^{n+1}} |a_k|, \quad B_n := \max_{2^{n-2\nu} \leqslant k \leqslant 2^{n+2\nu}} |a_k|,$$
$$M_n := \Big\{ k \in [2^{n-\nu}, 2^{n+\nu}] : |a_k| > \frac{A_n}{8C2^{2\nu}} \Big\},$$

where C and ν are constants from the definition of general monotone sequences. A natural number n is called *good* if either $n \leqslant 2\nu$ or $B_n \leqslant 2^{4\nu}A_n$. All other natural numbers are *bad*.

Let

$$M_n^+ := \{k \in M_n : a_k > 0\} \quad \text{and} \quad M_n^- := M_n \setminus M_n^+.$$

We will use the following lemma.

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Lemma 2.3 *Let a vanishing sequence* $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$*, where* $\beta_n = \frac{1}{n}\sum_{k=n/2^{\nu}}^{2^{\nu}n} |a_k|$*. Then for any good n such that* $2^n > C^3 2^{10\nu+8}$*, there exists an interval* $[l_n, m_n] \subseteq [2^{n-\nu}, 2^{n+\nu}]$ *such that at least one of the following conditions holds:*

(i) *for any* $k \in [l_n, m_n]$*, we have* $a_k \geqslant 0$ *and*

$$|M_n^+ \cap [l_n, m_n]| \geqslant \frac{2^n}{C^3 2^{15\nu+8}};$$

(ii) *for any* $k \in [l_n, m_n]$*, we have* $a_k \leqslant 0$ *and*

$$|M_n^- \cap [l_n, m_n]| \geqslant \frac{2^n}{C^3 2^{15\nu+8}}.$$

The proof is given in [6, Lemma 2.2] and is based on the fact that, for any good n such that $2^n > C2^{2\nu+3}$, we have

$$|M_n| \geqslant \frac{2^n}{C2^{5\nu+3}}.$$

Proof of Theorems 1.7 and 1.8 We will prove only the case of the sine series of Theorem 1.7. For the case of the cosine series in Theorem 1.7 and for both cases in Theorem!1.8, the proof is similar.

First, we prove the part "⇒". Let $\varepsilon > 0$; then there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\|g - S_n(g)\|_{C[0,2\pi]} \leqslant \frac{\varepsilon}{n^{\alpha}}.$$

Let n be a good number and $2^n > \max\{C^3 2^{15\nu+11}, 2^{\nu} N\}$. Assume Lemma 2.3(i) is valid and consider

$$Q_n(t) = \sum_{k=l_n+1}^{m_n} a_k \sin kt.$$

Then $|Q_n(t)| \leqslant 2\varepsilon/(2^{(n-2\nu)\alpha})$ for all $t \in [0, 2\pi]$. Setting $t = 1/(2^{n+2\nu})$, we obtain

$$\begin{aligned} \frac{2\varepsilon}{2^{(n-2\nu)\alpha}} &\geqslant \sum_{k=l_n+1}^{m_n} a_k \sin \frac{k}{2^{n+2\nu}} \geqslant \frac{2}{\pi} \frac{1}{2^{4\nu}} \frac{A_n}{8C2^{2\nu}} \Big(\frac{2^n}{C^3 2^{15\nu+8}} - 1 \Big) \\ &\geqslant \frac{1}{2} \frac{1}{2^{4\nu}} \frac{A_n}{8C2^{2\nu}} \frac{2^n}{C^3 2^{15\nu+9}} = \frac{2^n A_n}{C^4 2^{21\nu+13}}. \end{aligned}$$

Therefore,

$$A_n \leqslant \frac{L_1 \varepsilon}{2^{(\alpha+1)n}}.$$

Then

$$A_n \leqslant \frac{L_2 \varepsilon}{2^{(\alpha+1)n}}$$

holds for all good numbers, where $L_2 \geqslant L_1$ is another constant.

Let n be a bad number. Then $A_n < B_n 2^{-4\nu}$. Note that $B_n = A_{s_1}$, where $|s_1 - n| \leqslant 2\nu$.

Assume first that $s_1 < n$. Then either s_1 is a good number or there exists s_2 such that $|s_1 - s_2| \leqslant 2\nu$ and $A_{s_1} < A_{s_2} 2^{-4\nu}$. Also, we have

$$[2^{s_1}, 2^{s_1+2\nu}] \cap \mathbb{Z} \subset [2^{n-2\nu}, 2^{n+2\nu}] \cap \mathbb{Z}. \tag{2.10}$$

Then there is no a_k, $k \in [2^{s_1}, 2^{s_1+2\nu}] \cap \mathbb{Z}$, such that $|a_k| > A_{s_1}$. Hence, the case $s_2 > s_1$ is not possible.

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Repeating the process, since s_j is a decreasing sequence, we arrive at a finite sequence $n = s_0 > s_1 > \cdots > s_{i-1} > s_i$, where numbers $s_0, s_1, \dots, s_{i-1}$ are bad, and s_i is good. Moreover, $s_j - s_{j+1} \leqslant 2\nu$ and $A_{s_j} < A_{s_{j+1}} 2^{-4\nu}$ for any j. Since s_i is a good number, using the proof above, we have $A_{s_i} \leqslant L_2\varepsilon/2^{(\alpha+1)s_i}$, which implies

$$A_n = A_{s_0} \leqslant \frac{A_{s_1}}{2^{4\nu}} \leqslant \cdots \leqslant \frac{L_2\varepsilon}{2^{4\nu i} 2^{(\alpha+1)s_i}}. \tag{2.11}$$

Now, since $n \leqslant s_i + 2i\nu$, we have

$$A_n \leqslant \frac{L_2\varepsilon}{2^{4\nu i} 2^{(\alpha+1)s_i}} = \frac{L_2\varepsilon}{2^{(\alpha+1)n}} \frac{2^{(\alpha+1)n}}{2^{(1+\alpha)(2\nu i+s_i)}} \frac{1}{2^{2\nu i(1-\alpha)}} \leqslant \frac{L_2\varepsilon}{2^{(\alpha+1)n}}. \tag{2.12}$$

Let now $s_1 > n$. Then either s_1 is a good number or there exists $s_2 > s_1$ such that $s_2 - s_1 \leqslant 2\nu$ and $A_{s_1} < A_{s_2} 2^{-4\nu}$. Continuing this process and taking into account that the sequence of the Fourier coefficients vanishes at infinity, we arrive at the finite sequence $n = s_0 < s_1 < \cdots < s_{i-1} < s_i$, where the numbers $s_0, s_1, \dots, s_{i-1}$ are bad, and s_i is good. Then $A_{s_i} \leqslant L_2\varepsilon/2^{(\alpha+1)s_i}$ implies

$$A_n < A_{s_1} < A_{s_2} < \dots A_{s_i} \leqslant \frac{L_2\varepsilon}{2^{(\alpha+1)s_i}} \leqslant \frac{L_2\varepsilon}{2^{(\alpha+1)n}}.$$

Then we have

$$A_n \leqslant \frac{L_3\varepsilon}{2^{(1+\alpha)n}}$$

for any n. Let $k \in \mathbb{N}$ such that $k \in [2^l, 2^{l+1}]$ and $2^l \geqslant N$. Then

$$|a_k| \leqslant A_l \leqslant \frac{L_2\varepsilon}{2^{(1+\alpha)l}} \leqslant \frac{L_2\varepsilon}{2^{(1+\alpha)l}} \leqslant \frac{L_3\varepsilon}{k^{1+\alpha}}.$$

We would like to remark that for certain sequences $\{a_k\}$ the number of good points is finite. In this case the proof of the "$\Rightarrow$" part follows the same lines as above for all n being bad numbers. We repeat the procedure for $s_1 < n$; see (2.10)–(2.12), since the case $s_1 > n$ is impossible.

Now we prove the "$\Leftarrow$" part. Let $\varepsilon > 0$; then the inequality

$$\|g - S_n(g)\|_{C[0,2\pi]} \leqslant \sum_{k=n+1}^{\infty} |a_k| \leqslant \varepsilon \sum_{k=n+1}^{\infty} \frac{1}{k^{\alpha+1}} \leqslant \varepsilon \frac{C(\alpha)}{n^\alpha}$$

holds for all $n \geqslant N$, where N is sufficiently large integer number depending on ε. ∎

3 Several Examples of General Monotone Sequences

To compare [8, Theorem 3.1] and Theorem 1.2, we construct several examples of sequences $\{a_k\}_{k=1}^\infty \in GM(\beta^3) \setminus GM(\beta^2)$. First, for convenience, we recall the definitions of $GM(\beta^2)$ and $GM(\beta^3)$ classes.

A sequence $\{a_k\}_{k=1}^\infty$ is in $GM(\beta^2)$ if there exist $C > 0$, $\gamma > 1$ such that for all $n \in \mathbb{N}$,

$$\sum_{k=n}^{2n} |\Delta a_k| \leqslant \frac{C}{n} \sum_{s=\frac{n}{\gamma}}^{\gamma n} |a_s|. \tag{3.1}$$

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A sequence $\{a_k\}_{k=1}^\infty$ is in $GM(\boldsymbol{\beta}^3)$ if there exist $C>0$, $\gamma>1$ such that for all $n\in\mathbb{N}$,

$$\sum_{k=n}^{2n}|\Delta a_k| \leqslant \frac{C}{n}\max_{k\geqslant\frac{n}{\gamma}}\sum_{s=k}^{2k}|a_s|. \tag{3.2}$$

We set

$$N_1=1,\quad N_{j+1}=N_j+2M_j,$$

where $M_j>N_j$ and $\{M_j\}_{j=1}^\infty$ is an increasing sequence of integers. Consider the sequence

$$a_k=\begin{cases}\frac{(-1)^k}{C_j}, & N_j\leqslant k<2N_j,\\ \frac{1}{C_j}, & 2N_j\leqslant k<2N_j+M_j,\\ 0, & 2N_j+M_j\leqslant k<N_{j+1},\end{cases}$$

where $\{C_j\}_{j=1}^\infty$ is an increasing sequence.

1. We show that $\mathbf{a}\notin GM(\boldsymbol{\beta}^2)$. Let $k=N_j$; then on the one hand we have

$$\sum_{s=k}^{2k}|\Delta a_s| = \sum_{s=N_j}^{2N_j}|\Delta a_s| \asymp \sum_{s=N_j}^{2N_j}\frac{2}{C_j}\asymp\frac{N_j}{C_j}.$$

On the other hand, we have

$$\frac{1}{k}\sum_{s=\frac{k}{\gamma}}^{\gamma k}|a_s| \leqslant \frac{1}{N_j}\sum_{s=\frac{N_j}{\gamma}}^{\gamma N_j}\frac{1}{C_j}\asymp\frac{1}{N_j}N_j\frac{1}{C_j}=\frac{1}{C_j}.$$

Therefore, condition (3.1) does not hold.

2. Now we obtain sufficient conditions on $\{M_j\}_{j=1}^\infty$ for the sequence $\mathbf{a}$ to belong to the class $GM(\boldsymbol{\beta}^3)$. It is clear that it is enough to verify condition (3.2) for $k=N_j$. We have

$$\begin{aligned}\frac{1}{k}\max_{s\geqslant\frac{k}{\gamma}}\sum_{i=s}^{2s}|a_i| &= \frac{1}{N_j}\max_{s\geqslant\frac{N_j}{\gamma}}\sum_{i=s}^{2s}|a_i| \geqslant \frac{1}{N_j}\sum_{i=N_j+M_j/2}^{2N_j+M_j}|a_i|\\ &\asymp \frac{1}{N_j}\frac{1}{C_j}\big(N_j+M_j/2\big)=\frac{1+\frac{M_j}{2N_j}}{C_j}.\end{aligned}$$

Comparing the expressions $\frac{1+\frac{M_j}{2N_j}}{C_j}$ and $\frac{N_j}{C_j}$, we obtain that if

$$N_j^2=O(M_j)\quad\text{as}\quad j\to\infty,$$

then (3.2) holds, *i.e.*, $\{a_k\}_{k=1}^\infty\in GM(\boldsymbol{\beta}^3)$.

3. Now we study the uniform boundedness of the sums $\sum_{s=k}^{2k}|a_s|$. Let $2k=2N_j+M_j$; then

$$\sum_{s=k}^{2k}|a_s| = \sum_{s=N_j+M_j/2}^{2N_j+M_j}|a_s|\asymp\frac{N_j+M_j}{C_j}.$$

Hence, the following hold:

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(a) The condition

$$N_j + M_j = O(C_j) \quad \text{as} \quad j \to \infty.$$

implies the uniform boundedness of the sums $\sum_{s=k}^{2k} |a_s|$. In particular, the sequence $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ belongs to $GM(\boldsymbol{\beta}^3)$, where

$$a_k = \begin{cases} \frac{(-1)^k}{2^{N_j} N_j} & N_j \leqslant k < 2N_j, \\ \frac{1}{2^{N_j} N_j} & 2N_j \leqslant k < 2N_j + N_j 2^{N_j}, \\ 0 & 2N_j + N_j 2^{N_j} \leqslant k < N_{j+1}. \end{cases}$$

and $\sum_{k=n}^{2n} |a_k| \leqslant 2$, $n \geqslant 1$. But by Theorem 1.2, the series $\sum_{k=1}^{\infty} a_k \sin kx$ is not uniformly convergent, since $ka_k \nrightarrow 0$.

(b) the condition

$$N_j + M_j = o(C_j) \quad \text{as} \quad j \to \infty.$$

implies $\sum_{s=k}^{2k} |a_s| \to 0$ as $k \to \infty$. In particular, the series $\sum_{k=1}^{\infty} a_k \sin kx$ with coefficients $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ converges uniformly, where

$$a_k = \begin{cases} \frac{(-1)^k}{j^{\alpha} 2^{N_j} N_j} & N_j \leqslant k < 2N_j, \\ \frac{1}{j^{\alpha} 2^{N_j} N_j} & 2N_j \leqslant k < 2N_j + N_j 2^{N_j}, \\ 0 & 2N_j + N_j 2^{N_j} \leqslant k < N_{j+1}, \end{cases}$$

and $\alpha > 0$. Notice that $\{a_k\}_{k=1}^{\infty} \in GM(\boldsymbol{\beta}^3) \setminus GM(\boldsymbol{\beta}^2)$.

4. Note that if $\{C_j\}_{j=1}^{\infty}$ increases fast enough, then the uniform convergence of $\sum_{k=1}^{\infty} a_k \sin kx$ simply follows from the absolute convergence of $\sum_{k=1}^{\infty} a_k$, since

$$\sum_{k=1}^{\infty} |a_k| = \sum_{j=1}^{\infty} \sum_{k=N_j}^{N_{j+1}-1} |a_k| = \sum_{j=1}^{\infty} \sum_{k=N_j}^{2N_j+M_j} |a_k| = \sum_{j=1}^{\infty} \frac{1}{C_j}(N_j + M_j + 1).$$

In particular, the condition

$$j^{\alpha}(N_j + M_j) = O(C_j) \quad \text{as} \quad j \to \infty,$$

where $\alpha > 1$, implies convergence of the series $\sum_{k=1}^{\infty} a_k \sin kx$.

4 Theorem 1.2 Does not Hold for Unbounded $\widetilde{\mathbf{a}}$

Theorem 4.1 *There exists a uniformly convergent sine series $\sum_{k=1}^{\infty} a_k \sin kx$ such that*

(i) $\sum_{k=2^{n-1}}^{2^n} |a_k| \geqslant 2^{\frac{n}{2}-1} d_n$, $n \geqslant 1$,

(ii) $k|a_k| \geqslant 2^{n-1}|a_{2^{n-1}}| = 2^{\frac{n}{2}-1} d_n$, $2^{n-1} \leqslant k < 2^n$, $n \geqslant 1$,

where $\{d_n\}_{n=1}^{\infty}$ is arbitrary positive sequence such that

(a) $\sum_{n=1}^{\infty} d_n < \infty$;

(b) $2^{\frac{n}{2}} d_n \to \infty$ *as* $n \to \infty$.

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Note that formally speaking, the constructed sequence $\{a_n\}_{n=1}^{\infty}$ is in $GM(\boldsymbol{\beta}^3)$. We will use the Rudin–Shapiro sequence; see [14, Theorem 1] and [16].

Lemma A (Rudin–Shapiro) *There exists a sequence* $\{\varepsilon_k\}_{k=0}^{\infty}$, $\varepsilon_k = \pm 1$, $k \geqslant 0$ *such that*

$$\Big|\sum_{k=0}^{N} \varepsilon_k e^{ikt}\Big| < 5\sqrt{N+1}$$

for all $t \in [0, 2\pi]$ *and* $N = 0, 1, \ldots$

Proof of Theorem 4.1 Let $\{d_n\}_{n=1}^{\infty}$ be a positive sequence satisfying conditions (a) and (b). Also let $\{\varepsilon_k\}_{k=0}^{\infty}$ be the Rudin–Shapiro sequence. Consider the series

$$\sum_{n=1}^{\infty} c_n \sum_{k=2^{n-1}}^{2^n-1} \varepsilon_k e^{ikt}, \tag{4.1}$$

with $c_n \in \mathbb{R}$ such that $|c_n| = 2^{-\frac{n}{2}} d_n$, $n \in \mathbb{N}$. By using the Rudin–Shapiro theorem, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\Big| c_n \sum_{k=2^{n-1}}^{2^n-1} \varepsilon_k e^{ikt}\Big| &\leqslant \sum_{n=1}^{\infty} |c_n| \Bigg(\Big|\sum_{k=0}^{2^n-1} \varepsilon_k e^{ikt}\Big| + \Big|\sum_{k=0}^{2^{n-1}-1} \varepsilon_k e^{ikt}\Big| \Bigg) \\ &\leqslant C \sum_{n=1}^{\infty} |c_n| 2^{\frac{n}{2}} \leqslant C \sum_{n=1}^{\infty} d_n.\end{aligned}$$

Hence, the convergence of the series $\sum_{n=1}^{\infty} d_n$ implies the uniform convergence of series (4.1). Then the series

$$\sum_{n=1}^{\infty} c_n \sum_{k=2^{n-1}}^{2^n-1} \varepsilon_k \sin kt$$

converges uniformly. Denote by f its sum and by $a_k(f)$ the Fourier coefficients of f. Then

$$\sum_{k=2^{n-1}}^{2^n} |a_k(f)| \geqslant \sum_{k=2^{n-1}}^{2^n-1} |c_n| = |c_n| 2^{n-1} = 2^{\frac{n}{2}-1} d_n.$$

Condition (ii) is clear. ∎

Remark 4.2 (i) As mentioned in the introduction, the widest class of general monotone sequences satisfying condition (1.4) is when $\beta_n = \sum_{k=n}^{2n+1} |a_k|$. All sequences of the form

$$a_k = c_n \varepsilon_k, \qquad 2^{n-1} \leqslant k < 2^n, \quad n \in \mathbb{N},$$

where $c_n \in \mathbb{R}$ and $\{\varepsilon_k\}_{k=1}^{\infty}$ is the Rudin–Shapiro sequence (see the example in Theorem 4.1), belong to this extreme class. Moreover, such sequences do not belong to any smaller class, since we always have

$$\sum_{k=n}^{2n} |\Delta a_k| \asymp \sum_{k=n}^{2n+1} |a_k|, \quad n \geqslant 6.$$

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This follows from the fact that for any k, the sequence $\varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}, \varepsilon_{k+3}, \varepsilon_{k+4}$ changes its sign at least once. Therefore, for any integer $s \geqslant 6$ such that $2^{n-1} < s \leqslant 2^n$, $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=s}^{2s} |\Delta a_k| &\geqslant \Big(\sum_{k=s}^{2^n} + \sum_{k=2^n+1}^{2s} \Big) |\Delta a_k| \\ &\geqslant \frac{2^n - s + 1}{5} |c_n| + \frac{2s - 2^n}{5} |c_{n+1}| + |c_n \pm c_{n+1}| \\ &\geqslant C \Big(\sum_{k=s}^{2^n} + \sum_{k=2^n+1}^{2s+1} \Big) |a_k| = C \sum_{k=s}^{2s+1} |a_k|.\end{aligned}$$

(ii) Taking $d_n = 2^{-\frac{n}{2}}$, $n \in \mathbb{N}$, in Theorem 4.1 and following the construction, we see that $|c_n| = |a_k| \asymp 2^{-n}$, $2^{n-1} \leqslant k < 2^n$. In other words, there is a uniformly convergent series $\sum_{k=1}^{\infty} a_k \sin kx$ such that

$$m|a_m| \asymp \sum_{k=m}^{2m} |a_k| \asymp 1.$$

Moreover, in view of part (i) of this remark, $\{a_k\}$ satisfies the following condition

$$\sum_{k=m}^{2m} |\Delta a_k| \asymp 1 \asymp \frac{C}{m} \sum_{k=m}^{m+\lambda_m} |a_k|, \qquad \lambda_m = m2^m.$$

In other words, $\{a_n\}_{n=1}^{\infty} \in GM(\boldsymbol{\beta})$, where $\beta_n = \frac{1}{n} F_n(\widetilde{\mathbf{a}})$ with *non-admissible* functionals $F_n(\mathbf{x}) = \sum_{k=n}^{n+\lambda_n} \frac{|x_k|}{k}$. This shows that Theorem 1.2 does not hold for non-admissible functionals.

5 Final Remarks

1. Regarding the Lebesgue and Salem–Zygmund estimates stated in Subsection 1.4, see (1.6) and (1.7) respectively, it is worth mentioning that if a function h belongs to the Lipschitz space $\operatorname{Lip} \alpha$, then

$$\|h(x) - \sigma_n(h,x)\|_{C[0,2\pi]} = O\Big(\frac{1}{n^\alpha}\Big), \qquad \alpha < 1, \tag{5.1}$$

$$\|h(x) - \sigma_n(h,x)\|_{C[0,2\pi]} = O\Big(\frac{\ln n}{n^\alpha}\Big), \qquad \alpha = 1, \tag{5.2}$$

where $\sigma_n(h,x)$ is the first arithmetic mean of the Fourier series of h. These results were obtained by Bernstein [1]. Note that (5.1) implies that $E_n(h)_C = O(\frac{1}{n^\alpha})$, which is equivalent to $\operatorname{Lip} \alpha$ for $\alpha < 1$. Moreover, the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ belongs to $\operatorname{Lip} 1$, but

$$\|f(x) - \sigma_n(f,x)\|_{C[0,2\pi]} \geqslant C \frac{\ln n}{n}.$$

It is important to note that there is a crucial difference between the results (1.6)–(1.7) and (5.1)–(5.2) which becomes apparent only when we consider these relations for a particular value of x. Indeed, the relation $h(x) - \sigma_n(h,x) = O(\frac{1}{n^\alpha})$ depends only on the behavior of x in the neighborhood of the particular point x concerned but the relation $h(x) - S_n(h,x) = O(\frac{1}{n^\alpha})$ depends on the behavior of x in the entire interval $[0, 2\pi]$; see the discussion in [9].

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2. The natural extension of the Lipschitz space is

$$\operatorname{Lip}\omega(\cdot)=\Big\{f\in C[0,2\pi]:\omega(f,\delta)_C=O(\omega(\delta))\Big\},$$

where $\omega(\cdot)$ is a nondecreasing continuous function on $[0,2\pi]$ such that $\omega(0)=0$ and $\omega(\delta_1+\delta_2)\leqslant\omega(\delta_1)+\omega(\delta_2)$. In particular, it is known that if ω satisfies the so-called Bary–Stechkin condition

$$\int_\delta^\pi \omega(t)\frac{dt}{t^2}=O\Big(\frac{\omega(\delta)}{\delta}\Big),$$

then $h\in\operatorname{Lip}\omega(\cdot)$ implies that

$$\|h-S_n(h)\|_{C[0,2\pi]}=O\big(\omega(1/n)\big),$$

provided that h is of monotonic type.

It would be interesting to obtain criteria for the generalized Lipschitz classes $\operatorname{Lip}\omega(\cdot)$ similar to the ones derived in Corollaries 1.10 and 1.11.

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