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A NOTE ON THE GAUSSIAN CARDINAL-INTERPOLATION OPERATOR

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Suppose λ is a positive number, and let $\varphi_\lambda^{[d]}(x) := \exp(-\lambda\|x\|_2^2)$, $x \in \mathbf{R}^d$, denote the d-dimensional Gaussian. Basic theory of cardinal interpolation asserts the existence of a unique function $\chi_\lambda^{[d]}(x) = \sum_{j\in\mathbf{Z}^d} c_j\varphi_\lambda^{[d]}(x-j)$, $x \in \mathbf{R}^d$, satisfying the interpolatory conditions $\chi_\lambda^{[d]}(k) = \delta_{0k}$, $k \in \mathbf{Z}^d$, and decaying exponentially for large argument. In particular, the *Gaussian cardinal-interpolation operator* $\mathcal{L}_\lambda^{[d]}$, given by $(\mathcal{L}_\lambda^{[d]}\mathrm{y})(x) := \sum_{j\in\mathbf{Z}^d} y_j\chi_\lambda^{[d]}(x-j)$, $x \in \mathbf{R}^d$, $\mathrm{y} = (y_j)_{j\in\mathbf{Z}^d}$, is a well-defined linear map from $\ell^2(\mathbf{Z}^d)$ into $L^2(\mathbf{R}^d)$. It is shown here that its associated operator-norm is $\left[(\sum_{l\in\mathbf{Z}}\exp(-2\pi^2l^2/\lambda))/(\sum_{l\in\mathbf{Z}}\exp(-\pi^2l^2/\lambda))^2\right]^d$, implying, in particular, that $\mathcal{L}_\lambda^{[d]}$ is contractive. Some sidelights are also presented.

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1. Introduction

Suppose λ is a positive constant, and let φ_λ denote the univariate Gaussian

$$\varphi_\lambda(x) := e^{-\lambda x^2}, \quad x \in \mathbf{R}. \tag{1.1}$$

The *symbol* σ_λ associated with the Gaussian is the even, continuous, 2π-periodic function defined by the equation

$$\sigma_\lambda(u) := \sum_{j\in\mathbf{Z}} \varphi_\lambda(j)e^{-iju}, \quad u \in \mathbf{R}, \tag{1.2}$$

which, according to Poisson's summation formula, can also be written as follows:

$$\sigma_\lambda(u) = \sum_{k\in\mathbf{Z}} \widehat{\varphi}_\lambda(u+2\pi k) = \left(\frac{\pi}{\lambda}\right)^{1/2}\sum_{k\in\mathbf{Z}} e^{-(u+2\pi k)^2/(4\lambda)}, \quad u \in \mathbf{R}. \tag{1.3}$$

The latter equation reveals that $\sigma_\lambda(u)$ is positive for every real number u, so standard cardinal-interpolation theory (see, for example, **[10, 4]**) guarantees the existence of a unique *cardinal function*

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$$\chi_\lambda(x) := \sum_{k\in\mathbf{Z}} \rho_k \varphi_\lambda(x-k), \quad x \in \mathbf{R}, \tag{1.4}$$

where

$$\sum_{k\in\mathbf{Z}} \rho_k e^{-iku} = \frac{1}{\sigma_\lambda(u)}, \quad u \in [-\pi,\pi], \quad i.e., \quad \rho_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{iku}}{\sigma_\lambda(u)}\,du. \tag{1.5}$$

The cardinal function χ_λ enjoys the interpolatory property

$$\chi_\lambda(k) = \delta_{0k}, \quad k \in \mathbf{Z}, \tag{1.6}$$

and decays exponentially at infinity. Two consequences of the exponential decay of χ_λ are of moment to us: firstly, χ_λ is absolutely integrable on $\mathbf{R}$, and has a Fourier transform given by

$$\widehat{\chi_\lambda}(\xi) = \frac{\widehat{\varphi_\lambda}(\xi)}{\sigma_\lambda(\xi)} = \frac{\sqrt{(\pi/\lambda)}e^{-\xi^2/(4\lambda)}}{\sigma_\lambda(\xi)}, \quad \xi \in \mathbf{R}. \tag{1.7}$$

Secondly, the linear operator

$$\mathcal{L}_\lambda(\mathbf{y}) := \sum_{j\in\mathbf{Z}} y_j \chi_\lambda(\cdot - j), \quad \mathbf{y} = (y_j)_{j\in\mathbf{Z}}, \tag{1.8}$$

called the *Gaussian cardinal-interpolation operator*, is well defined as a map from $\ell^2(\mathbf{Z})$ to $L^2(\mathbf{R})$. The primary objective of this note is to determine its norm

$$\|\mathcal{L}_\lambda\| := \sup\left\{\|\mathcal{L}_\lambda \mathbf{y}\|_{L^2(\mathbf{R})} : \|\mathbf{y}\|_{\ell^2(\mathbf{Z})} \le 1\right\}. \tag{1.9}$$

The symbol σ_λ given by (1.2) is linked closely with Jacobi's *Theta function*

$$\vartheta(z) := \sum_{k\in\mathbf{Z}} q^{k^2} z^k, \quad z \in \mathbf{C}\backslash\{0\}, \quad q \in \mathbf{C}, \quad |q| < 1. \tag{1.10}$$

This connection between σ_λ and the Theta function of (1.10) has been put to good use in [**1**] and [**2**], and will be exploited here as well. Specifically, we shall rely on the following product formula (see [**11**, Section 21.3], [**3**, Section 32]):

$$\vartheta(z) = T(q)\prod_{k=0}^{\infty}(1+q^{2k+1}z)(1+q^{2k+1}z^{-1}), \quad z \in \mathbf{C}\backslash\{0\}, \quad T(q) := \prod_{l=1}^{\infty}(1-q^{2l}). \tag{1.11}$$

Impetus for the work reported in this article came from a reading of [**8**] and [**5**], where the 2-norm of the cardinal-spline-interpolation operator was explicitly computed. Detailed analysis of other p-norms of these spline-interpolation operators followed in [9] and [6, 7], but we are yet to begin such general studies for the Gaussian.

The paper is laid out in three sections, including the introduction. Section 2 describes the main results (all univariate), and their multivariate analogues round out the final section.

2. Main results: univariate

As stated in the introduction, our main goal in this section is to compute the 2-norm (1.9) of the linear operator $\mathcal{L}_\lambda : \ell^2(\mathbf{Z}) \to L^2(\mathbf{R})$ defined by (1.8). The following result is a first step towards that goal.

Theorem 2.1. *The norm of $\mathcal{L}_\lambda$ is given by the equation*

$$\|\mathcal{L}_\lambda\| = \max_{-\pi \le x \le \pi} H_\lambda(x), \tag{2.1}$$

where

$$H_\lambda(x) := \frac{\sqrt{(\pi/(2\lambda))}\sigma_{\lambda/2}(x)}{[\sigma_\lambda(x)]^2} = \frac{\sqrt{(\pi/(2\lambda))}\sum_{j\in\mathbf{Z}} e^{-(\lambda j^2/2)}e^{-ijx}}{\left[\sum_{j\in\mathbf{Z}} e^{-\lambda j^2}e^{-ijx}\right]^2}. \tag{2.2}$$

Proof. The Parseval-Plancherel theorem and equation (1.8) provide the relations

$$\begin{aligned}\|\mathcal{L}_\lambda \mathbf{y}\|^2_{L^2(\mathbf{R})} = (2\pi)^{-1}\|\widehat{\mathcal{L}_\lambda \mathbf{y}}\|^2_{L^2(\mathbf{R})} &= (2\pi)^{-1}\int_{-\infty}^{\infty}\left|\sum_{j\in\mathbf{Z}} y_j e^{-ij\xi}\right|^2 |\widehat{\chi_\lambda}(\xi)|^2 d\xi \\ &= (2\pi)^{-1}\int_{-\pi}^{\pi}\left|\sum_{j\in\mathbf{Z}} y_j e^{-ijx}\right|^2\left(\sum_{k\in\mathbf{Z}}|\widehat{\chi_\lambda}(x+2\pi k)|^2\right)dx,\end{aligned} \tag{2.3}$$

whereas (1.7), the periodicity of σ_λ, and (1.3) combine to give the equation

$$\sum_{k\in\mathbf{Z}}|\widehat{\chi_\lambda}(x+2\pi k)|^2 = H_\lambda(x), \quad -\pi \le x \le \pi. \tag{2.4}$$

The required result follows. □

Use of (1.3) in (2.2) leads to the identity

$$H_\lambda(x) = \frac{\sum_{k\in\mathbf{Z}} e^{-(x+2\pi k)^2/(2\lambda)}}{\left(\sum_{k\in\mathbf{Z}} e^{-(x+2\pi k)^2/(4\lambda)}\right)^2}, \quad x \in [-\pi, \pi]. \tag{2.5}$$

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Since

$$\left(\sum_{k\in Z} e^{-(x+2\pi k)^2/(4\lambda)}\right)^2 = \sum_{j,k\in Z} e^{-[(x+2\pi j)^2+(x+2\pi k)^2]/(4\lambda)} > \sum_{k\in Z} e^{-(x+2\pi k)^2/(2\lambda)}, \tag{2.6}$$

we have the estimate

$$0 < H_\lambda(x) < 1, \quad x \in [-\pi, \pi], \quad \lambda > 0; \tag{2.8}$$

in particular,

$$\| \mathcal{L}_\lambda \| < 1 \quad \text{for all } \lambda > 0. \tag{2.8}$$

Thus $\mathcal{L}_\lambda$ is a contraction, and we contrast this with the case of the cardinal-spline-interpolation operator, whose 2-norm is unity [**8**, **5**].

The uniform bound (2.8) notwithstanding, Theorem 2.1 is of limited interest unless the maximum value of H_λ can be identified. Our main finding is that this maximum is attained at $x = 0$, and this is the content of Theorem 2.4 (*vide infra*). Its proof will require some preludial work which we take up first.

Remark 2.2. (i) If $B > 2A > 0$, then the quadratic polynomial $At^2 - Bt + A$ has two real zeroes, one of which lies in the interval $(0, 1)$ and the other in the interval $(1, \infty)$.

(ii) Let $p_1(t) := t^4 - 2t^3 - 2t^2 - 2t + 1$. Then $p_1(t) \geq 1 - 2[(0.3)^3 + (0.3)^2 + (0.3)] > 0$ for every t in the interval $[0, 0.3]$.

Lemma 2.3. *Let $r(t)$ be defined as follows:*

$$r(t) := \begin{cases} \dfrac{(1-t^2)^2 - \sqrt{(1-t^2)^4 - 4t^2(1+t^2)^2}}{2t(1+t^2)}, & \text{if } 0 < t \leq 0.3; \\ 0, & \text{if } t = 0. \end{cases} \tag{2.9}$$

Then the following hold:

(i) *r is well defined and continuous on the interval* $[0, 0.3]$;
(ii) $0 < r(t) < 1$ *for every* $0 < t < 0.3$;
(iii) *r increases monotonically with t in* $[0, 0.3]$;
(iv) $r(t^3) \leq t$ *for* $0 \leq t \leq 0.1$.

Proof. The first three statements are quite easy to verify, with the aid of Remark 2.2 and the fact that

$$r'(t) = \frac{(1-t)(1+t)(1+6t^2+t^4)[(1-t^2)^2 - \sqrt{(1-t^2)^4 - 4t^2(1+t^2)^2}]}{2t^2(1+t^2)^2\sqrt{(1-t^2)^4 - 4t^2(1+t^2)^2}}, \quad 0 < t < 0.3.$$

(iv) The assertion being clearly true for $t = 0$, we assume $0 < t \le 0.1$. Since

$$r(t) = \frac{(1-t^2)^2}{2t(1+t^2)}\left[1 - \sqrt{1 - \frac{4t^2(1+t^2)^2}{(1-t^2)^4}}\right], \tag{2.10}$$

we have

$$r(t^3) = \frac{(1-t^6)^2}{2t^3(1+t^6)}\left[1 - \sqrt{1 - \frac{4t^6(1+t^6)^2}{(1-t^6)^4}}\right]. \tag{2.11}$$

The function $t \mapsto \dfrac{4t^6(1+t^6)^2}{(1-t^6)^4}$ increases on the interval $[0, 1)$, so

$$\frac{4t^6(1+t^6)^2}{(1-t^6)^4} \le \frac{4t(1+t)^2}{(1-t)^4} \le \frac{4(0.1)(1+0.1)^2}{(1-0.1)^4} = \frac{4840}{6561} =: y_0, \quad 0 < t \le 0.1. \tag{2.12}$$

Consider the function $\phi(y) := \sqrt{1-y}$, $0 \le y \le y_0$, where y_0 is the number defined in (2.12). By the Mean Value Theorem,

$$1 - \sqrt{1-y} = \phi(0) - \phi(y) < \frac{y}{2\sqrt{1-y_0}} < \frac{81y}{82}, \tag{2.13}$$

where the last inequality follows from observing that $1 - y_0 > (41/81)^2$. Putting $y = \dfrac{4t^6(1+t^6)^2}{(1-t^6)^4}$ and using (2.13) in (2.11) provides the inequality

$$r(t^3) \le \left(\frac{(1-t^6)^2}{2t^3(1+t^6)}\right)\left(\frac{81}{82}\right)\left(\frac{4t^6(1+t^6)^2}{(1-t^6)^4}\right) = \frac{81t^3(1+t^6)}{41(1-t^6)^2}, \tag{2.14}$$

and hence the estimate

$$\frac{r(t^3)}{t} \le \frac{81t^2(1+t^6)}{41(1-t^6)^2}, \quad 0 < t \le 0.1. \tag{2.15}$$

Since the function $t \mapsto \dfrac{81t^2(1+t^6)}{41(1-t^6)^2}$ increases with t in $[0, 1)$, we find from (2.15) that

$$\frac{r(t^3)}{t} \le \frac{81(0.1)^2(1+(0.1)^6)}{41(1-(0.1)^6)^2} < 1, \quad 0 < t \le 0.1. \tag{2.16}$$

□

With our preparations now completed, we proceed to the focal result, already advertised prior to Remark 2.2.

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Theorem 2.4. *Let H_λ be defined by* (2.2) (*equivalently,* (2.5)). *Then*

$$\max_{-\pi\le x\le\pi} H_\lambda(x) = H_\lambda(0).$$

Proof. Since H_λ is an even function, it suffices to consider the interval $[0,\pi]$. We divide the proof into two cases: "large" λ and "small" λ.

Case I: Assume

$$\lambda > -2\log(0.3), \quad \text{and let } q := e^{-\lambda}. \tag{2.17}$$

Let H_λ be given by (2.2), and define $\tilde{H}_\lambda(x) := \sqrt{(2\lambda/\pi)}H_\lambda(x)$. It is enough to show that the maximum value of $\tilde{H}_\lambda(x)$ on the interval $[0,\pi]$ is attained at $x=0$. According to (1.10) and (1.11),

$$\begin{aligned}\tilde{H}_\lambda(x) &= \left[\frac{\prod_{l=1}^\infty(1-q^l)}{\prod_{l=1}^\infty(1-q^{2l})^2}\right]\prod_{k=0}^\infty \frac{(1+q^{(2k+1)/2}e^{-ix})(1+q^{(2k+1)/2}e^{ix})}{[(1+q^{2k+1}e^{-ix})(1+q^{2k+1}e^{ix})]^2}\\ &= \left[\frac{\prod_{l=1}^\infty(1-q^l)}{\prod_{l=1}^\infty(1-q^{2l})^2}\right]\prod_{k=0}^\infty \frac{1+2q^{(2k+1)/2}\cos x+q^{2k+1}}{[1+2q^{2k+1}\cos x+q^{4k+2}]^2}\\ &=: \left[\frac{\prod_{l=1}^\infty(1-q^l)}{\prod_{l=1}^\infty(1-q^{2l})^2}\right]\prod_{k=0}^\infty f_k(x), \quad 0\le x\le\pi.\end{aligned} \tag{2.18}$$

We shall show that each f_k decreases on the interval $[0,\pi]$. Define

$$\alpha_k := q^{(2k+1)/2}, \quad \text{and note that } \alpha_k \le \sqrt{q} < 0.3, \quad k\ge 0, \tag{2.19}$$

by (2.17). A straightforward computation shows that

$$-f_k'(x) = \frac{(2\alpha_k\sin x)(1-2\alpha_k-2\alpha_k^3+\alpha_k^4-2\alpha_k^2\cos x)}{(1+2\alpha_k^2\cos x+\alpha_k^4)^3}, \quad 0<x<\pi. \tag{2.20}$$

The denominator of (2.20) is bounded below by the positive quantity $(1-\alpha_k^2)^6$, whilst $2\alpha_k\sin x > 0$ for $0<x<\pi$. Further, the remaining term in (2.20) satisfies the inequalities

$$1-2\alpha_k-2\alpha_k^3+\alpha_k^4-2\alpha_k^2\cos x \ge 1-2\alpha_k-2\alpha_k^3+\alpha_k^4-2\alpha_k^2 > 0, \tag{2.21}$$

where the final bound obtains from Remark 2.2(ii), via (2.19). Thus $f_k'(x)<0$ for $0<x<\pi$, that is f_k decreases on $[0,\pi]$.

Case II: Assume

$$0<\lambda\le -2\log(0.3) < 5/2, \tag{2.22}$$

where the last inequality stems from the following:

$$\frac{3}{10}e^{5/4} > \frac{3}{10}\left[1+\frac{5}{4}+\frac{25}{32}+\frac{125}{384}\right] = \frac{1289}{1280} > 1. \tag{2.23}$$

We use (2.5) to write

$$H_\lambda(x) = \frac{\sum_{k\in Z}(e^{-(2\pi^2/\lambda)})^{k^2}(e^{-(2\pi x/\lambda)})^k}{\left[\sum_{k\in Z}(e^{-(\pi^2/\lambda)})^{k^2}(e^{-(\pi x/\lambda)})^k\right]^2} = \frac{\sum_{k\in Z}(q^2)^{k^2}(t^2)^k}{\left[\sum_{k\in Z}q^{k^2}t^k\right]^2}, \tag{2.24}$$

where

$$q := e^{-(\pi^2/\lambda)} > 0 \quad \text{and} \quad t := e^{-(\pi x/\lambda)}. \tag{2.25}$$

The assumption that x belongs to the interval $[0, \pi]$ is tantamount to

$$q \le t \le 1; \tag{2.26}$$

in addition, we also note that

$$q = e^{-(\pi^2/\lambda)} < 0.1, \tag{2.27}$$

because $\lambda < 5/2$ and

$$e^{(2\pi^2/5)}(0.1) > \frac{1}{10}\left[1+\frac{2\pi^2}{5}+\frac{2\pi^4}{25}\right] > \frac{1}{10}\left[1+\frac{18}{5}+\frac{162}{25}\right] = \frac{277}{250} > 1. \tag{2.28}$$

Use of (1.10) and (1.11) in equation (2.24) yields

$$\begin{aligned} H_\lambda(x) &= \left[\frac{\prod_{l=1}^\infty(1-q^{4l})}{\prod_{l=1}^\infty(1-q^{2l})^2}\right]\prod_{k=0}^\infty\frac{(1+q^{4k+2}t^2)(1+q^{4k+2}t^{-2})}{[(1+q^{2k+1}t)(1+q^{2k+1}t^{-1})]^2} \\ &=: \left[\frac{\prod_{l=1}^\infty(1-q^{4l})}{\prod_{l=1}^\infty(1-q^{2l})^2}\right]\prod_{k=0}^\infty g_k(t), \qquad t = e^{-(\pi x/\lambda)}, \end{aligned} \tag{2.29}$$

so it is sufficient to show that

$$g_k(t) \le g_k(1), \quad q \le t \le 1, \quad k \ge 0. \tag{2.30}$$

Set

$$0 < \beta_k := q^{2k+1}, \quad \text{and observe that} \quad \beta_k \le \beta_0 = q < 0.1, \quad k \ge 0. \tag{2.31}$$

An elementary, albeit somewhat tedious, computation leads to the expression

$$g_k'(t) = \frac{2\beta_k(t^2-1)(\beta_k(1+\beta_k^2)t^2-(1-\beta_k^2)^2t+\beta_k(1+\beta_k^2))}{(\beta_k t^2+(1+\beta_k^2)t+\beta_k)^3}$$
$$=: \frac{2\beta_k(t^2-1)P_k(t)}{(\beta_k t^2+(1+\beta_k^2)t+\beta_k)^3}, \qquad 0<t<1. \tag{2.32}$$

Plainly

$$2\beta_k(t^2-1)<0<(\beta_k t^2+(1+\beta_k^2)t+\beta_k)^3 \quad \text{for } 0<t<1. \tag{2.33}$$

Moreover, by virtue of (2.27) and Remark 2.2, there exist positive numbers r_k and $\tilde{r}_k$ such that $P_k(r_k)=0=P_k(\tilde{r}_k)$ and

$$0<r_k=r(\beta_k)<1<\tilde{r}_k, \tag{2.34}$$

where r is the function defined by equation (2.9). It follows that P_k is positive on the interval $[0,r_k)$ and negative on $(r_k,1]$. This fact, taken in conjunction with (2.33) and (2.32), proves that g_k decreases on $[0,r_k]$ and increases on $[r_k,1]$.

Now if $k\geq 1$, then $\beta_k=q^{2k+1}\leq q^3$, so from (2.27) and parts (iii) and (iv) of Lemma 2.3,

$$r_k=r(\beta_k)\leq r(q^3)\leq q. \tag{2.35}$$

Since g_k increases on the interval $[r_k,1]$, equation (2.35) ensures that

$$g_k(t)\leq g_k(1), \quad q\leq t\leq 1, \quad k\geq 1. \tag{2.36}$$

The foregoing argument fails for $k=0$ because $r_0=r(q)$ may exceed q. Nevertheless, the general analysis (carried out in the last paragraph but one) still applies, allowing the estimate

$$g_0(t)\leq \max\{g_0(q),g_0(1)\}, \quad q\leq t\leq 1. \tag{2.37}$$

But it is a simple matter to check that

$$g_0(1)-g_0(q)=\frac{1-4q+2q^2-4q^3+10q^4-4q^5+2q^6-4q^7+q^8}{2(1+q)^4(1+q^2)^2}$$
$$>\frac{1-4[(0.1)+(0.1)^3+(0.1)^5+(0.1)^7]}{2(1+q)^4(1+q^2)^2}>0,$$

where the first inequality above is consequent upon the fact that $0<q<0.1$. Ergo,

$$g_0(t)\leq g_0(1), \quad q\leq t\leq 1, \tag{2.38}$$

and the proof is complete. □

An immediate consequence of Theorems 2.1 and 2.4 is the following:

Corollary 2.5. *Suppose $\mathcal{L}_\lambda$ is the linear operator defined by* (1.8), *and let* $\|\mathcal{L}_\lambda\|$ *be its norm defined via* (1.9). *Then*

$$\|\mathcal{L}_\lambda\| = \frac{\sum_{k\in\mathbf{Z}} \exp(-2\pi^2 k^2/\lambda)}{\left(\sum_{k\in\mathbf{Z}} \exp(-\pi^2 k^2/\lambda)\right)^2} \tag{2.39}$$

Proof. Put $x = 0$ in (2.5). □

We close this section with a supplementary line of enquiry which was prompted by some studies undertaken in [**6**, **7**]. Let W denote the Whittaker operator (or, perhaps more properly, the Whittaker-Shannon-Kotel'nikov (WSK) operator – see [**12**, p. 4]) given by

$$(W\mathbf{y})(x) := \sum_{j\in\mathbf{Z}} y_j \frac{\sin \pi(x-j)}{\pi(x-j)}, \quad x \in \mathbf{R}, \quad \mathbf{y} = (y_j)_{j\in\mathbf{Z}} \in \ell^2(\mathbf{Z}). \tag{2.40}$$

For every $\mathbf{y} \in \ell^2(\mathbf{Z})$, $W\mathbf{y}$ can be realized as the L^2-Fourier transform of the square-integrable function

$$\frac{1}{2\pi} I(u) \sum_{j\in\mathbf{Z}} y_j e^{iju}, \quad u \in \mathbf{R}, \tag{2.41}$$

where I is the characteristic (indicator) function of the interval $(-\pi, \pi)$. Therefore the linear operator W maps $\ell^2(\mathbf{Z})$ into $L^2(\mathbf{R})$. Furthermore, from Parseval's theorem and the Parseval-Plancherel theorem, one deduces that

$$\|W\mathbf{y}\|_{L^2(\mathbf{R})} = \|\mathbf{y}\|_{\ell^2(\mathbf{Z})} \ \forall \mathbf{y} \in \ell^2(\mathbf{Z}); \quad \text{in particular,} \quad \|W\| = 1. \tag{2.42}$$

Some connections between the cardinal-interpolation operators $\mathcal{L}_\lambda$ and the WSK operator W will be brought out in the pair of results below (cf. [**6**, Theorems 3.3 and 3.4] and [**7**]):

Theorem 2.6. *Let $\mathcal{L}_\lambda$ and W be the linear operators defined by* (1.8) *and* (2.40), *respectively. The following hold:*

(i) $\|\mathcal{L}_\lambda\| \to \|W\|$ *as* $\lambda \to 0^+$;
(ii) $\lim_{\lambda\to 0^+} \|(\mathcal{L}_\lambda - W)\mathbf{y}\|_{L^2(\mathbf{R})} = 0$ *for every* $\mathbf{y} \in \ell^2(\mathbf{Z})$.

Proof. (i) This follows from (2.39), (2.42), and the fact that $\lim_{\rho\to 0^+} \sum_{k\in\mathbf{Z}} e^{-(k^2/\rho)} = 1$.
(ii) Since $\mathcal{L}_\lambda - W$ is linear, and $\|\mathcal{L}_\lambda - W\| < 2$ by (2.8) and (2.42), it suffices to prove the assertion for sequences $\mathbf{y}^{(\nu)}$, $\nu \in \mathbf{Z}$, given by

$$\mathbf{y}^{(\nu)} := (y_j^{\nu})_{j\in\mathbf{Z}}, \quad \text{where} \quad y_j^{\nu} = \delta_{\nu j}. \tag{2.54}$$

But

$$\|(\mathcal{L}_\lambda - W)\mathbf{y}^{(\nu)}\| = \left\|\chi_\lambda(\cdot - \nu) - \frac{\sin\pi(\cdot - \nu)}{\pi(\cdot - \nu)}\right\|_{L^2(\mathbf{R})} = \left\|\chi_\lambda(\cdot) - \frac{\sin\pi(\cdot)}{\pi(\cdot)}\right\|_{L^2(\mathbf{R})}, \tag{2.43}$$

and the last term in (2.43) approaches zero as $\lambda \to 0^+$, by the "if" part of [**2**, Theorem 3.7]. □

We remark that the validity of assertion (ii) in the theorem above may also be gleaned from [**2**], for the uniform boundedness of the quantities $\|\mathcal{L}_\lambda\|$ was already observed in Proposition 3.5 of that paper.

Theorem 2.7. *The following classes of functions are equivalent:*

(i) $\{f \in L^2(\mathbf{R}) : f(x) = \int_{-\pi}^{\pi} e^{ixt}d\beta(t), \beta \in C[-\pi,\pi]\}$.

(ii) $\{f : f(x) = (W\mathbf{y})(x), \mathbf{y} \in \ell^2(\mathbf{Z})\}$.

(iii) $\{f : \lim_{\lambda\to 0^+} \|f - \mathcal{L}_\lambda \mathbf{y}\|_{L^2(\mathbf{R})} = 0, \mathbf{y} \in \ell^2(\mathbf{Z})\}$.

Proof. The equivalence of (i) and (ii) is known (see [7]), whereas Theorem 2.6(ii) supplies the equivalence of (ii) and (iii). □

3. Multivariate analogues

We turn now to multidimensional analogues of results given previously. Proofs will be withheld for the most part, because they derive from predictable tensor-product arguments.

Suppose λ is a positive number. Let $\varphi_\lambda^{[d]}$ and $\sigma_\lambda^{[d]}$ denote the d-dimensional Gaussian and its symbol, respectively:

$$\varphi_\lambda^{[d]}(x) := e^{-\lambda\|x\|^2} = \prod_{j=1}^{d} \varphi_\lambda(x_j), \quad x = (x_1, \ldots, x_d) \in \mathbf{R}^d, \tag{3.1}$$

and

$$\sigma_\lambda^{[d]}(u) := \sum_{k\in\mathbf{Z}^d} e^{-\lambda\|k\|^2} e^{-ik^T u} = (\pi/\lambda)^{d/2} \sum_{k\in\mathbf{Z}^d} e^{-\|u+2\pi k\|^2/(4\lambda)}$$
$$= \prod_{j=1}^{d} \sigma_\lambda(u_j), \quad u = (u_1, \ldots, u_d) \in \mathbf{R}^d, \tag{3.2}$$

where φ_λ and σ_λ are the univariate functions defined in (1.1) and (1.2), respectively, and $\|\|$ denotes the Euclidean norm in $\mathbf{R}^d$. Denote by $\chi_\lambda^{[d]}$ the corresponding cardinal function, to wit,

$$\chi_\lambda^{[d]}(x) := \sum_{k\in\mathbf{Z}^d} \rho_k^{[d]}\varphi_\lambda^{[d]}(x-k), \quad x \in \mathbf{R}^d, \tag{3.3}$$

where

$$\rho_k^{[d]} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{e^{ik^T u}}{\sigma_\lambda^{[d]}(u)}\,du, \quad \text{and} \quad \chi_\lambda^{[d]}(k) = \delta_{0k}, \quad k \in \mathbf{Z}^d. \tag{3.4}$$

We note that

$$\rho_k^{[d]} = \prod_{j=1}^{d} \rho_{k_j} \quad \text{and} \quad \chi_\lambda^{[d]}(x) = \prod_{j=1}^{d} \chi_\lambda(x_j), \quad x = (x_1, \ldots, x_d) \in \mathbf{R}^d, \quad k = (k_1, \ldots, k_d) \in \mathbf{Z}^d, \tag{3.5}$$

with $(\rho_l)_{l\in\mathbf{Z}}$ being given by equation (1.5) and χ_λ the univariate cardinal function of (1.4). Define the linear operator $\mathcal{L}_\lambda^{[d]} : \ell^2(\mathbf{Z}^d) \to L^2(\mathbf{R}^d)$ by

$$(\mathcal{L}_\lambda^{[d]}\mathbf{y})(x) := \sum_{k\in\mathbf{Z}^d} y_k \chi_\lambda^{[d]}(x-k), \quad x \in \mathbf{R}^d, \quad \mathbf{y} = (y_j)_{j\in\mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d), \tag{3.6}$$

and denote by $\|\mathcal{L}_\lambda^{[d]}\|$ its norm

$$\|\mathcal{L}_\lambda^{[d]}\| := \sup\left\{\|\mathcal{L}_\lambda^{[d]}\mathbf{y}\|_{L^2(\mathbf{R}^d)} : \|\mathbf{y}\|_{\ell^2(\mathbf{Z}^d)} \le 1\right\}. \tag{3.7}$$

The following result is the multidimensional version of Theorem 2.4/Corollary 2.5.

Theorem 3.1. *Let $\mathcal{L}_\lambda^{[d]}$ and $\|\mathcal{L}_\lambda^{[d]}\|$ be given as above, and let H_λ be the univariate function defined via* (2.2) (*equivalently,* (2.5)). *Then*

$$\begin{aligned}\|\mathcal{L}_\lambda^{[d]}\| &= \max\left\{\prod_{j=1}^{d} H_\lambda(x_j) : x = (x_1, \ldots, x_d) \in [-\pi, \pi]^d\right\} \\ &= \left[\frac{\sum_{k\in\mathbf{Z}} \exp(-2\pi^2 k^2/\lambda)}{\left(\sum_{k\in\mathbf{Z}} \exp(-\pi^2 k^2/\lambda)\right)^2}\right]^d.\end{aligned} \tag{3.8}$$

In analogy with the second part of Section 2, we define the linear operator $W^{[d]} : \ell^2(\mathbf{Z}^d) \to L^2(\mathbf{R}^d)$ by the equation

$$(W^{[d]}\mathbf{y})(x) := \sum_{k\in\mathbf{Z}^d} y_k \left(\prod_{j=1}^{d} \frac{\sin \pi(x_j - k_j)}{\pi(x_j - k_j)}\right), \quad x = (x_1, \ldots, x_d) \in \mathbf{R}^d, \quad k = (k_1, \ldots, k_d) \in \mathbf{Z}^d. \tag{3.9}$$

(According to [**12**, p. 56], the operator $W^{[d]}$ was first used in the context of sampling theory by E. Parzen.)

For every $\mathbf{y} \in \ell^2(\mathbf{Z}^d)$, $W^{[d]}\mathbf{y}$ is realizable as the L^2-Fourier transform of the square-integrable function

$$\frac{1}{(2\pi)^d} I^{[d]}(u) \sum_{k \in Z^d} y_k e^{ik^T u}, \quad u \in \mathbf{R}^d, \tag{3.10}$$

where $I^{[d]}$ is the characteristic (indicator) function of the cube $(-\pi, \pi)^d$. Furthermore

$$\| W^{[d]}\mathbf{y} \|_{L^2(\mathbf{R}^d)} = \| \mathbf{y} \|_{\ell^2(\mathbf{Z}^d)} \ \forall \mathbf{y} \in \ell^2(\mathbf{Z}^d); \quad \text{in particular,} \quad \| W^{[d]} \| = 1. \tag{3.11}$$

We conclude with the following multivariate extensions of Theorems 2.6 and 2.7.

Theorem 3.2. *Let* $\mathcal{L}_\lambda^{[d]}$ *and* $W^{[d]}$ *be the linear operators defined by* (3.6) *and* (3.9), *respectively. The following hold:*

(i) $\| \mathcal{L}_\lambda^{[d]} \| \to \| W^{[d]} \|$ as $\lambda \to 0^+$;

(ii) $\lim_{\lambda \to 0^+} \| (\mathcal{L}_\lambda^{[d]} - W^{[d]})\mathbf{y} \|_{L^2(\mathbf{R}^d)} = 0$ *for every* $\mathbf{y} \in \ell^2(\mathbf{Z}^d)$.

Theorem 3.3. *The following classes of functions are equivalent:*

(i) $\{ f \in L^2(\mathbf{R}^d) : supp\, \hat{f} \subset [-\pi, \pi]^d \}$.

(ii) $\{ f : \lim_{\lambda \to 0^+} \| f - \mathcal{L}_\lambda^{[d]}\mathbf{y} \|_{L^2(\mathbf{R}^d)} = 0, \mathbf{y} \in \ell^2(\mathbf{Z}^d) \}$.

(iii) $\{ f : f(x) = (W^{[d]}\mathbf{y})(x), \mathbf{y} \in \ell^2(\mathbf{Z}^d) \}$.

Proof. The equivalence of (i) and (ii) is a special case of [**2**, Theorem 3.7], whilst that of (ii) and (iii) follows from Theorem 3.2(ii). □

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