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Vector Fields and the Cohomology Ring of Toric Varieties

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Abstract. Let X be a smooth complex projective variety with a holomorphic vector field with isolated zero set Z. From the results of Carrell and Lieberman there exists a filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$, the ring of $\mathbb{C}$-valued functions on Z, such that $\operatorname{Gr} A(Z) \cong H^*(X, \mathbb{C})$ as graded algebras. In this note, for a smooth projective toric variety and a vector field generated by the action of a 1-parameter subgroup of the torus, we work out this filtration. Our main result is an explicit connection between this filtration and the polytope algebra of X.

1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$ with a holomorphic vector field $\mathcal{V}$ such that $\operatorname{Zero}(\mathcal{V})$ is non-trivial and isolated. In [3, 4], using the Koszul complex of the vector field $\mathcal{V}$, Carrell and Lieberman prove that the coordinate ring $A(Z)$ of the zero scheme Z of $\mathcal{V}$ admits a filtration $F_0 \subset F_1 \subset \cdots$ such that the associated graded $\operatorname{Gr} A(Z)$ is isomorphic to $H^*(X, \mathbb{C})$ as graded algebra. In this paper, for a smooth projective toric variety, we work out this filtration. Our main result is a natural isomorphism between $\operatorname{Gr} A(Z)$ and Brion's description of the polytope algebra (see [1]). We also give direct proofs that the usual relations in the cohomology of a toric variety hold in $\operatorname{Gr} A(Z)$.

For the vector field $\mathcal{V}$, in the toric case, we take the generating vector field of a 1-parameter subgroup γ, in general position, of the torus T, so that the fixed point set Z of γ is the same as the fixed point set of T. Any lattice polytope Δ normal to the fan of X, determines a line bundle L_Δ on X. We show that $c_1(L_\Delta)$, the first Chern class of this line bundle, under the isomorphism $H^*(X, \mathbb{C}) \cong \operatorname{Gr} A(Z)$, is represented by the the function $f_\Delta \in A(Z)$ given by the simple formula:

$$f_\Delta(z) = \langle \gamma, v_z \rangle \quad \forall z \in Z,$$

where v_z is the vertex of Δ corresponding to a fixed point z. Multiplication by f_Δ is the Lefschetz operator in $H^*(X, \mathbb{C}) \cong \operatorname{Gr} A(Z)$. From these functions f_Δ we obtain the functions f_ρ in $\operatorname{Gr} A(Z)$ corresponding to D_ρ, the cohomology classes of the orbit closures of codimension 1. It is known that these span $H^2(X, \mathbb{C})$ as a vector space and generate $H^*(X, \mathbb{C})$ as an algebra. Using this, we then construct the filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$ (Theorem 4.3).

From [1], $H^*(X, \mathbb{C})$ can be realized as a quotient of the algebra of continuous functions on the fan of X whose restriction to each cone is a polynomial. Let p be

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a continuous function on the fan whose restriction to each cone of maximal dimension is a homogeneous polynomial of degree k, representing a cohomology class in $H^{2k}(X,\mathbb{C})$. We show that p corresponds to the function $f \in F_kA(Z)$ defined by

$$f(z) = p_{|\sigma_z}(\gamma),$$

where σ_z is the cone of maximal dimension corresponding to a fixed point z (Theorem 5.2).

This paper is motivated in part by a comment of T. Oda. In [8, p. 417], Oda briefly comments about how to explain the results of Carrell–Lieberman in the toric case: as Khovanskii has shown in [6], composition of γ and the moment map of the toric variety X defines a Morse function on X whose critical points are the fixed points (see Remark 4.8). Since the number of critical points of index i is the i-th Betti number, Oda suggests that the grading on the fixed point set induced by the Morse index is the grading in Carrell–Lieberman and hence gives the cohomology algebra. It is not difficult to see that this is not necessarily correct.[1] (See also Example 6.1.)

The present note is closely related to the work of V. Puppe [10] which gives a similar filtration in the topological setting.

In Section 2, we discuss Carrell–Lieberman results on the connection between the zeros of holomorphic vector fields and cohomology. In Section 3 we discuss the classical results on the cohomology of toric varieties. In Section 4, we work out the Carrell–Lieberman filtration in the toric case. The main result of the paper, that is the connection between the polytope algebra and the Carrell–Lieberman filtration, is discussed in Section 5. In Section 6 we see two examples in dimension 2.

2 Zeros of Holomorphic Vector Fields and Cohomology

Let X be a smooth projective variety over $\mathbb{C}$. The purpose of this section is to give a brief survey of the results of Carrell and Lieberman on the connection between the zeros of vector fields and cohomology. We start with the Koszul complex of a holomorphic vector field $\mathcal{V}$ on X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. The vector field $\mathcal{V}$ defines a derivation $\mathcal{V}\colon \mathcal{O}_X \to \mathcal{O}_X$, which extends to give a contraction operator $i(\mathcal{V})\colon \Omega^p \to \Omega^{p-1}$ on the sheaves of holomorphic p-forms on X such that $i(\mathcal{V})^2 = 0$. In addition, for all $\phi, \omega \in \Omega^*$,

$$i(\mathcal{V})(\phi \wedge \omega) = i(V)\phi \wedge \omega + (-1)^p \phi \wedge i(\mathcal{V})\omega$$

if $\phi \in \Omega^p$. Thus we get a complex K^* of sheaves

$$0 \to \Omega^n \to \Omega^{n-1} \to \cdots \to \Omega^1 \to \mathcal{O}_X \to 0$$

where $n = \dim X$, and, in turn, a spectral sequence whose first term is $E_1^{-p,q} = H^q(X,\Omega^p)$ with first differential $i(\mathcal{V})$. In [3] Carrell and Lieberman prove that if

[1] In fact, if F_i denotes the space of functions on Z consisting of the constant functions and the functions supported on the fixed points of index less than or equal to $2i$, then for $i \le j$ we have $F_iF_j \subset F_i$. This means that in the corresponding graded algebra, the multiplication is zero, which is certainly not the case in the cohomology algebra.

$\mathcal{V}$ has zeros, then every differential in this spectral sequence is zero. Consequently $E_1 = E_\infty$, and we obtain a $\mathbb{C}$-algebra isomorphism

$$\bigoplus_s H^{q+s}\big(X, \Omega^q\big) \cong \bigoplus_s F_s H^q(K^*)/F_{s-1}H^q(K^*),$$

Here $H^q(K^*)$ denotes the hypercohomology of this Koszul complex and $F_\bullet$ is its canonical filtration.

They also prove that when $\mathcal{V}$ has isolated zeros the hypercohomology groups $H^q(K^*)$ vanish for $q > 0$ (see [3]). Zero($\mathcal{V}$) can be viewed as the scheme Z defined by the sheaf of ideals $i(\mathcal{V})\Omega^1 \subset \mathcal{O}_X$, so when this scheme is finite (and non trivial), we get the following result:

Theorem 2.1 ([2, Theorem 5.4]) *Suppose X admits a holomorphic vector field $\mathcal{V}$ with Zero($\mathcal{V}$) isolated but non-trivial, then $H^p(X, \Omega^q) = \{0\}$ for all $p \neq q$ (hence $H^p(X, \Omega^p) = H^{2p}(X, \mathbb{C})$). Moreover, the coordinate ring $A(Z)$ of the zero scheme Z of $\mathcal{V}$ admits an increasing filtration $F_\bullet = F_\bullet A(Z)$ such that*

(1) *$F_i F_j \subset F_{i+j}$, and*
(2) $H^*(X, \mathbb{C}) = \bigoplus_{i\geq 0} H^{2i}(X, \mathbb{C}) \cong \bigoplus_{i \geq 0} \mathrm{Gr}_i(A(Z)) = \mathrm{Gr}\, A(Z)$,

where the displayed summands are isomorphic over $\mathbb{C}$. Here

$$\mathrm{Gr}_i(A(Z)) := F_i A(Z)/F_{i-1}A(Z).$$

For the rest of this section we assume that the zero set Z is isolated (but non-empty). Let $E \to X$ be a holomorphic vector bundle and $\mathcal{E}$ its sheaf of holomorphic sections. One says that E is $\mathcal{V}$-equivariant if the derivation $\mathcal{V}$ of $\mathcal{O}_x$ lifts to $\mathcal{E}$. That is, there exists a $\mathbb{C}$-linear sheaf homomorphism $\tilde{\mathcal{V}} \colon \mathcal{E} \to \mathcal{E}$ such that if $\sigma \in \mathcal{E}_x$ and $f \in \mathcal{O}_{X,x}$ then

$$\tilde{\mathcal{V}}(f\sigma) = \mathcal{V}(f)\sigma + f\tilde{\mathcal{V}}(\sigma).$$

Hence $\tilde{\mathcal{V}}$ defines an $\mathcal{O}_Z$-linear map $\tilde{\mathcal{V}}_Z \colon \mathcal{E}_Z \to \mathcal{E}_Z$, where $\mathcal{E}_Z = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$.

Let us recall the Chern–Weil construction. Let

$$c(\mathcal{E}) \in H^1(X, \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \otimes \Omega^1)$$

denote the Atiyah–Chern class of $\mathcal{E}$, and let $p\colon \mathrm{Hom}(\mathcal{E}, \mathcal{E})^{\otimes l} \to \mathcal{O}_X$ be any $\mathcal{O}_X$-linear map. Then $p(c(\mathcal{E}))$ is a well-defined element of $H^l(X, \Omega^l)$. On the other hand, p also defines a map $p_Z\colon \mathrm{Hom}(\mathcal{E}_Z, \mathcal{E}_Z)^{\otimes l} \to \mathcal{O}_Z$. This means that $p_Z(\tilde{\mathcal{V}}_Z^{\otimes l})$ gives a well-defined global section of $\mathcal{O}_Z$, that is, $p_Z(\tilde{\mathcal{V}}_Z^{\otimes l}) \in A(Z)$. We have,

Theorem 2.2 ([2, Theorem 5.5]) *If p has degree l, then $p(\tilde{\mathcal{V}}_Z^{\otimes l}) \in F_l A(Z)$, and in the associated graded algebra,* i.e., *in $\mathrm{Gr}_l\, A(Z)$, $p(\tilde{\mathcal{V}}_Z^{\otimes l})$ corresponds to $p(c(\mathcal{E})) \in H^l(X, \Omega^l) = H^{2l}(X, \mathbb{C})$, where $c(\mathcal{E})$ denotes the Atiyah–Chern class of $\mathcal{E}$.*

Later, we will use the above theorem to identify a function in $A(Z)$ corresponding to the Chern class of a line bundle on a toric variety. As the vector field $\mathcal{V}$ we take the generating vector field of a 1-parameter subgroup, in general position, of the torus.

3 Preliminaries on the Cohomology of Toric Varieties

Let T be the algebraic torus $(\mathbb{C}^*)^d$. As usual, N denotes the lattice of 1-parameter subgroups of T, $N_{\mathbb{R}}$ the real vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, M the dual lattice of N which is the lattice of characters of T, and $M_{\mathbb{R}}$ the real vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$. A vector $n = (n_1, \dots, n_d) \in \mathbb{Z}^d \cong N$ corresponds to the 1-parameter subgroup $t^n = (t^{n_1}, \dots, t^{n_d})$. Similarly, a covector $m = (m_1, \dots, m_d) \in (\mathbb{Z}^d)^* \cong M$ corresponds to the character $x^m = x_1^{m_1} \cdots x_d^{m_d}$. We use $\langle\, ,\, \rangle \colon N \times M \to \mathbb{Z}$ for the natural pairing between N and M.

Let X be a d-dimensional smooth projective toric variety. Let $\Sigma \subset N_{\mathbb{R}}$ be the simplicial fan corresponding to X. We denote by $\Sigma(i)$ the set of all i-dimensional cones in Σ. For each $\rho \in \Sigma(1)$, let ξ_ρ be the primitive vector along ρ, *i.e.,* the smallest integral vector on ρ.

There is a one-to-one correspondence between the orbits of dimension i in X and the cones in $\Sigma(d-i)$. The fixed points of T correspond to the cones in $\Sigma(d)$. In a smooth toric variety all the orbit closures are smooth; the cohomology class dual to the closure of the orbit corresponding to $\rho \in \Sigma(1)$ is denoted by $D_\rho \in H^2(X, \mathbb{C})$. It is well known that the cohomology algebra of a toric variety is generated by the classes D_ρ. More precisely, we have,

Theorem 3.1 (see [5, p. 106]) *Let X be a smooth projective toric variety. Then*

$$H^*(X, \mathbb{C}) = \mathbb{Z}[D_\rho, \rho \in \Sigma(1)]/I,$$

where I is the ideal generated by all

(i) $D_{\rho_1} \cdots D_{\rho_k}$, $\forall \rho_1, \dots, \rho_k$ *not in a cone of* Σ, *and*
(ii) $\sum_{\rho \in \Sigma(1)} \langle \xi_\rho, u \rangle D_\rho$, $\forall u \in M$.

Now, let $\Delta \subset M_{\mathbb{R}}$ be a simple rational polytope normal to the fan Σ. The polytope Δ defines a diagonal representation $\pi \colon T \to GL(V)$ where $\dim_{\mathbb{C}}(V) =$ the number of lattice points in Δ. Fix a basis for V so that T acts by diagonal matrices. If the mutual differences of the lattice points in Δ generate M, then we get an embedding of X in $\mathbb{P}(V)$ as the closure of the orbit of $(1 : 1 : \cdots : 1)$. In the rest of the paper, we assume that the above condition holds for Δ.

The set of faces of dimension i in Δ is denoted by $\Delta(i)$. There is a one-to-one correspondence between the faces in $\Delta(i)$ and the cones in $\Sigma(d-i)$ which in turn correspond to the orbits of dimension i in X. Hence the fixed points of T on X correspond to the vertices of Δ.

The support function $l_\Delta \colon N_{\mathbb{R}} \to \mathbb{R}$ is defined by $l_\Delta(\xi) = \max_{x \in \Delta} \langle \xi, x \rangle$.

Let L_Δ be the line bundle on X obtained by restricting the dual of the universal subbundle on $\mathbb{P}(V)$ to X. We will need the following classical theorem which tells us how the first Chern class $c_1(L_\Delta)$ is represented as a linear combination of the classes D_ρ.

Theorem 3.2 *With notation as above we have*

$$c_1(L_\Delta) = \sum_{\rho \in \Sigma(1)} l_\Delta(\xi_\rho) D_\rho.$$

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4 The Filtration on $A(Z)$ in the Toric Case

As before, let X be a smooth projective toric variety with fan Σ and a lattice polytope Δ normal to the fan which gives rise to a representation $\pi\colon T \to GL(V)$ and a T-equivariant embedding of X in $\mathbb{P}(V)$, for a vector space V over $\mathbb{C}$. Let $\gamma \in N$ be a 1-parameter subgroup of T. We can choose γ so that the set of fixed points of γ is the same as the set of fixed points of T. We denote the set of fixed points by Z.

In this section, we construct a filtration $F_0 \subset F_1 \subset \cdots$ for $A(Z)$ such that $H^*(X,\mathbb{C}) \cong \operatorname{Gr} A(Z)$.

Notation In the following, z denotes a fixed point, σ_z the corresponding d-dimensional cone in Σ and v_z the corresponding vertex in Δ. A 1-dimensional cone in Σ is denoted by ρ and the corresponding facet of Δ by F_ρ.

From Theorem 2.1 applied to the generating vector field of γ, there exists a filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$, the ring of $\mathbb{C}$ valued functions on Z, so that $H^*(X,\mathbb{C}) \cong \bigoplus_{i=0}^{\infty} F_{i+1}/F_i$, as graded algebras. In particular, we have $H^2(X,\mathbb{C}) \cong F_1/F_0$. The subspace $F_0A(Z)$ is just the set of constant functions. To determine the image of $H^2(X,\mathbb{C})$ in $\operatorname{Gr} A(Z)$ we need to determine F_1. We start by finding the representatives in F_1 for the Chern classes of the line bundles.

The 1-parameter subgroup $\gamma\colon \mathbb{C}^* \to T$ acts on V via π and hence the action of γ on X lifts to an action of γ on the line bundle L_Δ. Thus the generating vector field of γ has a lift to L_Δ. If we view L_Δ as $\{(x,l) \in X \times V \mid x = [l]\}$ then the action of γ on L_Δ is given by:

$$\gamma(t)\cdot(x,l) = (\pi(t^\gamma)x, \pi(t^\gamma)l).$$

Now, from Theorem 2.2 we have,

Proposition 4.1 *Under the isomorphism $F_1/F_0 \cong H^2(X,\mathbb{C})$, the first Chern class $c_1(L_\Delta)$ is represented by the function f_Δ defined by*

$$f_\Delta(z) = \langle \gamma, v_z\rangle, \quad \forall z \in Z,$$

where v_z is the vertex of Δ corresponding to the fixed point z.

Proof In Theorem 2.2, take E to be L_Δ and p be the identity polynomial. The derivation $\tilde{\nabla}$ is just the derivation given by the $\mathbb{G}_m$-action of γ on L_Δ. Let z be a fixed point and $(z,l) \in (L_\Delta)_z$ a point in the fiber of z. We have:

$$\begin{aligned}\gamma(t)\cdot(z,l) &= (z, \pi(t^\gamma)l),\\ &= (z, \langle\gamma, v_z\rangle l).\end{aligned}$$

and hence $f_\Delta(z) = \langle \gamma, v_z\rangle$. ■

Next, we wish to determine the images of the classes D_ρ, $\rho \in \Sigma(1)$, in F_1/F_0. Fix a 1-dimensional cone ρ in $\Sigma(1)$. Let F_ρ be the facet of Δ orthogonal to ρ.

Let us assume that we can move the facet F_ρ of Δ parallel to itself to obtain a new integral polytope Δ' (Figure 1). The polytope Δ' is still normal to the fan Σ. If

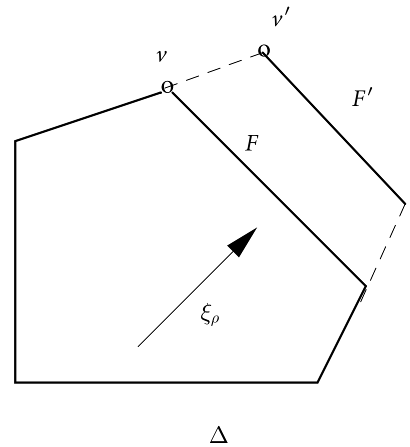

Figure 1: Moving a facet F_ρ

the facet can not be moved, we can replace Δ with $k\Delta$, for a big enough integer k, to make this moving of a facet possible. Replacing Δ with $k\Delta$ does not affect the formula we are going to obtain for D_ρ. Let F'_ρ denote the facet of Δ' obtained by moving F_ρ. The maximum of the function $\langle \xi_\rho, \cdot \rangle$ on Δ and Δ' is obtained on the facets F_ρ and F'_ρ respectively. For support functions of these polytopes we can write,

$$\begin{aligned} l_\Delta(\xi_\rho) &= \langle \xi_\rho, \text{ some point in } F_\rho \rangle, \\ l_{\Delta'}(\xi_\rho) &= \langle \xi_\rho, \text{ some point in } {F'}_\rho \rangle \\ l_\Delta(\xi_{\rho'}) &= l_{\Delta'}(\xi_{\rho'}), \quad \forall \rho' \neq \rho. \end{aligned}$$

We also have:

$$c_1(L_\Delta) = l_\Delta(\xi_\rho) D_\rho + \sum_{\substack{\rho' \in \Sigma(1) \\ \rho' \neq \rho}} l_\Delta(\xi_{\rho'}) D_{\rho'},$$

$$c_1(L_{\Delta'}) = l_{\Delta'}(\xi_\rho) D_\rho + \sum_{\substack{\rho' \in \Sigma(1) \\ \rho' \neq \rho}} l_{\Delta'}(\xi_{\rho'}) D_{\rho'}.$$

Hence

$$c_1(L_\Delta) - c_1(L_{\Delta'}) = (l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)) D_\rho.$$

So

$$D_\rho = \frac{c_1(L_\Delta) - c_1(L_{\Delta'})}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}.$$

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Now, let z be a torus fixed point, σ_z the corresponding d-dimensional cone, and v_z and v'_z the corresponding vertices in Δ and Δ' respectively. From Proposition 4.1, D_ρ corresponds to the function $f_\rho \in F_1A(Z)$ given by,

$$f_\rho(z) = \frac{f_\Delta(z) - f_{\Delta'}(z)}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)} = \frac{\langle \gamma, v_z - v'_z \rangle}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}.$$

If $v_z \notin F_\rho$ then $v_z = v'_z$ and hence $f_\rho(z) = 0$. If $v_z \in F_\rho$ then $l_\Delta(\xi_\rho) = \langle \xi_\rho, v_z \rangle$ and $l_{\Delta'}(\xi_\rho) = \langle \xi_\rho, v'_z \rangle$. We obtain that

$$f_\rho(z) = \begin{cases} \dfrac{\langle \gamma, v_z - v'_z \rangle}{\langle \xi_\rho, v_z - v'_z \rangle} & \text{if } v_z \in F_\rho, \\ 0 & \text{if } v_z \notin F_\rho. \end{cases}$$

Since Δ is a simple polytope, there are d edges at the vertex v_z. If $v_z \in F_\rho$, then there is only one edge e at v_z which does not belong to F_ρ. The vector $v_z - v'_z$, in fact, is along this edge. Note that the above formula for $f_\rho(z)$ does not depend on the length of the vector $v_z - v'_z$ (*i.e.*, how much we move the facet F_ρ to obtain the new polytope Δ'). Let $u_{\sigma_z,\rho}$ be the vector along the edge e normalized such that $\langle u_{\sigma_z,\rho}, \xi_\rho \rangle = 1$. Then we have,

Proposition 4.2 *With notation as above, the cohomology class D_ρ is represented by the function f_ρ in $F_1A(Z)$ defined by*

$$f_\rho(z) = \begin{cases} \langle \gamma, u_{\sigma_z,\rho} \rangle & \text{if } v_z \in F_\rho, \\ 0 & \text{if } v_z \notin F_\rho. \end{cases}$$

Since $H^2(X, \mathbb{C})$ is spanned by the classes $D_\rho, \rho \in \Sigma(1)$ and $H^*(X, \mathbb{C})$ is generated in degree 2, from Theorem 2.2 we obtain,

Theorem 4.3 *$F_1A(Z)/F_0A(Z) = \mathrm{Span}_{\mathbb{C}}\langle f_\rho, \rho \in \Sigma(1) \rangle$. Moreover, $F_iA(Z) =$ all polynomials of degree $\leq i$ in the f_ρ.*

One can prove directly that the functions $f_\rho, \rho \in \Sigma(1)$, satisfy the relations in the statement of Theorem 3.1. More precisely,

Theorem 4.4 *The functions $f_\rho, \rho \in \Sigma(1)$, satisfy the following relations:*

(i) $f_{\rho_1} \cdots f_{\rho_k} = 0$, $\forall \rho_1, \dots, \rho_k$ *not in a cone of* Σ, *and*
(ii) $\sum_{\rho \in \Sigma(1)} \langle \xi_\rho, u \rangle f_\rho =$ *some constant function on* Z, $\forall u \in M$.

Proof (i) is straightforward because every f_ρ is non-zero only at z such that the corresponding vertex lies in the facet F_ρ corresponding to ρ. Now, if $\rho_1, \dots, \rho_k$ are not in a cone of Σ, it means that the intersection of the corresponding facets F_{ρ_i} is empty, *i.e.*, the product of the f_{ρ_i} is zero.

For (ii), let z be a torus fixed point and, σ_z and v_z the corresponding d-dimensional cone and vertex, respectively. Let A be the $d \times d$ matrix whose rows are vectors ξ_ρ and let B be the $d \times d$ matrix whose columns are vectors $u_{\sigma_z,\rho}$, where ρ is an edge of σ_z. Since the cone at the vertex v_z, which is generated by the vectors $u_{\sigma_z,\rho}$, is dual to the cone σ_z, we get $AB = \mathrm{id}$. Now, we have

$$\begin{aligned}\sum_{\rho\in\Sigma(1)} \langle \xi_\rho, u\rangle f_\rho(z) &= \sum_{\rho \text{ an edge of } \sigma_z} \langle \xi_\rho, u\rangle f_\rho(z)\\ &= \sum_{\rho \text{ an edge of } \sigma_z} \langle \xi_\rho, u\rangle \langle \gamma, u_{\sigma_z,\rho}\rangle\\ &= (A\cdot u)^t \cdot (\gamma \cdot B)^t\\ &= \langle \gamma, u\rangle,\end{aligned}$$

where $\cdot$ means product of matrices and γ is regarded as a row vector and u is regarded as a column vector. So we proved that the expression (ii) is equal to $\langle \gamma, u\rangle$ which is independent of z and hence is a constant function on Z. ■

Remark 4.5 One can introduce a finite subset $\mathbb{Z}$ of the affine space $\mathbb{A}^{\Sigma(1)}$ such that $\mathbb{Z}$ is isomorphic to Z, and the natural grading on the coordinate ring $A(\mathbb{Z})$ induced from the grading on $\mathbb{A}^{\Sigma(1)}$ coincides with the above filtration $F_\bullet$ given by the f_ρ. Define the function $\Theta\colon Z \to \mathbb{R}^{\Sigma(1)} \subset \mathbb{C}^{\Sigma(1)}$ by

$$\Theta(z)_\rho = f_\rho(z),$$

and let $\mathbb{Z} = \Theta(Z)$.

Proposition 4.6 *With the grading on* $A(\mathbb{Z})$ *as above,* $\mathrm{Gr}\, A(\mathbb{Z}) \cong H^*(X,\mathbb{C})$ *as graded algebras.*

Proof Immediate. ■

Remark 4.7 (Lefschetz operator) A lattice polytope Δ normal to the fan of X gives rise to an embedding of X in a projective space. The Lefschetz operator in $H^*(X,\mathbb{C}) \cong \mathrm{Gr}\, A(Z)$ corresponding to this embedding is given by the multiplication by the function f_Δ (see Proposition 4.1).

Remark 4.8 Let $\mu\colon X \to M_{\mathbb{R}}$ be the moment map of the toric variety and, as before, $\gamma \in N$ a 1-parameter subgroup in general position. In [6] Khovanskii shows that the composition of γ and μ defines a Morse function on X whose critical points are the fixed points of X. The Morse index of a fixed point corresponding to a vertex v_z is twice the number of edges at v_z on which the linear function γ is decreasing. Returning to the definition of the functions f_ρ (Proposition 4.2), the linear function γ

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is decreasing on the edge e at v_z if and only if $f_\rho(z) < 0$. That is, the Morse index of a fixed point z is equal to twice the number of negative coordinates of the point $\Theta(z) \in \mathbb{R}^{\Sigma(1)}$. Since the number of critical points of index $2i$ is the $2i$-th Betti number of X, we conclude the non-trivial relation that the number of points in $\mathcal{Z}$ exactly i of their coordinates are negative is equal to $\dim \operatorname{Gr}_i A(\mathcal{Z})$.

5 Relation with the Polytope Algebra

Consider the abelian group generated by all the convex polytopes in a vector space subject to the relation

$$[P \cup Q] + [P \cap Q] - [P] - [Q] = 0,$$

whenever P, Q and $P \cup Q$ are convex polytopes. This group can be equipped with a ring structure, product of two polytopes being their Minkowski sum. The ring we obtain is McMullen's *polytope algebra* (see [7, p. 86]). The polytope algebra plays an important role in the study of finitely additive measures on the convex polytopes. To each simplicial polytope Δ, one can associate a subalgebra of the polytope algebra generated by all the polytopes whose facets are parallel to the facets of Δ. It is called the *polytope algebra* of Δ. For an integrally simple polytope Δ, its polytope algebra coincides with the cohomology algebra of the corresponding toric variety X. There is a description of the polytope algebra of Δ as a quotient of the algebra of differential operators (see [9], and for more details [11]).

In [1], Brion gives a description of the polytope algebra of a polytope as a quotient of the algebra of *continuous conewise polynomial functions*. Let $\Sigma \subset N_\mathbb{R}$ be the normal fan of the polytope Δ. Let R be the algebra of all continuous functions on $N_\mathbb{R}$ which restricted to each cone of Σ are given by a polynomial. Let I be the ideal of R generated by all the linear functions on $N_\mathbb{R}$. Then the polytope algebra of Δ is isomorphic to R/I.

There is a good set of generators for R parameterized by the set of 1-dimensional cones $\Sigma(1)$. For each $\rho \in \Sigma(1)$, define $g_\rho \colon N_\mathbb{R} \to \mathbb{R}$ as a conewise linear function, supported on the cones containing ρ, as follows:

(i) $g_\rho = 0$ on any cone not containing ρ, and

(ii) for a d-dimensional cone σ containing ρ, the function g_ρ restricted to σ is the unique linear function defined by $g_\rho(x) = 0$ for $x \in \rho' \neq \rho, \rho' \in \Sigma(1)$ and $g_\rho(\xi_\rho) = 1$.

One can show that the g_ρ are a set of generators for R. Moreover, by sending g_ρ to D_ρ, we get an isomorphism between R/I and $H^*(X, \mathbb{C})$. In particular, the images of the g_ρ in R/I satisfy the relations in Theorem 3.1.

In what follows, we show how this description of the cohomology is related to the $\operatorname{Gr} A(Z)$ description. Let γ be a 1-parameter subgroup in general position. Take $p \in R$. For a cone σ of maximum dimension, let p_σ denote the restriction of p to σ. Then p_σ is a polynomial on σ. Since the vector space spanned by σ is all of $N_\mathbb{R}$, the function p_σ can be considered as a polynomial on all of $N_\mathbb{R}$. Thus it makes sense to evaluate p_σ at γ. We define a homomorphism $\Phi \colon R \to A(Z)$ by $\Phi(p) = f$ where the function f is defined by

$$f(z) = p_{\sigma_z}(\gamma) \quad z \in Z,$$

here σ_z is the cone corresponding to the fixed point z.

Proposition 5.1 *We can write $R = \bigoplus_{k=0}^{\infty} R_k$ where R_k is the subspace of all conewise polynomial functions on $N_{\mathbb{R}}$ whose restriction to each cone is a homogeneous polynomial of degree k.*

Proof Let $p \in R$ and let p_k denote the function on $N_{\mathbb{R}}$ whose restriction to each cone σ is the degree k part of p_σ. We need to prove that p_k belongs to R, that is, p_k is continuous. Let σ and τ be two adjacent cones and let $v \in \sigma \cap \tau$. Since p is continuous, we have

$$(*) \qquad p_\sigma(tv) = p_\tau(tv), \quad \forall t \in \mathbb{R}.$$

Let $p_{\sigma,k}(v)$ (respectively $p_{\tau,k}(v)$) denote the coefficient of t^k in $p_\sigma(tv)$ (respectively $p_\tau(tv)$). From $(*)$, we have

$$p_{\sigma,k}(v) = p_{\tau,k}(v).$$

But $p_{\sigma,k} = p_{k|\sigma}$ and $p_{\tau,k} = p_{k|\tau}$. Thus $p_{k|\sigma}$ and $p_{k|\tau}$ agree at the intersection of σ and τ and hence p_k is continuous, that is, $p_k \in R$. ∎

Let $F_\bullet = F_0 \subset F_1 \subset \cdots$ be the filtration in the Carrell–Lieberman theorem (Theorem 2.1) where the vector field $\mathcal{V}$ is the generating vector field of a 1-parameter subgroup γ in general position. The following theorem gives an explicit connection between the grading in R by the R_k and the Carrell–Lieberman filtration $F_\bullet$.

Theorem 5.2

(i) $\Phi(g_\rho) = f_\rho$;

(ii) $\Phi(\oplus_{i=0}^{k} R_i) = F_k$.

Proof (i) Let $\rho \in \Sigma(1)$ and let σ be a d-dimensional cone containing ρ. Since σ is simplicial, the set $\{\xi_{\rho'} \mid \rho' \in \Sigma(1), \rho' \subset \sigma\}$ form a basis for $N_{\mathbb{R}}$. Consider the linear function l defined by $l(\xi_\rho) = 1$ and $l(\xi_{\rho'}) = 0, \rho' \subset \sigma$ and $\rho' \neq \rho$. Let A be the $d \times d$ matrix whose rows are vectors ξ_ρ and B be the $d \times d$ matrix whose columns are vectors $u_{\sigma,\rho'}$, where ρ' is an edge of σ. Let v be the vertex of Δ corresponding to σ. The cone at v is dual to σ and hence we have $AB = \mathrm{id}$. View γ as a row vector. Then γ in the basis $\xi_{\rho'}, \rho' \subset \sigma$ is $\gamma A^{-1} = \gamma B$. Thus, one sees that $l(\gamma)$ is equal to the ρ-th component of γB. But this is the same as $f_\rho(z)$.

(ii) Each $p \in \bigoplus_{i=0}^{k} R_i$ can be written as a polynomial of degree $\leq k$ in the g_ρ. Also, each $f \in F_k$ can be written as a polynomial of degree $\leq k$ in the f_ρ. Now, (ii) follows from (i). ∎

Now, let us define an algebra homomorphism $\Psi\colon R \to \operatorname{Gr} A(Z) = \bigoplus_{i=1}^{\infty} F_i/F_{i-1}$ as follows. For $p \in R_k$, let $\Psi(p) = \Phi(p) \in F_k/F_{k-1}$, and extend the definition of Ψ to all of R by linearity.

Theorem 5.3 *Ψ induces an isomorphism between R/I and* $\mathrm{Gr}\,A(Z)$.

Proof It follows from the definition that Ψ is an algebra homomorphism. Since R is generated by the g_ρ and $\mathrm{Gr}\,A(Z)$ is generated by the images of the f_ρ, from Theorem 5.2(i) it follows that Ψ is surjective. To prove the theorem we need to show that $\ker(\Psi) = I$. Let l be a (global) linear function on $N_{\mathbb{R}}$. Then $\Phi(l) = f$ is a constant function on Z defined by $f(z) = l(\gamma), \forall z \in Z$. Thus $\Phi(l)$ belongs to F_0, that is $\Psi(l)$, as an element of F_1/F_0, is zero. Thus, l belongs to $\ker(\Psi)$. Since I is the ideal of R generated by the (global) linear functions, we see that $I \subset \ker(\Psi)$. But both of R/I and $\mathrm{Gr}\,A(Z)$ are isomorphic, as graded algebras, to $H^*(X, \mathbb{C})$. Hence the dimensions of the graded pieces of R/I and $\mathrm{Gr}\,A(Z)$ are the same. Since $I \subset \ker(\Psi)$, by comparing dimensions, we conclude that $I = \ker(\Psi)$ and the theorem is proved. ■

6 Examples

In this section we consider two examples in dimension 2, namely, $\mathbb{C}P^2$ and the Hirzebruch surface $\mathbb{F}_a$. For each example, we compute the functions f_ρ and the finite affine set $\mathcal{Z}$.

Example 6.1 ($\mathbb{C}P^2$) The fan of $\mathbb{C}P^2$, and a polytope normal to it, is shown in Figure 2. There are three 1-dimensional cones denoted by ρ_1, ρ_2 and ρ_3 along the primitive vectors $\xi_1 = (1, 0), \xi_2 = (0, 1)$ and $\xi_3 = (-1, -1)$. There are three 2-dimensional cones σ_1, σ_2 and σ_3 corresponding to the three fixed points z_1, z_2, z_3. To each cone σ_i, there corresponds a vertex of the normal polytope and two vectors $u_{\sigma_i,\rho}$ at this vertex along the edges. For σ_1, these vectors are $\{(1, 0), (0, 1)\}$, for σ_2 they are $\{(-1, 0), (-1, 1)\}$ and finally, for σ_3 they are $\{(0, -1), (1, -1)\}$.

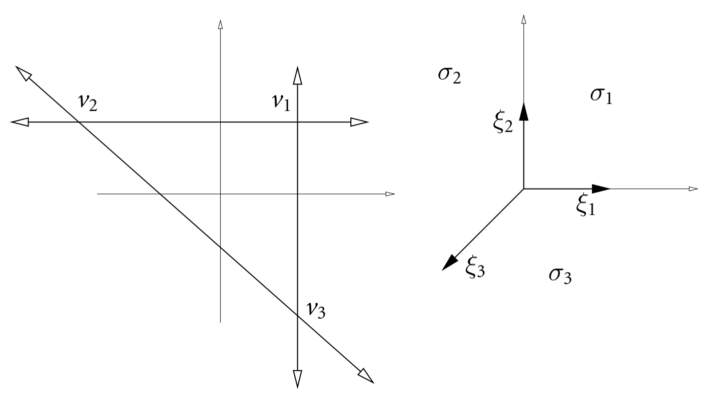

Figure 2: Fan of $\mathbb{C}P^2$ (right) and a polytope normal to the fan together with the vectors $u_{\sigma_i,\rho}$ (left).

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Let $\gamma = (\gamma_1, \gamma_2)$ be a 1-parameter subgroup. From the definition of the functions f_ρ (Proposition 4.2), we get the following table for their values:

	z_1	z_2	z_3
f_1	γ_2	$\gamma_2 - \gamma_1$	0
f_2	γ_1	0	$\gamma_1 - \gamma_2$
f_3	0	$-\gamma_1$	$-\gamma_2$

and hence, $\mathcal{Z} = \{(\gamma_2, \gamma_1, 0), (\gamma_2 - \gamma_1, 0, -\gamma_1), (0, \gamma_1 - \gamma_2, -\gamma_2)\} \subset \mathbb{C}^3$. Note that the points in $\mathcal{Z}$ lie on the same line parallel to $(1, 1, 1)$. One can see that $\mathrm{Gr}_i A(\mathcal{Z}) \cong \mathbb{C}, 0 \le i \le 2$ and $\mathrm{Gr}_i A(\mathcal{Z}) = \{0\}, i > 2$. If x is a non-zero element of $\mathrm{Gr}_1 A(\mathcal{Z})$ then, $H^*(\mathbb{C}P^2, \mathbb{C}) \cong \mathrm{Gr}\, A(\mathcal{Z}) \cong \mathbb{C}[x]/\langle x^3 \rangle$.

The above calculation can be carried out in general for $\mathbb{C}P^n$. One can show that all the points in the set $\mathcal{Z}$ lie on the same line parallel to $(1, \dots, 1)$, and $\mathrm{Gr}_i \cong \mathbb{C}$ for $0 \le i \le n$ and $\mathrm{Gr}_i \cong 0$ for $i > n$ and thus $H^*(\mathbb{C}P^n, \mathbb{C}) \cong \mathrm{Gr}\, A(\mathcal{Z}) \cong \mathbb{C}[x]/\langle x^{n+1} \rangle$. In fact, the associate graded algebra of the coordinate ring of any set of $n + 1$ points lying on the same line (in the affine space) gives the cohomology algebra of $\mathbb{C}P^n$.

Example 6.2 (Hirzebruch surface) For each $a \in \mathbb{N} \cup \{0\}$, one can construct a toric surface $\mathbb{F}_a$, called a *Hirzebruch surface* whose fan, and a normal polytope to it, is shown in Figure 3. There are four 1-dimensional cones denoted by ρ_1, ρ_2, ρ_3 and ρ_4 along the primitive vectors $\xi_1 = (1, 0), \xi_2 = (0, 1), \xi_3 = (-1, a)$ and $\xi_4 = (0, -1)$. There are four 2-dimensional cones denoted by σ_1 to σ_4. They correspond to the four fixed points z_1, z_2, z_3 and z_4. To each σ_i there corresponds a vertex of the normal polytope to the fan and two vectors $u_{\sigma_i,\rho}$ along the edges. For σ_1 these vectors are $\{(1, 0), (0, 1)\}$, for σ_2 they are $\{(-1, 0), (a, 1)\}$, for σ_3 they are $\{(-1, 0), (-a, -1)\}$ and finally, for σ_4 they are $\{(1, 0), (0, -1)\}$.

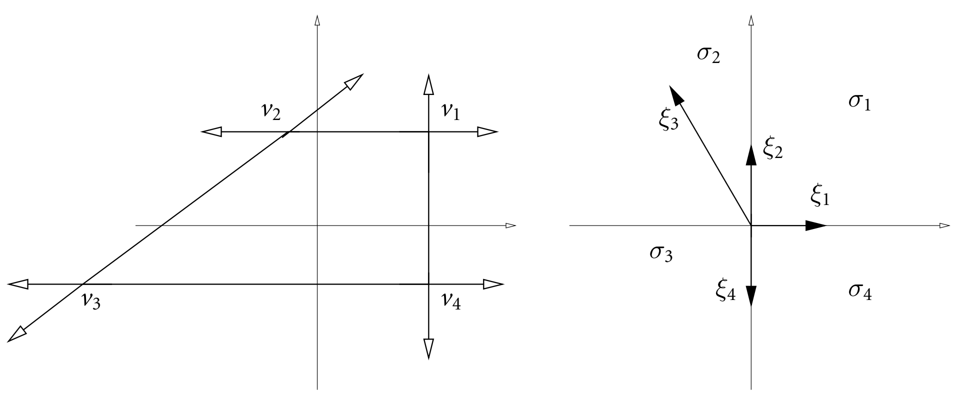

Figure 3: Fan of $\mathbb{F}_a$ (right) and a polytope normal to the fan together with the vectors $u_{\sigma_i,\rho}$ (left).

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Let $\gamma = (\gamma_1, \gamma_2)$ be a 1-parameter subgroup. We get the following table for the values of the f_ρ:

	z_1	z_2	z_3	z_4
f_1	γ_1	0	0	γ_1
f_2	γ_2	$a\gamma_1 + \gamma_2$	0	0
f_3	0	$-\gamma_1$	$-\gamma_1$	0
f_4	0	0	$-a\gamma_1 - \gamma_2$	$-\gamma_2$

and hence,

$$\mathcal{Z} = \{(\gamma_1, \gamma_2, 0, 0), (0, a\gamma_1 + \gamma_2, -\gamma_1, 0), (0, 0, -\gamma_1, -a\gamma_1 - \gamma_2), (\gamma_1, 0, 0, -\gamma_2)\} \subset \mathbb{C}^4.$$

Note that the points in $\mathcal{Z}$ lie on the same 2-plane defined by $f_1 - f_3 = \gamma_1$ and $af_1 + f_2 - f_4 = a\gamma_1 + \gamma_2$. Also, no three of them are collinear. Thus, one can see that $\mathrm{Gr}_0 \cong \mathbb{C}, \mathrm{Gr}_1 \cong \mathbb{C}^2, \mathrm{Gr}_2 \cong \mathbb{C}$ and $\mathrm{Gr}_i = \{0\}, i > 2$. One can see that there are two polynomials l_1 and l_2 of degree 1 on $\mathbb{C}^4$ such that they form a basis for $\mathrm{Gr}_1 A(\mathcal{Z})$ and, ${l_1}^2 = {l_2}^2 = 0$ in $\mathrm{Gr}_2 A(\mathcal{Z})$. Hence $H^*(\mathbb{F}_a) \cong \mathrm{Gr}\, A(\mathcal{Z}) \cong \mathbb{C}[l_1, l_2]/\langle {l_1}^2, {l_2}^2\rangle$. In fact, any set of four points lying on the same 2-plane such that no three are collinear can give the cohomology of $\mathbb{F}_a$.

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