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HYDRODYNAMIC LIMIT FOR A TYPE OF EXCLUSION PROCESS WITH SLOW BONDS IN DIMENSION $d \geq 2$

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Abstract

Let Λ be a connected closed region with smooth boundary contained in the d-dimensional continuous torus $\mathbb{T}^d$. In the discrete torus $N^{-1}\mathbb{T}_N^d$, we consider a nearest-neighbor symmetric exclusion process where occupancies of neighboring sites are exchanged at rates depending on Λ in the following way: if both sites are in Λ or Λ^c, the exchange rate is 1; if one site is in Λ and the other site is in Λ^c, and the direction of the bond connecting the sites is e_j, then the exchange rate is defined as N^{-1} times the absolute value of the inner product between e_j and the normal exterior vector to $\partial\Lambda$. We show that this exclusion-type process has a nontrivial hydrodynamical behavior under diffusive scaling and, in the continuum limit, particles are not blocked or reflected by $\partial\Lambda$. Thus, the model represents a system of particles under hard-core interaction in the presence of a permeable membrane which slows down the passage of particles between two complementary regions.

Keywords: Hydrodynamic limit; exclusion process; nonhomogeneous media

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1. Introduction

The exclusion process is a continuous-time interacting particle system where particles move as independent random walks on a graph except for the exclusion rule that prevents two particles from occupying the same site, or vertex. In the symmetric case, the process evolves as follows: to each bond we associate a waiting exponential time, which is independent of the waiting time for any other bond; at the waiting time the occupancies of the sites connected by the bond are exchanged; the parameter of the exchange times, or exchange rate, depends only on the bond. The specification of the exchange rates determines the environment for the exclusion process. In our case, as the underlying graph, we consider the discrete torus with N^d points and nearest-neighbor bonds. The variable N is the scaling parameter.

In this paper we study the hydrodynamical behavior of symmetric exclusion processes in nonhomogeneous environments, where the nonhomogeneity is due to the presence of slow bonds. While a usual bond has exchange rate 1, a slow bond has a lower exchange rate. With respect to the scaling parameter, we assume that a slow bond has an exchange rate of the order N^{-1}.

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When the environment is homogeneous, the exclusion process has a well-known hydrodynamical behavior under diffusive scaling. Results in nonhomogeneous environments have been obtained in several cases, even when the environment is random and consists of only slow bonds. For one-dimensional processes, in [5], the exchange rate over a bond $[x/N, (x+1)/N]$ is given by $[N(W(x+1/N) - W(x/N))]^{-1}$, where W is an α-stable subordinator of a Lévy process. Faggionato *et al.* [5] obtained a quenched hydrodynamic limit. In papers previous to [5], for example [3] and [9], the randomness or nonhomogeneity did not survive in the continuum limit. Another one-dimensional result, following [5], was obtained in [6], for more general, but nonrandom, increasing functions W. The techniques used in those papers were strongly based on theorems about the convergence of one-dimensional continuous-time stochastic processes. In fact, even the d-dimensional case treated in [10] considered a class of nonhomogeneous environments that could be decomposed, in a proper sense, into d one-dimensional cases. Recently, different approaches have been examined to deal with d-dimensional environments; see [4] and [7].

We now describe the exclusion processes that we are concerned with. Let $\{e_j : j = 1, \dots, d\}$ be the canonical basis of $\mathbb{R}^d$ and $\Lambda \subset \mathbb{T}^d$ be a simple connected region with smooth boundary $\partial\Lambda$. If the bond $[x/N, (x+e_j)/N] \in N^{-1}\mathbb{T}_N^d$ has vertices in each of the regions Λ and Λ^{c}, its exchange rate is defined as N^{-1} times the absolute value of the inner product between e_j and the normal exterior vector to $\partial\Lambda$. For others edges, the exchange rate is defined as 1. This means that the slow bonds are among those crossing the boundary of Λ. We call this process the exclusion process with slow bonds over $\partial\Lambda$.

We can interpret $\partial\Lambda$ as a permeable membrane, which slows down the passage of particles between the regions Λ and Λ^{c}. For this type of exclusion process, the membrane does not completely prevent the passage of particles, and still survives in the continuum limit, appearing explicitly in the hydrodynamic equation. The exchange rate of particles for a bond crossing $\partial\Lambda$ is smaller if the bond is close to a tangent line of $\partial\Lambda$. Note that this assumption has a physical meaning; consider, for example, cases of reflections in several physical models: the partial reflection of light crossing a medium with different refractive indices, mechanical systems where particles try to cross some interface, etc. However, the direction of the velocity of particles is not changed as usually occurs in physical reflection. Our definition of the exchange rates also allows a strong convergence result for the empirical measures associated to the exclusion process, making the proof of the hydrodynamic limit simpler.

The hydrodynamical equation of the exclusion process with slow bonds over $\partial\Lambda$ is a parabolic partial differential equation $\partial_t\rho = \mathcal{L}_\Lambda\rho$, where the operator $\mathcal{L}_\Lambda$ is a sort of d-dimensional Krein–Feller operator. Without the presence of slow bonds, the operator $\mathcal{L}_\Lambda$ would be replaced by the Laplacian operator acting on C^2 functions and the hydrodynamical equation is therefore the heat equation. Here, the existence of the membrane modifies the domain, and, thus, the operator itself. In fact, we observe that the proper domain for $\mathcal{L}_\Lambda$ contains functions that are discontinuous over $\partial\Lambda$. Geometrically, $\mathcal{L}_\Lambda$ glues the discontinuity of a function around $\partial\Lambda$ and then behaves like the Laplacian.

One possible approach to prove the hydrodynamic limit for the exclusion process with slow bonds over $\partial\Lambda$ is through gamma convergence. In [7], this approach and the conditions for it to hold were discussed; see also [3]. There, the coersiveness condition would require some kind of Rellich–Kondrachov's theorem (namely, the compact embedding in L^2 of some sort of Sobolev space supporting an extension of $\mathcal{L}_\Lambda$; see [2, p. 272]). In the method presented here, we go in this direction, but instead of reaching the hypotheses of [7], we use similar analytical tools in order to obtain a short and simple proof of the uniqueness of the hydrodynamic equation.

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We also show that the extension of $\mathcal{L}_\Lambda$ satisfies the Hille–Yoshida theorem. On the other hand, the convergence from discrete to continuous that we present here is made in a very direct way, and it was inspired by the convergence of the discrete Laplacian to the continuous Laplacian.

The paper is presented as follows. In Section 2 we define the model and state all results contained in the paper; Section 3 is devoted to proving results concerning the continuous operator $\mathcal{L}_\Lambda$; in Section 4, the hydrodynamic limit is proved.

2. Notation and results

Let $\mathbb{T}^d$ be the d-dimensional torus, which is $[0, 1)^d$ with periodic boundary conditions, and let $\mathbb{T}_N^d$ be the discrete torus with N^d points, i.e. $\{0, \dots, N-1\}^d$ with periodic boundary conditions. We denote by $\eta = (\eta(x))_{x\in\mathbb{T}_N^d}$ a typical configuration in the state space $\Omega_N = \{0, 1\}^{\mathbb{T}_N^d}$, for which $\eta(x) = 0$ means that site x is vacant and $\eta(x) = 1$ means that site x is occupied. If a bond of $N^{-1}\mathbb{T}_N^d$ has vertices x/N and y/N, it will be denoted by $[x/N, y/N]$.

Recall that $\{e_j : j = 1, \dots, d\}$ is the canonical basis of $\mathbb{R}^d$. The symmetric nearest-neighbor exclusion process with exchange rates $\xi^N_{x,y} > 0$, $x, y \in \mathbb{T}_N^d$, $|x - y| = 1$, is a Markov process with configuration space Ω_N, whose generator L_N acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} \xi^N_{x,x+e_j}[f(\eta^{x,x+e_j}) - f(\eta)],$$

where $\eta^{x,x+e_j}$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x + e_j)$:

$$(\eta^{x,x+e_j})(y) = \begin{cases} \eta(x+e_j) & \text{if } y = x, \\ \eta(x) & \text{if } y = x + e_j, \\ \eta(y) & \text{otherwise.} \end{cases}$$

Let ν^N_α, $\alpha \in [0, 1]$, be the Bernoulli product measure Ω_N, i.e. the product measure whose marginals have Bernoulli distribution with parameter α. Then $\{\nu^N_\alpha : 0 \le \alpha \le 1\}$ is a family of invariant, in fact reversible, measures for any symmetric exclusion process.

Now, fix a simple connected region $\Lambda \subset \mathbb{T}^d$ with smooth boundary $\partial\Lambda$. Denote by $\vec{\zeta}(u)$ the normal unitary exterior vector to the smooth surface $\partial\Lambda$ in the point $u \in \partial\Lambda$. If $x/N \in \Lambda$ and $(x + e_j)/N \in \Lambda^{\mathrm{c}}$, or $x/N \in \Lambda^{\mathrm{c}}$ and $(x + e_j)/N \in \Lambda$, we define $\vec{\zeta}_{x,j}$ as a vector $\vec{\zeta}(u)$ evaluated at an arbitrary but fixed point $u \in \partial\Lambda \cap [x, x + e_j]$. The exclusion process with slow bonds over $\partial\Lambda$ is a symmetric nearest-neighbor exclusion process with exchange rates

$$\xi^N_{x,x+e_j} = \xi^N_{x+e_j,x} = \begin{cases} \dfrac{|\vec{\zeta}_{x,j} \cdot e_j|}{N} & \text{if } x/N \in \Lambda \text{ and } (x+e_j)/N \in \Lambda^{\mathrm{c}}, \\ & \text{or } x/N \in \Lambda^{\mathrm{c}} \text{ and } (x+e_j)/N \in \Lambda, \\ 1 & \text{otherwise,} \end{cases} \tag{2.1}$$

for $j = 1, \dots, d$ and every $x \in \mathbb{T}_N^d$. In this case, the exchange rate of a bond crossing the boundary $\partial\Lambda$ is also of order N^{-1}, but it depends on the angle of incidence: the crossing of $\partial\Lambda$ by a particle becomes less frequent as the direction of entrance gets closer to the tangent plane to the surface $\partial\Lambda$. For a picture illustrating ideas, see Figure 1.

From now on, the rates in the definition of L_N will always be given by (2.1). Denote by $\{\eta^N_t : t \ge 0\}$ a Markov process with state space Ω_N and generator L_N speeded up by N^2.

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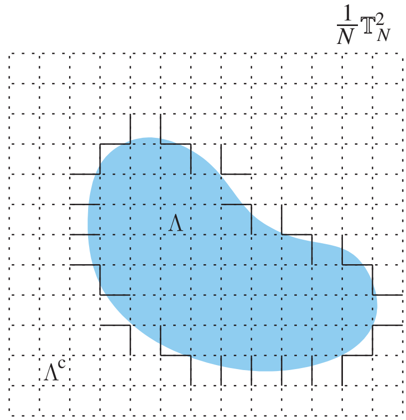

FIGURE 1: The darker region corresponds to Λ. The thick lines represent bonds with exchange rates $|\vec{\zeta}_{x,j} \cdot e_j|/N$; any other bond has exchange rate 1.

Let $D(\mathbb{R}_+, \Omega_N)$ be the Skorokhod space of càdlàg trajectories (those that are continuous from the right with left limits) taking values in Ω_N. For a measure μ on Ω_N, denote by P^N_μ the probability measure on $D(\mathbb{R}_+, \Omega_N)$ induced by the initial state μ and the Markov process $\{\eta^N_t : t \geq 0\}$. The expectation with respect to P^N_μ is going to be denoted by E^N_μ.

A sequence of probability measures $\{\mu_N : N \geq 1\}$ is said to be associated to a profile $\gamma : \mathbb{T}^d \to [0, 1]$ if μ_N is a probability measure on Ω_N for every N, and

$$\lim_{N\to\infty} \mu_N \left\{ \left| \frac{1}{N^d} \sum_{x\in\mathbb{T}^d_N} H\left(\frac{x}{N}\right)\eta(x) - \int H(u)\gamma(u)\,\mathrm{d}u \right| > \delta \right\} = 0 \tag{2.2}$$

for every $\delta > 0$, and every continuous function $H : \mathbb{T}^d \to \mathbb{R}$.

The exclusion process with slow bonds over $\partial\Lambda$ has a related random walk on $N^{-1}\mathbb{T}^d_N$ that describes the evolution of the system with a single particle. Thus, particles in the exclusion process evolve independently as such a random walk except for the hard-core interaction. To simplify notation later, we introduce here the generator of this random walk, which is given by

$$(\mathbb{L}_N H)\left(\frac{x}{N}\right) = \sum_{j=1}^{d} \left\{ \xi^N_{x,x+e_j} \left[H\left(\frac{x+e_j}{N}\right) - H\left(\frac{x}{N}\right) \right] + \xi^N_{x,x-e_j} \left[H\left(\frac{x-e_j}{N}\right) - H\left(\frac{x}{N}\right) \right] \right\} \tag{2.3}$$

for every $H : N^{-1}\mathbb{T}^d_N \to \mathbb{R}$ and every $x \in \mathbb{T}^d_N$. We will not differentiate between the notation for functions H defined on $\mathbb{T}^d$ and on $N^{-1}\mathbb{T}^d_N$.

2.1. The operator $\mathcal{L}_\Lambda$

Here we define the operator $\mathcal{L}_\Lambda$ and state its main properties. First, its domain is defined as a set of functions that are two times continuously differentiable inside and outside Λ, and satisfy some additional conditions related to their behavior at $\partial\Lambda$. Such conditions are imposed in order to have good properties of $\mathcal{L}_\Lambda$ that allows us to conclude the uniqueness of solutions of the hydrodynamic equation, and obtain a strong convergence result for the empirical measures in the proof of the hydrodynamic limit. The necessity of these conditions will be made clear later.

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Definition 2.1. Recall that $\vec{\zeta}$ denotes the normal exterior vector to the surface $\partial\Lambda$. The domain $\mathfrak{D}_\Lambda \subset L^2(\mathbb{T}^d)$ will be the set of functions $H \in L^2(\mathbb{T}^d)$ such that $H(u) = h(u) + \lambda \mathbf{1}_\Lambda(u)$, where

(a) $\lambda \in \mathbb{R}$;

(b) $h \in C^2(\mathbb{T}^d)$;

(c) $\nabla h|_{\partial\Lambda}(u) = -\lambda\vec{\zeta}(u)$.

Now, we define the operator $\mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ by $\mathcal{L}_\Lambda H = \Delta h$.

Geometrically, the operator $\mathcal{L}_\Lambda$ removes the discontinuity around the surface $\partial\Lambda$ and then acts like the Laplacian operator.

Remark 2.1. It is not entirely obvious why there exist functions $h \in C^2(\mathbb{T}^d)$ such that $\nabla h|_{\partial\Lambda}(u) = -\lambda\vec{\zeta}(u)$ for $\lambda \neq 0$. For an example of such a function, consider $g : \mathbb{T}^d \to \mathbb{R}$ defined by

$$g(u) = \begin{cases} \lambda \operatorname{dist}(u, \partial\Lambda) & \text{if } u \in \Lambda^{\mathrm{c}}, \\ -\lambda \operatorname{dist}(u, \partial\Lambda) & \text{if } u \in \Lambda. \end{cases}$$

Since $\partial\Lambda$ has no self intersection and is smooth, it is simple to check that there exists a sufficiently small $\varepsilon > 0$ such that

$$V = \{u \in \mathbb{T}^d : \operatorname{dist}(u, \partial\Lambda) < \varepsilon\}$$

has a smooth boundary and without self intersection. Thus, the function g is smooth in the open neighborhood V of $\partial\Lambda$, and satisfies the condition that $\nabla g|_{\partial\Lambda}(u) = -\lambda\vec{\zeta}(u)$. However, g is not differentiable in the space $\mathbb{T}^d$. To solve this problem, it is enough to multiply g by $\sum_i \Phi_i$, where $\{\Phi_i\}$ is a partition of unity such that the support of any Φ_i is contained in V and $\sum_i \Phi_i(u) = 1$ for all $u \in U \subset V$, where U is an open set containing $\partial\Lambda$. Finally, the function

$$h(u) = g(u) \sum_i \Phi_i(u)$$

satisfies the required conditions.

For the next result, we need to introduce some notation. We denote by $\mathbb{I}$ the identity operator in $L^2(\mathbb{T}^d)$, and by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\| \cdot \|$ its usual inner product and norm:

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathbb{T}^d} f(u) g(u)\, \mathrm{d}u \quad \text{and} \quad \|f\| = \sqrt{\langle\!\langle f, f \rangle\!\rangle}, \qquad f, g \in L^2(\mathbb{T}^d).$$

Theorem 2.1. *There exists a Hilbert space $(\mathcal{H}^1_\Lambda, \langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\Lambda})$ which is compactly embedded in $L^2(\mathbb{T}^d)$ such that $\mathfrak{D}_\Lambda \subset \mathcal{H}^1_\Lambda$ and $\mathcal{L}_\Lambda$ can be extended to $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ in such a way that the extension enjoys the following properties.*

(a) *The domain $\mathcal{H}^1_\Lambda$ is dense in $L^2(\mathbb{T}^d)$.*

(b) *The operator $\mathcal{L}_\Lambda$ is self-adjoint and nonpositive: $\langle\!\langle H, -\mathcal{L}_\Lambda H \rangle\!\rangle \geq 0$ for all H in $\mathcal{H}^1_\Lambda$.*

(c) *The operator $\mathbb{I} - \mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ is bijective and $\mathfrak{D}_\Lambda$ is a core for it.*

(d) *The operator $\mathcal{L}_\Lambda$ is dissipative, i.e.*

$$\|\mu H - \mathcal{L}_\Lambda H\| \geq \mu \|H\|$$

for all $H \in \mathcal{H}^1_\Lambda$ and $\mu > 0$.

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(e) *The eigenvalues of* $-\mathcal{L}_\Lambda$ *form a countable set* $0 = \mu_0 \le \mu_1 \le \cdots$ *with* $\lim_{n\to\infty} \mu_n = \infty$, *and all these eigenvalues have finite multiplicity.*

(f) *There exists a complete orthonormal basis of* $L^2(\mathbb{T}^d)$ *composed of eigenvectors of* $-\mathcal{L}_\Lambda$.

In view of (a), (c), and (d), by the Hille–Yoshida theorem, $\mathcal{L}_\Lambda$ *is the generator of a strongly continuous contraction semigroup in* $L^2(\mathbb{T}^d)$.

The space $\mathcal{H}^1_\Lambda$ will be defined in Section 3. The name has been chosen in analogy to the notation used for Sobolev spaces.

2.2. The hydrodynamic equation

Consider a bounded Borel measurable profile $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}$. A bounded function $\rho \colon \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation

$$\partial_t \rho = \mathcal{L}_\Lambda \rho, \qquad \rho(0, \cdot) = \rho_0(\cdot) \tag{2.4}$$

if, for all functions H in $\mathcal{H}^1_\Lambda$ and all $t > 0$, ρ satisfies the integral equation

$$\langle\langle \rho_t, H \rangle\rangle - \langle\langle \rho_0, H \rangle\rangle - \int_0^t \langle\langle \rho_s, \mathcal{L}_\Lambda H \rangle\rangle \, \mathrm{d}s = 0,$$

where we use the notation ρ_t for $\rho(t, \cdot)$. We prove in Subsection 4.3 the uniqueness of weak solutions of (2.4). Existence follows from the convergence result for the empirical measures associated to the diffusively rescaled exclusion processes with slow bonds over Λ; this is discussed in Section 4. Here we do not use time-dependent test functions as usual in the definition of the weak solution, but we have a well-posed problem and we do not need a solution in a stronger sense to prove the hydrodynamic limit which is the next stated theorem.

Theorem 2.2. *Fix a Borel measurable initial profile* $\gamma \colon \mathbb{T}^d \to [0, 1]$ *and consider a sequence of probability measures* μ_N *on* Ω_N *associated to* γ. *Then, for any* $t \ge 0$,

$$\lim_{N\to\infty} \mathrm{P}^N_{\mu_N} \Bigg[\Bigg| \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H\Big(\frac{x}{N}\Big) \eta_t(x) - \int_{\mathbb{T}^d} H(u) \rho(t, u) \, \mathrm{d}u \Bigg| > \delta \Bigg] = 0$$

for every $\delta > 0$ *and every function* $H \in C(\mathbb{T}^d)$, *where* ρ *is the unique weak solution of the differential equation (2.4) with* $\rho_0 = \gamma$.

3. The operator $\mathcal{L}_\Lambda$

We begin by studying properties of $\mathcal{L}_\Lambda$ defined on the domain $\mathfrak{D}_\Lambda$ and we consider the extension afterwards.

Lemma 3.1. *The domain* $\mathfrak{D}_\Lambda$ *is dense in* $L^2(\mathbb{T}^d)$.

Proof. It is enough to prove that there exists a subset of $\mathfrak{D}_\Lambda$ which is dense in $L^2(\mathbb{T}^d)$. All smooth functions with support contained in $\mathbb{T}^d \setminus \partial\Lambda$ belong to $\mathfrak{D}_\Lambda$, which is clearly a dense subset of $L^2(\mathbb{T}^d)$, since $\partial\Lambda$ is a smooth zero Lebesgue measure surface that divides $\mathbb{T}^d \setminus \partial\Lambda$ into two disjoint open regions.

From now on, we use ℓ_d to denote the d-dimensional Lebesgue measure on $\mathbb{T}^d$.

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Lemma 3.2. *The operator* $-\mathcal{L}_\Lambda \colon \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ *is symmetric and nonnegative. Furthermore, it satisfies a Poincaré inequality, which means that there exists a finite constant* $C > 0$ *such that*

$$\|H\|^2 \le C\langle\!\langle -\mathcal{L}_\Lambda H, H\rangle\!\rangle + \left(\int_{\mathbb{T}^d} H(x)\,\mathrm{d}x\right)^2$$

for all functions $H \in \mathfrak{D}_\Lambda$.

Proof. Let $H, G \in \mathfrak{D}_\Lambda$. Write $H = h + \lambda_h \mathbf{1}_\Lambda$ and $G = g + \lambda_g \mathbf{1}_\Lambda$, as in Definition 2.1. By the first Green identity and Definition 2.1(c), we have

$$\begin{aligned}\lambda_h \int_\Lambda \Delta g\,\mathrm{d}u &= \lambda_h \int_{\partial\Lambda} (\nabla g \cdot \vec{\zeta})\,\mathrm{d}S \\ &= -\lambda_h\lambda_g \mathrm{Vol}_{d-1}(\partial\Lambda) \\ &= \lambda_g \int_{\partial\Lambda} (\nabla h \cdot \vec{\zeta})\,\mathrm{d}S \\ &= \lambda_g \int_\Lambda \Delta h\,\mathrm{d}u, \end{aligned} \tag{3.1}$$

where $\mathrm{d}S$ is an infinitesimal volume element of $\partial\Lambda$ and $\mathrm{Vol}_{d-1}(\partial\Lambda)$ is its $(d-1)$-dimensional volume. Thus,

$$\begin{aligned}\langle\!\langle H, -\mathcal{L}_\Lambda G\rangle\!\rangle &= \langle\!\langle h + \lambda_h \mathbf{1}_\Lambda, -\Delta g\rangle\!\rangle \\ &= -\int_{\mathbb{T}^d} h\Delta g\,\mathrm{d}u - \lambda_h \int_\Lambda \Delta g\,\mathrm{d}u \\ &= -\int_{\mathbb{T}^d} g\Delta h\,\mathrm{d}u - \lambda_g \int_\Lambda \Delta h\,\mathrm{d}u \\ &= \langle\!\langle -\mathcal{L}_\Lambda H, G\rangle\!\rangle. \end{aligned}$$

For the nonnegativeness, using (3.1),

$$\langle\!\langle H, -\mathcal{L}_\Lambda H\rangle\!\rangle = -\int_{\mathbb{T}^d} h\Delta h\,\mathrm{d}u - \lambda_h \int_\Lambda \Delta h\,\mathrm{d}u = \int_{\mathbb{T}^d} |\nabla h|^2\,\mathrm{d}u + \lambda_h^2 \mathrm{Vol}_{d-1}(\partial\Lambda) \ge 0.$$

It remains to prove the Poincaré inequality. Write

$$\|H\|^2 - \left(\int_{\mathbb{T}^d} H(x)\,\mathrm{d}x\right)^2 = \int_{\mathbb{T}^d}\left[H(u) - \int_{\mathbb{T}^d} H(v)\,\mathrm{d}v\right]^2 \mathrm{d}u,$$

which can be rewritten as

$$\int_{\mathbb{T}^d}\left[\left(h(u) - \int_{\mathbb{T}^d} h(v)\,\mathrm{d}v\right) + \lambda_h(\mathbf{1}_\Lambda(u) - \ell_d(\Lambda))\right]^2 \mathrm{d}u.$$

Now apply the inequality $(a+b)^2 \le 2(a^2+b^2)$ to the previous expression to show that it is bounded by

$$2\int_{\mathbb{T}^d}\left(h(u) - \int_{\mathbb{T}^d} h(v)\,\mathrm{d}v\right)^2 \mathrm{d}u + 2\lambda_h^2(\ell_d(\Lambda) - (\ell_d(\Lambda))^2).$$

By the usual Poincaré inequality (see [2, p. 265]), the last expression is less than or equal to

$$2C_1 \int_{\mathbb{T}^d} |\nabla h(u)|^2\,\mathrm{d}u + 2\lambda_h^2(\ell_d(\Lambda) - (\ell_d(\Lambda))^2).$$

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Choosing a constant $C_2 > 0$ such that $\ell_d(\Lambda) - (\ell_d(\Lambda))^2 \le C_2 \mathrm{Vol}_{d-1}(\partial\Lambda)$, the previous expression is bounded above by

$$2\max\{C_1, C_2\}\langle\!\langle -\mathcal{L}_\Lambda H, H\rangle\!\rangle,$$

which completes the proof with $C = 2\max\{C_1, C_2\}$.

Denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$ the inner product on $\mathfrak{D}_\Lambda$ defined by

$$\langle\!\langle F, G\rangle\!\rangle_{1,\Lambda} = \langle\!\langle F, G\rangle\!\rangle + \langle\!\langle F, -\mathcal{L}_\Lambda G\rangle\!\rangle.$$

Let $\mathcal{H}^1_\Lambda$ be the set of all functions F in $L^2(\mathbb{T}^d)$ for which there exists a sequence $\{F_n : n \ge 1\}$ in $\mathfrak{D}_\Lambda$ such that F_n converges to F in $L^2(\mathbb{T}^d)$ and F_n is Cauchy for the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$. Such a sequence $\{F_n\}$ is called admissible for F. For F and G in $\mathcal{H}^1_\Lambda$, define

$$\langle\!\langle F, G\rangle\!\rangle_{1,\Lambda} = \lim_{n\to\infty}\langle\!\langle F_n, G_n\rangle\!\rangle_{1,\Lambda}, \tag{3.2}$$

where $\{F_n\}$ and $\{G_n\}$ are admissible sequences for F and G, respectively. By [11, Proposition 5.3.3], the limit exists and does not depend on the admissible sequence chosen. Moreover, $\mathcal{H}^1_\Lambda$ endowed with the scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$ just defined is a real Hilbert space. From now on, we consider $\mathcal{H}^1_\Lambda$ with the norm induced by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{1,\Lambda}$, unless we mention that we are going to use the L^2-norm.

Lemma 3.3. *The embedding $\mathcal{H}^1_\Lambda \subset L^2(\mathbb{T}^d)$ is compact.*

Proof. Let $\{H_n\}$ be a bounded sequence in $\mathcal{H}^1_\Lambda$. Fix $\{F_n\}$ as a sequence in $\mathfrak{D}_\Lambda$ such that $\|F_n - H_n\| \to 0$ and $\{F_n\}$ is also bounded in $\mathcal{H}^1_\Lambda$. Thus, to obtain a convergent subsequence of $\{H_n\}$, it is sufficient to find a convergent subsequence of $\{F_n\}$ in $L^2(\mathbb{T}^d)$. Write $F_n = f_n + \lambda_n \mathbf{1}_\Lambda$, with $f_n \in C^2(\mathbb{T}^d)$. Then,

$$\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda} = \langle\!\langle f_n + \lambda_n\mathbf{1}_\Lambda, f_n + \lambda_n\mathbf{1}_\Lambda\rangle\!\rangle + \langle\!\langle f_n + \lambda_n\mathbf{1}_\Lambda, -\Delta f_n\rangle\!\rangle.$$

Expanding the right-hand side and using (3.1), we obtain

$$\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda} = \|f_n\|^2 + \lambda_n^2\ell_d(\Lambda) + 2\lambda_n\int_\Lambda f_n(u)\,\mathrm{d}u + \|\nabla f_n\|^2 + \lambda_n^2\mathrm{Vol}_{d-1}(\partial\Lambda),$$

which is greater than or equal to

$$\begin{aligned}
&\|f_n\|^2 + \lambda_n^2\ell_d(\Lambda) - \lambda_n^2 - \ell_d(\Lambda)\int_\Lambda f_n^2(u)\,\mathrm{d}u + \|\nabla f_n\|^2 + \lambda_n^2\mathrm{Vol}_{d-1}(\partial\Lambda)\\
&\quad = (\ell_d(\Lambda) - 1 + \mathrm{Vol}_{d-1}(\partial\Lambda))\lambda_n^2 + (1-\ell_d(\Lambda))\int_\Lambda f_n^2(u)\,\mathrm{d}u\\
&\qquad + \int_{\Lambda^{\mathrm{c}}} f_n^2(u)\,\mathrm{d}u + \|\nabla f_n\|^2\\
&\quad \ge (\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda^{\mathrm{c}}))\lambda_n^2 + (1-\ell_d(\Lambda))\|f_n\|^2 + \|\nabla f_n\|^2.
\end{aligned}$$

If we set $\tilde f_n = f_n + \lambda_n$, and write $F_n = \tilde f_n - \lambda_n\mathbf{1}_{\Lambda^{\mathrm{c}}}$, an analogous computation shows that $\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda}$ is greater than or equal to

$$(\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda))\lambda_n^2 + (1-\ell_d(\Lambda^{\mathrm{c}}))\|\tilde f_n\|^2 + \|\nabla\tilde f_n\|^2.$$

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By the classical isoperimetric inequality on the torus (see [1, Lemma 4.6] for the statement and a direct proof), we have

$$\max\{\mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda^{\mathrm{c}}), \mathrm{Vol}_{d-1}(\partial\Lambda) - \ell_d(\Lambda)\} > 0.$$

Since $\{\langle\!\langle F_n, F_n\rangle\!\rangle_{1,\Lambda}\}$ is a bounded sequence, we conclude that $\{\lambda_n\}$ is bounded, as well as the sequence $\{\|f_n\|^2 + \|\nabla f_n\|^2\}$. By the Rellich–Kondrachov compactness theorem (see [2, Theorem 5.7.1]), $\{f_n\}$ has a convergent subsequence in $L^2(\mathbb{T}^d)$. From this subsequence, choosing a convergent subsequence of $\{\lambda_n\}$ completes the proof.

Lemma 3.4. *The image of* $\mathbb{I} - \mathcal{L}_\Lambda : \mathfrak{D}_\Lambda \to L^2(\mathbb{T}^d)$ *is dense in* $L^2(\mathbb{T}^d)$.

Proof. By a similar argument to that used in the proof of Lemma 3.1, it is enough to show that any smooth function f with support contained in $\mathbb{T}^d \setminus \partial\Lambda$ belongs to $(\mathbb{I} - \mathcal{L}_\Lambda)(\mathfrak{D}_\Lambda)$. Therefore, we need to find a function h in $C^2(\mathbb{T}^d)$ with support in $\mathbb{T}^d \setminus \partial\Lambda$ such that

$$h - \Delta h = f. \tag{3.3}$$

From the classical theory of second-order elliptic equations (see, e.g. [2, Theorem 5.7.1]), there exists $h \in C^2$ satisfying (3.3).

Proof of Theorem 2.1. (a) Since $\mathfrak{D}_\Lambda \subset \mathcal{H}^1_\Lambda$, it follows from Lemma 3.1 that $\mathcal{H}^1_\Lambda$ is dense in $L^2(\mathbb{T}^d)$.

(b) Define $\mathbb{I} - \mathcal{L}_\Lambda = \mathcal{A} : \mathfrak{D}_\Lambda \to \mathbb{L}^2(\mathbb{T}^d)$. From Lemma 3.2, $\mathcal{A}$ is linear, symmetric, and strongly monotone on the Hilbert space $L^2(\mathbb{T}^d)$. By strongly monotone we mean that there exists a $c > 0$ such that

$$\langle\!\langle \mathcal{A}H, H\rangle\!\rangle \geq c\|H\|^2 \quad \text{for all } H \in \mathfrak{D}_\Lambda.$$

In this case, $\mathcal{A}$ satisfies the inequality above with $c = 1$. By [11, Theorem 5.5a], in the conditions above, the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ is self-adjoint, bijective, and strongly monotone. With an abuse of notation, now define the extension $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ as $(\mathbb{I} - \mathcal{A})$. Since $\mathbb{I}$ and $\mathcal{A}$ are self-adjoint in $\mathcal{H}^1_\Lambda$, this property is inherited by $\mathcal{L}_\Lambda : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$.

For nonpositiveness, note that

$$\langle\!\langle -\mathcal{L}_\Lambda H, H\rangle\!\rangle = \langle\!\langle -(\mathbb{I} - \mathcal{A})H, H\rangle\!\rangle = -\langle\!\langle H, H\rangle\!\rangle + \langle\!\langle \mathcal{A}H, H\rangle\!\rangle \geq 0.$$

(c) As mentioned in the proof of part (b), the Friedrichs extension $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$ is bijective. So it remains to show that $\mathfrak{D}_\Lambda$ is a core of $\mathcal{A} : \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^2)$. For any operator B, denote by $\mathcal{G}(B)$ the graphic of B. Then $\mathfrak{D}_\Lambda$ is a core for $\mathcal{A}$, if the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})^{L^2\times L^2}$ in $L^2 \times L^2$ is equal to $\mathcal{G}(\mathcal{A})$. Since $\mathcal{A}$ is self-adjoint, $\mathcal{A}$ is a closed operator, otherwise, $\mathcal{G}(\mathcal{A})$ is a closed set. Thus, the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})$ is a subset of $\mathcal{G}(\mathcal{A})$. Let $H \in \mathcal{H}^1_\Lambda$. From Lemma 3.4, there exists a sequence $\{H_n\}$ in $\mathfrak{D}_\Lambda$ such that $\mathcal{A}H_n$ converges to $\mathcal{A}H$ in L^2. Hence, as proved in [11, Theorem 5.5.a], $\mathcal{A}^{-1}$ is a bounded linear operator, and H_n converges to H in L^2, from which it follows that the closure of $\mathcal{G}(\mathcal{A}|_{\mathfrak{D}_\Lambda})$ contains $\mathcal{G}(\mathcal{A})$.

(d) Fix a function H in $\mathcal{H}^1_\Lambda$ and $\mu > 0$. Set $G = (\mu\mathbb{I} - \mathcal{L}_\Lambda)H$. Taking the inner product with respect to H on both sides of this equality, we obtain

$$\mu\langle\!\langle H, H\rangle\!\rangle + \langle\!\langle -\mathcal{L}_\Lambda H, H\rangle\!\rangle = \langle\!\langle H, G\rangle\!\rangle \leq \langle\!\langle H, H\rangle\!\rangle^{1/2}\langle\!\langle G, G\rangle\!\rangle^{1/2}.$$

Since H belongs to $\mathcal{H}^1_\Lambda$, by (b), the second term on the left-hand side is positive. Therefore, $\mu\|H\| \leq \|G\| = \|(\mu\mathbb{I} - \mathcal{L}_\Lambda)H\|$.

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Now we will show that (e) and (f) hold. We have seen that the operator $(\mathbb{I} - \mathcal{L}_\Lambda)\colon \mathfrak{D}_\Lambda \to L^2(\mathbb{T})$ is symmetric and strongly monotone. By Lemma 3.3, the embedding $\mathcal{H}^1_\Lambda \subset L^2(\mathbb{T}^d)$ is compact. Therefore, by [11, Theorem 5.5.c], the Friedrichs extension $\mathcal{A}\colon \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$ satisfies claims (e) and (f) with $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_n \uparrow \infty$. In particular, the operator $-\mathcal{L}_\Lambda = (\mathcal{A} - \mathbb{I})$ has the same property with $0 \leq \mu_1 \leq \mu_2 \leq \cdots, \mu_n \uparrow \infty$. Since 0 is an eigenvalue of $-\mathcal{L}_\Lambda$, a constant function is an eigenfunction with eigenvalue 0, then (e) and (f) also hold.

4. Scaling limit

Let $\mathcal{M}$ be the space of positive Radon measures on $\mathbb{T}^d$ with total mass bounded by 1 endowed with the weak topology. For a measure $\pi \in \mathcal{M}$ and a measurable π-integrable function $H\colon \mathbb{T}^d \to \mathbb{R}$, we denote by $\langle \pi, H\rangle$ the integral of H with respect to π.

Recall that $\{\eta^N_t : t \geq 0\}$ denotes a Markov process with state space Ω_N and generator L_N speeded up by N^2. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time t associated to $\{\eta^N_t : t \geq 0\}$, which is the random measure in $\mathcal{M}$ given by

$$\pi^N_t = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta^N_t(x)\delta_{x/N}, \tag{4.1}$$

where δ_u is the Dirac measure concentrated on u.

Note that

$$\langle \pi^N_t, H\rangle = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} H\Big(\frac{x}{N}\Big)\eta^N_t(x)$$

for the empirical measures, and $\langle \pi, H\rangle = \langle\langle \rho, H\rangle\rangle$ for absolutely continuous measures π with L^2 bounded density ρ, and $H \in L^2(\mathbb{T}^d)$.

Fix $T > 0$. Let $D([0, T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi\colon [0, T] \to \mathcal{M}$ endowed with the Skorokhod topology. Then, the $\mathcal{M}$-valued process $\{\pi^N_t : t \geq 0\}$ is a random element of $D([0, T], \mathcal{M})$ whose distribution is determined by the initial distribution of $\{\eta^N_t : t \geq 0\}$. For each probability measure μ on Ω_N, denote by $\mathrm{Q}^{\Lambda,N}_\mu$ the distribution of $\{\pi^N_t : t \geq 0\}$ on the path space $D([0, T], \mathcal{M})$, when η^N_0 has distribution μ.

Proposition 4.1. *Fix a Borel measurable profile $\gamma\colon \mathbb{T}^d \to [0, 1]$, and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on Ω_N associated to γ in the sense of (2.2). Then there exists a unique weak solution ρ of (2.4) with initial condition γ, and the sequence of probability measures $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ converges weakly to $\mathrm{Q}^\gamma_\Lambda$ as $N \uparrow \infty$, where $\mathrm{Q}^\gamma_\Lambda$ is the probability measure on $D([0, T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, \mathrm{d}u) = \rho(t, u)\,\mathrm{d}u$.*

It is straightforward to obtain Theorem 2.2 as a corollary of the previous proposition. The proof of Proposition 4.1 follows directly from the uniqueness of weak solutions of (2.4), proved in Subsection 4.3, and the next two results.

Proposition 4.2. *For any sequence $\{\mu_N : N \geq 1\}$ of probability measures with μ_N concentrated on Ω_N, the sequence of measures $\{\mathrm{Q}^{\Lambda,N}_{\mu_N} : N \geq 1\}$ is tight.*

Proposition 4.3. *Fix a Borel measurable profile $\gamma\colon \mathbb{T}^d \to [0, 1]$, and consider a sequence $\{\mu_N : N \geq 1\}$ of probability measures on Ω_N associated to γ in the sense of (2.2). Then any limit point of $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ is concentrated on absolutely continuous trajectories that are weak solutions of (2.4) with initial condition γ.*

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Proof of Proposition 4.1. By Proposition 4.2, the set of measures $\{\mathrm{Q}^{\Lambda,N}_{\mu_N} : N \geq 1\}$ is tight. Since the Skorokhod space $D([0, T], \mathcal{M})$ is Polish, by Prohorov's theorem, tightness is equivalent to relative compactness (for the weak convergence). By the relative compactness, in order to prove the convergence of the sequence $(Q^{\Lambda,N}_{\mu_N})_{N\geq 1}$ to the probability measure $\mathrm{Q}^{\gamma}_{\Lambda}$, it is enough to show that any convergent subsequence of $(Q^{\Lambda,N}_{\mu_N})_{N\geq 1}$ has limit equal to $\mathrm{Q}^{\gamma}_{\Lambda}$. Let Q^* be a limit of a convergent subsequence. By Proposition 4.3, Q^* is concentrated on trajectories $\pi(t, \mathrm{d}u) = \rho(t, u)\,\mathrm{d}u$ such that $\rho(t, u)$ is a weak solution of (2.4) with initial condition γ. The uniqueness of weak solutions of (2.4) proved in Subsection 4.3 implies that $\mathrm{Q}^* = \mathrm{Q}^{\gamma}_{\Lambda}$.

In Subsection 4.1, we prove Proposition 4.2 and in Subsection 4.2 we prove Proposition 4.3. As a consequence, we have the existence of solutions of (2.4) with initial condition γ. We complete the proof in Subsection 4.3, showing the uniqueness of weak solutions of (2.4).

4.1. Tightness

Here we prove Proposition 4.2. Let $D([0, T], \mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg trajectories with domain $[0, T]$ endowed with the Skorokhod topology. To prove the tightness of $\{\pi^N_t : 0 \leq t \leq T\}$ in $D([0, T], \mathcal{M})$, it is enough to show tightness in $D([0, T], \mathbb{R})$ of the real-valued processes $\{\langle \pi^N_t, H\rangle : 0 \leq t \leq T\}$ for a set of functions $H : \mathbb{T}^d \to \mathbb{R}$ which is dense in the space of continuous real functions on $\mathbb{T}^d$ endowed with the uniform topology; see [8, p. 54]. Furthermore, if a sequence of distributions in $D([0, T], \mathbb{R})$ endowed with the uniform topology is tight, then it is also tight in $D([0, T], \mathbb{R})$ endowed with the Skorokhod topology. Here we prove the tightness of $\{\langle \pi^N_t, H\rangle : 0 \leq t \leq T\}$ in $D([0, T], \mathbb{R})$, endowed with the uniform topology, for $H \in C^2(\mathbb{T}^d)$.

Fix $H \in C^2(\mathbb{T}^d)$. By definition, $\{\langle \pi^N_t, H\rangle : 0 \leq t \leq T\}$ is tight in $D([0, T], \mathbb{R})$ endowed with the uniform topology if, for the boundedness,

$$\lim_{m\to\infty} \sup_N \mathrm{P}^N_{\mu_N}\Big[\sup_{0\leq t\leq T} |\langle \pi^N_t, H\rangle| > m\Big] = 0 \tag{4.2}$$

and, for the equicontinuity,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathrm{P}^N_{\mu_N}\Big[\sup_{|t-s|\leq\delta} |\langle \pi^N_t, H\rangle - \langle \pi^N_s, H\rangle| > \varepsilon\Big] = 0 \quad \text{for all } \varepsilon > 0. \tag{4.3}$$

The limit in (4.2) is trivial since

$$|\langle \pi^N_t, H\rangle| \leq \sup_{u\in\mathbb{T}^d} |H(u)|.$$

So we need to only prove (4.3). By Dynkyn's formula (see the appendix of [8]),

$$M^N_t = \langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t N^2 L_N \langle \pi^N_s, H\rangle\,\mathrm{d}s$$

is a martingale. By the previous expression, (4.3) follows from

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathrm{P}^N_{\mu_N}\Big[\sup_{|t-s|\leq\delta} |M^N_t - M^N_s| > \varepsilon\Big] = 0 \quad \text{for all } \varepsilon > 0,$$

and

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathrm{P}^N_{\mu_N}\Big[\sup_{0\leq t-s\leq\delta} \Big|\int_s^t N^2 L_N \langle \pi^N_r, H\rangle\,\mathrm{d}r\Big| > \varepsilon\Big] = 0 \quad \text{for all } \varepsilon > 0.$$

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Indeed, we show the stronger results below:

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathrm{E}^N_{\mu^N}\Big[\sup_{|t-s|\le\delta}|M^N_t - M^N_s|\Big] = 0 \tag{4.4}$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathrm{E}^N_{\mu^N}\bigg[\sup_{0\le t-s\le\delta}\bigg|\int_s^t N^2 L_N\langle\pi^N_r, H\rangle\,\mathrm{d}r\bigg|\bigg] = 0. \tag{4.5}$$

To verify (4.4), we use the quadratic variation of M^N_t that we denote by $\langle M^N_t\rangle$. By Doob's inequality, we have

$$\begin{aligned}
\mathrm{E}^N_{\mu^N}\Big[\sup_{|t-s|\le\delta}|M^N_t - M^N_s|\Big] &\le 2\,\mathrm{E}^N_{\mu^N}\Big[\sup_{0\le t\le T}|M^N_t|\Big]\\
&\le 2\,\mathrm{E}^N_{\mu^N}\Big[\sup_{0\le t\le T}|M^N_t|^2\Big]^{1/2}\\
&\le 4\,\mathrm{E}^N_{\mu^N}[\langle M^N_T\rangle]^{1/2}.
\end{aligned}$$

Since

$$\langle M^N_t\rangle = \int_0^t N^2[L_N\langle\pi^N_s, H\rangle^2 - 2\langle\pi^N_s, H\rangle L_N\langle\pi^N_s, H\rangle]\,\mathrm{d}s,$$

we obtain, by a straightforward computation,

$$\langle M^N_t\rangle = \int_0^t N^2\sum_{j=1}^d\sum_{x\in\mathbb{T}^d_N}\xi^N_{x,x+e_j}\frac{1}{N^{2d}}\bigg[(\eta_s(x)-\eta_s(x+e_j))\bigg(H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\bigg)\bigg]^2\mathrm{d}s.$$

Therefore, since $\xi^N_{x,x+e_j}\le 1$,

$$\begin{aligned}
\langle M^N_t\rangle &\le \frac{T}{N^{2d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}^d_N}\xi^N_{x,x+e_j}\bigg[H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\bigg]^2\\
&\le \frac{Td}{N^d}\Big(\sup_{u\in\mathbb{T}^d}|\nabla H(u)\cdot e_j|\Big)^2.
\end{aligned} \tag{4.6}$$

Thus, M^N_t converges to 0 in L^2 and (4.4) holds.

We complete the proof by verifying (4.5). Write

$$\begin{aligned}
N^2 L_N\langle\pi^N_s, H\rangle &= \frac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}^d_N}\xi^N_{x,x+e_j}(\eta_s(x)-\eta_s(x+e_j))\bigg(H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\bigg)\\
&= \frac{1}{N^{d-2}}\sum_{j=1}^d\sum_{x\in\mathbb{T}^d_N}\eta_s(x)\bigg[\xi^N_{x,x+e_j}\bigg(H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\bigg)\\
&\qquad\qquad + \xi^N_{x,x-e_j}\bigg(H\Big(\frac{x-e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\bigg)\bigg].
\end{aligned}$$

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Define $\Gamma_N \subset \mathbb{T}_N^d$ as the set of vertices that have some adjacent edge with exchange rate not equal to 1. Then $N^2 L_N \langle \pi_s^N, H\rangle$ is equal to the sum of

$$\frac{1}{N^{d-2}} \sum_{j=1}^{d} \sum_{x \notin \Gamma_N} \eta_s(x) \Big[H\Big(\frac{x+e_j}{N}\Big) + H\Big(\frac{x-e_j}{N}\Big) - 2H\Big(\frac{x}{N}\Big)\Big] \tag{4.7}$$

and

$$\begin{aligned} \frac{1}{N^{d-2}} \sum_{j=1}^{d} \sum_{x \in \Gamma_N} \eta_s(x) \Big[& \xi^N_{x,x+e_j} \Big(H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big) \\ & + \xi^N_{x,x-e_j} \Big(H\Big(\frac{x-e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big)\Big]. \end{aligned} \tag{4.8}$$

By the Taylor expansion (note that $H \in C^2$), the absolute value of the summand in (4.7) is bounded by $N^{-2} \sup_{u \in \mathbb{T}^d} |\Delta H(u)|$. Considering the factor N^{-d+2} in front of the sum, we conclude that (4.7) is bounded in absolute value by $d \sup_{u\in\mathbb{T}^d} |\Delta H(u)|$.

Since there are an order of N^{d-1} vertices in Γ_N, and $\xi_{x,x+e_j} \le 1$, the absolute value of (4.8) is bounded by

$$\begin{aligned} \frac{1}{N^{d-2}} \sum_{j=1}^{d} \sum_{x \in \Gamma_N} & \Big[\Big| H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big| + \Big| H\Big(\frac{x-e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big|\Big] \\ & \le 2d \sup_{u \in \mathbb{T}^d} |\nabla H(u) \cdot e_j|. \end{aligned}$$

By the boundedness of (4.7) and (4.8), there exists $C > 0$, depending only on H, such that $|N^2 L_N \langle \pi_s^N, H\rangle| \le C$, which yields

$$\Big| \int_r^t N^2 L_N \langle \pi_s^N, H\rangle \, \mathrm{d}s \Big| \le C(t-r),$$

and (4.5) holds.

4.2. Characterization of limit points

Let $\gamma \colon \mathbb{T}^d \to [0, 1]$ be a Borel measurable profile, and consider a sequence $\{\mu_N : N \ge 1\}$ of measures on Ω_N associated to γ in the sense of (2.2). We prove Proposition 4.3 in this subsection, i.e. that all limit points Q^* of the sequence $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ are concentrated on absolutely continuous trajectories $\pi(t, \mathrm{d}u) = \rho(t, u)\,\mathrm{d}u$, whose density $\rho(t, u)$ is a weak solution of the hydrodynamic equation (2.4) with γ as an initial condition.

Let Q^* be a limit point of the sequence $\mathrm{Q}^{\Lambda,N}_{\mu_N}$, and assume without loss of generality that $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ converges to Q^*.

Since there is at most one particle per site, Q^* is concentrated on trajectories $\pi_t(\mathrm{d}u)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_t(\mathrm{d}u) = \rho(t, u)\,\mathrm{d}u$, and whose density ρ is nonnegative and bounded by 1; see [8, Chapter 4].

Lemma 4.1. *Any limit point Q^* of $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ is concentrated on absolutely continuous trajectories $\pi_t(\mathrm{d}u) = \rho(t, u)\,\mathrm{d}u$ such that, for any $H \in \mathfrak{D}_\Lambda$,*

$$\langle\langle \rho_t, H\rangle\rangle - \langle\langle \gamma, H\rangle\rangle = \int_0^t \langle\langle \rho_s, \mathcal{L}_\Lambda H\rangle\rangle \, \mathrm{d}s. \tag{4.9}$$

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By this lemma we can prove Proposition 4.3.

Proof of Proposition 4.3. It just remains to extend equality (4.9) to functions $H \in \mathcal{H}^1_\Lambda$. By Theorem 2.1, the set $\mathfrak{D}_\Lambda$ is a core for the Friedrichs extension $\mathbb{I} - \mathcal{L}_\Lambda \colon \mathcal{H}^1_\Lambda \to L^2(\mathbb{T}^d)$. Thus, for any $H \in \mathcal{H}^1_\Lambda$, there exists a sequence $H_n \in \mathfrak{D}_\Lambda$ such that $H_n \to H$ and $(\mathbb{I} - \mathcal{L}_\Lambda)H_n \to (\mathbb{I} - \mathcal{L}_\Lambda)H$, both in $L^2(\mathbb{T}^d)$. This implies that $\mathcal{L}_\Lambda H_n \to \mathcal{L}_\Lambda H$ in $L^2(\mathbb{T}^d)$. Replacing H_n in equality (4.9), and taking the limit as $n \to \infty$ completes the proof.

The remainder of this section is devoted to the proof of Lemma 4.1. Fix a function $H \in \mathfrak{D}_\Lambda$, and define the martingale M_t^N by

$$\langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t N^2 L_N \langle \pi_s^N, H\rangle \,\mathrm{d}s. \tag{4.10}$$

We claim that, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathrm{P}^N_{\mu_N}\Big[\sup_{0\le t\le T} |M_t^N| > \delta\Big] = 0. \tag{4.11}$$

For $H \in C^2$, this follows from the Chebyshev inequality and the estimates given in the proof of the tightness, where we showed that

$$\lim_{N\to\infty} \mathrm{E}^N_\mu\Big[\sup_{0\le t\le T} |M_t^N|\Big] \le \lim_{N\to\infty} \mathrm{E}^N_\mu\Big[\sup_{0\le t\le T} \langle M_t^N\rangle\Big]^{1/2} = 0. \tag{4.12}$$

For $H = h + \lambda \mathbf{1}_\Lambda$ in $\mathfrak{D}_\Lambda$, the first inequality in (4.6) is still valid and

$$\begin{aligned}
\langle M_t^N\rangle &\le \frac{T}{N^{2d-2}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \xi^N_{x,x+e_j}\Big[H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big]^2 \\
&= \frac{T}{N^{2d-2}} \sum_{j=1}^d \sum_{x\notin\Gamma_N} \Big[h\Big(\frac{x+e_j}{N}\Big) - h\Big(\frac{x}{N}\Big)\Big]^2 \qquad (4.13)\\
&\quad + \frac{T}{N^{2d-2}} \sum_{j=1}^d \sum_{x\in\Gamma_N} \xi^N_{x,x+e_j}\Big[H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big]^2, \qquad (4.14)
\end{aligned}$$

where Γ_N is also defined in the proof of the tightness. Expression (4.13) goes to 0 as N increases, since the function h is Lipschitz. For the expression in (4.14), let $x \in \Gamma_N$. If $x/N \in \Lambda$ and $(x+e_j)/N \in \Lambda^\mathrm{c}$, then $\xi^N_{x,x+e_j} \le 1/N$. The same holds if $x/N \in \Lambda^\mathrm{c}$ and $(x+e_j)/N \in \Lambda$. If x/N, $(x+e_j)/N$ both belong to Λ or Λ^c, the exchange rate $\xi^N_{x,x+e_j}$ is 1, but

$$\Big|H\Big(\frac{x+e_j}{N}\Big) - H\Big(\frac{x}{N}\Big)\Big| = \Big|h\Big(\frac{x+e_j}{N}\Big) - h\Big(\frac{x}{N}\Big)\Big| \le \frac{1}{N} \sup_{u\in\mathbb{T}^d} |\nabla H(u)\cdot e_j|.$$

In both cases, expression (4.14) is of the order $O(N^{-d})$. Therefore, from (4.12), we obtain (4.11).

The next step is to show that we can replace $N^2\mathbb{L}_N$ by the continuous operator $\mathcal{L}_\Lambda$ in the martingale formula (4.10) and that the resulting expression still converges to 0 in probability. This will follow from the ensuing proposition. Recall the definition of $\mathbb{L}_N$ given in (2.3).

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Proposition 4.4. *For any $H \in \mathfrak{D}_\Lambda$,*

$$\lim_{N \leftarrow \infty} \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \left| N^2 \mathbb{L}_N H\left(\frac{x}{N}\right) - \mathcal{L}_\Lambda H\left(\frac{x}{N}\right) \right| = 0. \tag{4.15}$$

Proof. As usual, set $H = h + \lambda \mathbf{1}_\Lambda$, where $h \in C^2(\mathbb{T}^d)$. Rewrite the sum in (4.15) as

$$\frac{1}{N^d} \sum_{x \notin \Gamma_N} \left| N^2 \mathbb{L}_N H\left(\frac{x}{N}\right) - \mathcal{L}_\Lambda H\left(\frac{x}{N}\right) \right| + \frac{1}{N^d} \sum_{x \in \Gamma_N} \left| N^2 \mathbb{L}_N H\left(\frac{x}{N}\right) - \mathcal{L}_\Lambda H\left(\frac{x}{N}\right) \right|.$$

The first term above is equal to

$$\frac{1}{N^d} \sum_{x \notin \Gamma_N} \left| N^2 \left(h\left(\frac{x+e_j}{N}\right) + h\left(\frac{x-e_j}{N}\right) - 2h\left(\frac{x}{N}\right)\right) - \Delta h\left(\frac{x}{N}\right) \right|,$$

which converges to 0 because $h \in C^2$. The second term is less than or equal to the sum of

$$\frac{1}{N^d} \sum_{x \in \Gamma_N} \left| \Delta h\left(\frac{x}{N}\right) \right| \tag{4.16}$$

and

$$\frac{1}{N^{d-1}} \sum_{x \in \Gamma_N} \sum_{j=1}^{d} \left| N \xi^N_{x,x+e_j} \left(H\left(\frac{x+e_j}{N}\right) - H\left(\frac{x}{N}\right)\right) + N \xi^N_{x,x-e_j} \left(H\left(\frac{x-e_j}{N}\right) - H\left(\frac{x}{N}\right)\right) \right|. \tag{4.17}$$

Since there are $O(N^{d-1})$ terms in Γ_N, the expression in (4.16) converges to 0 as $N \to \infty$. Since $\partial\Lambda$ is smooth, the quantity of points $x \in \Gamma_N$ for which $\xi^N_{x,x+e_j} \neq 1$ and $\xi^N_{x,x-e_j} \neq 1$ is negligible. Therefore, we must only worry about points $x \in \Gamma_N$ such that, for some j, only one of $\xi^N_{x,x+e_j}$ and $\xi^N_{x,x-e_j}$ is of order N^{-1}. This occurs in one of the following four cases: $x/N \in \Lambda$, $(x-e_j)/N \in \Lambda$, and $(x+e_j)/N \in \Lambda^{\mathrm{c}}$; $x/N \in \Lambda$, $(x-e_j)/N \in \Lambda^{\mathrm{c}}$, and $(x+e_j)/N \in \Lambda$; $x/N \in \Lambda^{\mathrm{c}}$, $(x-e_j)/N \in \Lambda$, and $(x+e_j)/N \in \Lambda^{\mathrm{c}}$; $x/N \in \Lambda^{\mathrm{c}}$, $(x-e_j)/N \in \Lambda^{\mathrm{c}}$, and $(x+e_j)/N \in \Lambda$. The analysis of these cases are analogous; thus we consider only the first case. Suppose that $x/N \in \Lambda$, $(x-e_j)/N \in \Lambda$, and $(x+e_j)/N \in \Lambda^{\mathrm{c}}$. In this case, the summand in (4.17) can be rewritten as

$$\begin{aligned} N \xi^N_{x,x+e_j} &\left(H\left(\frac{x+e_j}{N}\right) - H\left(\frac{x}{N}\right)\right) + N \xi^N_{x,x-e_j} \left(H\left(\frac{x-e_j}{N}\right) - H\left(\frac{x}{N}\right)\right) \\ &= |\vec{\zeta}_{x,j} \cdot e_j| \left[H\left(\frac{x+e_j}{N}\right) - H\left(\frac{x}{N}\right)\right] + N \left[H\left(\frac{x-e_j}{N}\right) - H\left(\frac{x}{N}\right)\right], \end{aligned}$$

which becomes uniformly (in $x \in \Gamma_N$) close to

$$-\lambda |\vec{\zeta}_{x,j} \cdot e_j| \operatorname{sgn}(\vec{\zeta}_{x,j} \cdot e_j) - \frac{\partial h}{\partial u_j}\left(\frac{x}{N}\right) = -\lambda \vec{\zeta}_{x,j} \cdot e_j - \frac{\partial h}{\partial u_j}\left(\frac{x}{N}\right).$$

The condition $\nabla h|_{\partial\Lambda}(u) = -\lambda \vec{\zeta}(u)$, which was imposed in the definition of $\mathfrak{D}_\Lambda$, implies that

$$\lim_{N \to \infty} N \xi^N_{x,x+e_j} \left(H\left(\frac{x+e_j}{N}\right) - H\left(\frac{x}{N}\right)\right) + N \xi^N_{x,x-e_j} \left(H\left(\frac{x-e_j}{N}\right) - H\left(\frac{x}{N}\right)\right) = 0.$$

Therefore, the terms in (4.17) converge uniformly to 0, and the same holds for the whole sum.

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Corollary 4.1. *For $H \in \mathfrak{D}_\Lambda$ and every $\delta > 0$,*

$$\lim_{N\to\infty} \mathrm{Q}^{\Lambda,N}_{\mu_N}\left[\sup_{0\le t\le T}\left|\langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t \langle \pi^N_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\right| > \delta\right] = 0.$$

Proof. By a simple calculation, the martingale defined in (4.10) can be rewritten as

$$M^N_t = \langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t \langle \pi^N_s, N^2\mathbb{L}_N H\rangle\,\mathrm{d}s.$$

The result follows from Proposition 4.4 and (4.11).

At this point we have all the ingredients needed to prove Lemma 4.1, which says that, under Q^*, with probability 1, (4.9) holds for any $H \in \mathfrak{D}_\Lambda$. In order to prove this, it is enough to show that, for any $\delta > 0$ and any $H \in \mathfrak{D}_\Lambda$,

$$\mathrm{Q}^*\left[\sup_{0\le t\le T}\left|\langle \pi_t, H\rangle - \langle \pi_0, H\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\right| > \delta\right] = 0. \tag{4.18}$$

So let H be a fixed function in $\mathfrak{D}_\Lambda$. The idea to estimate the probability in (4.18) is to apply Portmanteau's theorem to replace Q^* by $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ and then use Corollary 4.1. But to obtain an appropriate inequality we need the set

$$\left\{\sup_{0\le t\le T}\left|\langle \pi_t, H\rangle - \langle \pi_0, H\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\right| > \delta\right\}$$

to be open in $D([0,T],\mathcal{M})$. In order to guarantee this, we need H to be continuous, which is not the case. To solve this problem, we use approximations of H by smooth functions.

For $\varepsilon > 0$, define

$$(\partial\Lambda)^\varepsilon = \{u \in \mathbb{T}^d;\, \mathrm{dist}(u,\partial\Lambda) \le \varepsilon\}.$$

Let H^ε be a smooth function which coincides with H on $\mathbb{T}^d \setminus (\partial\Lambda)^\varepsilon$ and $\sup_{\mathbb{T}^d}|H^\varepsilon| \le \sup_{\mathbb{T}^d}|H|$. Fix $\delta > 0$. By the triangular inequality,

$$\begin{aligned}
&\mathrm{Q}^*\left[\sup_{0\le t\le T}\left|\langle \pi_t, H\rangle - \langle \pi_0, H\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\right| > \delta\right]\\
&\quad\le \mathrm{Q}^*\left[\sup_{0\le t\le T}\left|\langle \pi_t, H^\varepsilon\rangle - \langle \pi_0, H^\varepsilon\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\right| > \frac{\delta}{3}\right]\\
&\qquad + 2\,\mathrm{Q}^*\left[\sup_{0\le t\le T}|\langle \pi_t, H^\varepsilon - H\rangle| > \frac{\delta}{3}\right].
\end{aligned} \tag{4.19}$$

Recall that Q^* is concentrated on trajectories $\pi_t(\mathrm{d}u) = \rho(t,u)\,\mathrm{d}u$ whose density ρ is nonnegative and bounded above by 1. Then, under Q^*,

$$\begin{aligned}
\sup_{0\le t\le T}|\langle \pi_t, H^\varepsilon - H\rangle| &\le \sup_{0\le t\le T}\int_{(\partial\Lambda)^\varepsilon}\rho(t,u)|H^\varepsilon(u) - H(u)|\,\mathrm{d}u\\
&\le 2\ell_d((\partial\Lambda)^\varepsilon)\sup_{u\in\mathbb{T}^d}|H(u)|.
\end{aligned}$$

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Therefore, for small enough ε, the second probability on the right-hand side of inequality (4.19) is null. So it remains to show that

$$\mathrm{Q}^*\biggl[\sup_{0\le t\le T}\biggl|\langle \pi_t, H^\varepsilon\rangle - \langle \pi_0, H^\varepsilon\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\biggr| > \frac{\delta}{3}\biggr] = 0.$$

If G_1, G_2, and G_3 are continuous functions, the application from $D([0,T],\mathcal{M})$ to $\mathbb{R}$ that associates to a trajectory $\{\pi_t,\ 0\le t\le T\}$ the number

$$\sup_{0\le t\le T}\biggl|\langle \pi_t, G_1\rangle - \langle \pi_0, G_2\rangle - \int_0^t \langle \pi_s, G_3\rangle\,\mathrm{d}s\biggr|$$

is continuous in the Skorokhod metric. Note that H^ε and $\mathcal{L}_\Lambda H$ are continuous functions. By Portmanteau's theorem,

$$\begin{aligned}\mathrm{Q}^*\biggl[\sup_{0\le t\le T}\biggl|\langle \pi_t, H^\varepsilon\rangle - \langle \pi_0, H^\varepsilon\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\biggr| > \frac{\delta}{3}\biggr]\\ \le \varliminf_{N\to\infty} \mathrm{Q}^{\Lambda,N}_{\mu_N}\biggl[\sup_{0\le t\le T}\biggl|\langle \pi^N_t, H^\varepsilon\rangle - \langle \pi^N_0, H^\varepsilon\rangle - \int_0^t \langle \pi^N_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\biggr| > \frac{\delta}{3}\biggr],\end{aligned} \tag{4.20}$$

since $\mathrm{Q}^{\Lambda,N}_{\mu_N}$ converges weakly to Q^* and the above set is open.

Now we replace H^ε by H. This may be confusing to the reader; however, the previous introduction of H^ε was a necessary step in the proof. From this point, to deal with the right-hand side of (4.20), we need Corollary 4.1. Hence, H^ε should be replaced by H.

By definition,

$$\begin{aligned}\sup_{0\le t\le T}|\langle \pi^N_t, H^\varepsilon - H\rangle| &\le \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\biggl|H^\varepsilon\Bigl(\frac{x}{N}\Bigr) - H\Bigl(\frac{x}{N}\Bigr)\biggr|\\ &\le \biggl(\ell_d((\partial\Lambda)^\varepsilon) + O\Bigl(\frac{1}{N}\Bigr)\biggr) 2\sup_{u\in\mathbb{T}}|H(u)|,\end{aligned}$$

because H^ε coincides with H in $\mathbb{T}\setminus(\partial\Lambda)^\varepsilon$. Using the same argument as before, we obtain

$$\begin{aligned}\varliminf_{N\to\infty} \mathrm{Q}^{\Lambda,N}_{\mu_N}\biggl[\sup_{0\le t\le T}\biggl|\langle \pi_t, H^\varepsilon\rangle - \langle \pi_0, H^\varepsilon\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\biggr| > \frac{\delta}{3}\biggr]\\ \le \varlimsup_{N\to\infty} \mathrm{Q}^{\Lambda,N}_{\mu_N}\biggl[\sup_{0\le t\le T}\biggl|\langle \pi_t, H\rangle - \langle \pi_0, H\rangle - \int_0^t \langle \pi_s, \mathcal{L}_\Lambda H\rangle\,\mathrm{d}s\biggr| > \frac{\delta}{9}\biggr]\\ + 2\varlimsup_{N\to\infty} \mathrm{Q}^{\Lambda,N}_{\mu_N}\biggl[\sup_{0\le t\le T}\bigl|\langle \pi_t, H^\varepsilon - H\rangle\bigr| > \frac{\delta}{9}\biggr].\end{aligned}$$

Again, for small enough ε, the second probability in the sum above is null. From Corollary 4.1 we finally conclude that (4.18) holds. Therefore, Q^* is concentrated on absolutely continuous paths $\pi_t(\mathrm{d}u) = \rho(t,u)\,\mathrm{d}u$ with positive density bounded by 1, and, Q^*-almost surely,

$$\langle\!\langle \rho_t, H\rangle\!\rangle - \langle\!\langle \rho_0, H\rangle\!\rangle = \int_0^t \langle\!\langle \rho_s, \mathcal{L}_\Lambda H\rangle\!\rangle\,\mathrm{d}s$$

for any $H\in\mathfrak{D}_\Lambda$. Hence, we have proved Lemma 4.1.

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4.3. Uniqueness of weak solutions

Now we prove that the solution of (2.4) is unique. It suffices to check that the only solution of (2.4) with $\rho_0 \equiv 0$ is $\rho \equiv 0$, because of the linearity of $\mathcal{L}_\Lambda$. Let $\rho\colon \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ be a weak solution of the parabolic differential equation

$$\partial_t \rho = \mathcal{L}_\Lambda \rho, \qquad \rho(0,\cdot) = 0.$$

By definition,

$$\langle\!\langle \rho_t, H \rangle\!\rangle = \int_0^t \langle\!\langle \rho_s, \mathcal{L}_\Lambda H \rangle\!\rangle \, \mathrm{d}s \tag{4.21}$$

for all functions H in $\mathcal{H}^1_\Lambda$ and all $t > 0$. From Theorem 2.1, the operator $-\mathcal{L}_\Lambda$ has countable eigenvalues $\{\mu_n : n \geq 0\}$ and eigenvectors $\{F_n\}$. All eigenvalues have finite multiplicity, $0 = \mu_0 \leq \mu_1 \leq \cdots$, and $\lim_{n\to\infty} \mu_n = \infty$. Besides, the eigenvectors $\{F_n\}$ form a complete orthonormal system in the $L^2(\mathbb{T}^d)$. Define

$$R(t) = \sum_{n\in\mathbb{N}} \frac{1}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$$

for all $t > 0$. Note that $R(0) = 0$ and $R(t)$ is well defined because ρ_t belongs to $L^2(\mathbb{T}^d)$. Since ρ satisfies (4.21), we have $\mathrm{d}\langle\!\langle \rho_t, F_n \rangle\!\rangle^2/\mathrm{d}t = -2\mu_n \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$. Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} R\right)(t) = -\sum_{n\in\mathbb{N}} \frac{2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2,$$

because

$$\sum_{n\leq N} \frac{-2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2 \quad \text{converges uniformly to} \quad \sum_{n\in\mathbb{N}} \frac{-2\mu_n}{n^2(1+\mu_n)} \langle\!\langle \rho_t, F_n \rangle\!\rangle^2$$

as N increases to ∞. Thus, $R(t) \geq 0$ and $(\mathrm{d}R/\mathrm{d}t)(t) \leq 0$ for all $t > 0$ and $R(0) = 0$. From this, we obtain $R(t) = 0$ for all $t > 0$. Since $\{F_n\}$ is a complete orthonormal system, $\langle\!\langle \rho_t, \rho_t \rangle\!\rangle = 0$ for all $t > 0$, which implies that $\rho \equiv 0$.

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