



Geometric Invariants of Cuspidal Edges

*Dedicated to Professor María del Carmen Romero-Fuster
on the occasion of her sixtieth birthday*

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Abstract. We give a normal form of the cuspidal edge that uses only diffeomorphisms on the source and isometries on the target. Using this normal form, we study differential geometric invariants of cuspidal edges that determine them up to order three. We also clarify relations between these invariants.

1 Introduction

A generic classification of singularities of wave fronts was given by Arnol'd and Zakalyukin (see [1], for example). They showed that the generic singularities of wave fronts in R^3 are cuspidal edges and swallowtails. Recently, there have been numerous studies of wave fronts from the viewpoint of differential geometry, for example, [7, 11, 12, 15, 17]. Cuspidal edges are fundamental singularities of wave fronts in R^3 . The singular curvature and the limiting normal curvature for cuspidal edges are defined in [15] by a limit of geodesic curvatures and a limit of normal curvatures, respectively. On the other hand, the umbilic curvature is defined in [9] for surfaces in Euclidean 3-space with corank 1 singularities by using the first and second fundamental forms. So the umbilic curvature is defined for cuspidal edges. It is shown in [9] that if the umbilic curvature κ_u is non-zero at a singular point, then there exists a unique sphere having contact not of type A_n (for example, D_4 , E_6 , etc.) with the surface in that point: the sphere with center in the normal plane of the surface at the point, with radius equal to $1/\kappa_u$, and in a well-defined direction of the normal plane (see Section 4).

Therefore, the singular, the limiting normal, and the umbilic curvatures are invariants defined by using fundamental tools of differential geometry of surfaces and singularity theory, and they are fundamental invariants of cuspidal edges. Needless to say, the curvature and torsion of a cuspidal edge locus as a space curve in R^3 are also fundamental invariants.

In this paper we clarify the relations amongst these invariants and also make a list of invariants that determine cuspidal edges up to order three. We show that, in the

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case of cuspidal edges, the umbilic curvature κ_u coincides with the absolute value of limiting normal curvature κ_n (Theorem 4.3). In this sense, the umbilic curvature is a generalization of the normal curvature for surfaces with corank 1 singularities. It should be remarked that the umbilic curvature does not require a well-defined unit normal vector, and it is meaningful as a geometric invariant of surfaces with corank 1 singularities in general. We show that the singular curvature κ_s and the limiting normal curvature κ_n at a singular point of a surface M in R^3 with singularities consisting of cuspidal edges are equivalent to the principal curvatures of a regular surface in the following sense. When M is a regular surface in R^3 given in the Monge form, that is, by the equation $z = f(x, y)$ for some smooth function f having first derivatives with respect to x and y vanishing at $(0, 0)$ (or, equivalently, its tangent plane at the origin is given by $z = 0$), then, taking the x and y axes to be in principal directions at the origin, the surface M assumes the local form

$$f(u, v) = \frac{1}{2}(\alpha_1 u^2 + \alpha_2 v^2) + \text{h.o.t.},$$

where α_1, α_2 are the principal curvatures at the origin and h.o.t. represents terms whose degrees are greater than two.

Now, if M is a surface in R^3 with singularities consisting of cuspidal edges, we show that M can be parametrized by an equation of Monge type (just using changes of coordinates in the source and isometries in the target, which do not change the geometry of the surface) that we call *normal form*, given by

$$f(u, v) = \frac{1}{2}(2u, \kappa_s u^2 + v^2 + \text{h.o.t.}, \kappa_n u^2 + \text{h.o.t.})$$

(see Section 3 and Theorem 4.4). Therefore, κ_s and κ_n can be considered as the principal curvatures of M at singular points. But we can say even more about these two invariants. While for a regular curve in a regular surface, it holds that $\kappa^2 = \kappa_n^2 + \kappa_g^2$, where κ_n and κ_g are the normal and geodesic curvatures of the curve, respectively, and κ is the curvature of the curve as a space curve, for the singular curve consists of cuspidal edges, the relation

$$\kappa^2 = \kappa_n^2 + \kappa_s^2$$

holds (see Corollary 4.5).

Furthermore, using the normal form, we detect (Section 5) invariants up to order three, and show (Section 6) that the torsion of the curve consisting of cuspidal edges as a space curve, κ_s, κ_n and these three invariants determine the cuspidal edge up to order three (see Theorem 6.1). In a joint work of M. Umehara, K. Yamada and the authors [10], we consider intrinsic properties of these invariants and the relation between boundedness of Gaussian curvature near cuspidal edges.

The Whitney umbrella (or cross-cap) is the only singularity of a generic map from a surface to R^3 ([21]). Its normal form was given by J. M. West in [20], and it was shown to be useful for considering the differential geometry of surfaces near the singular point; see also [2]. For instance, using this normal form, the authors in [6] showed that there are three fundamental intrinsic invariants for cross-caps and proved the existence of extrinsic invariants. Some other works where this normal form was important are [3–6, 9, 13, 14, 19]. The normal form for cuspidal edges is fundamental in this paper for finding geometric invariants of cuspidal edges, and the

authors believe that this normal form can be used for other problems similar to those considered in the references mentioned just above.

2 Preliminaries

The unit cotangent bundle $T_1^*R^3$ of R^3 has the canonical contact structure and can be identified with the unit tangent bundle T_1R^3 . Let α denote the canonical contact form on it. A map $i: M \rightarrow T_1R^3$ is said to be *isotropic* if $\dim M = 2$ and the pull-back $i^*\alpha$ vanishes identically. An isotropic immersion is called a *Legendrian immersion*. We call the image of $\pi \circ i$ the *wave front set* of i , where $\pi: T_1R^3 \rightarrow R^3$ is the canonical projection, and we denote it by $W(i)$. Moreover, i is called the *Legendrian lift* of $W(i)$. With this framework, we define the notion of fronts as follows. A map-germ $f: (R^2, 0) \rightarrow (R^3, 0)$ is called a *wave front* or a *front* if there exists a unit vector field ν of R^3 along f such that $L = (f, \nu): (R^2, 0) \rightarrow (T_1R^3, 0)$ is a Legendrian immersion by an identification $T_1R^3 = R^3 \times S^2$, where S^2 is the unit sphere in R^3 (cf. [1]; see also [8]). A point $q \in (R^2, 0)$ is a singular point if f is not an immersion at q .

A singular point p of a map f is called a *cuspidal edge* if the map-germ f at p is \mathcal{A} -equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at 0. (Two map-germs $f_1, f_2: (R^n, 0) \rightarrow (R^m, 0)$ are \mathcal{A} -equivalent if there exist diffeomorphisms $S: (R^n, 0) \rightarrow (R^n, 0)$ and $T: (R^m, 0) \rightarrow (R^m, 0)$ such that $f_2 \circ S = T \circ f_1$.) Therefore, if the singular point p of f is a cuspidal edge, then f at p is a front, and furthermore, they are one of two types of generic singularities of fronts (the other one is a *swallowtail*, which is a singular point p of f satisfying that f at p is \mathcal{A} -equivalent to $(u, v) \mapsto (u, u^2v + 3u^4, 2uv + 4u^3)$ at 0). So we establish notation and a fundamental property of singularities of fronts, which are used in the sequel.

Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a front and ν a unit normal vector field along f , and take (u, v) as a coordinate system of the source. The function

$$\lambda = \det(f_u, f_v, \nu)$$

is called the *signed area density*, where $f_u = \partial f / \partial u$ and $f_v = \partial f / \partial v$. A singular point q of f is called *non-degenerate* if $d\lambda(q) \neq 0$. If q is a non-degenerate singular point of f , then the set of singular points $S(f)$ is a regular curve, which we shall call the *singular curve* at q , and we shall denote by γ a parametrization for this curve. The tangential 1-dimensional vector space of the singular curve γ is called the *singular direction*. Furthermore, if q is a non-degenerate singular point, then a non-zero smooth vector field η on $(R^2, 0)$ such that $df(\eta) = 0$ on $S(f)$ is defined. We call η a *null vector field* and its direction the *null direction*; for details, see [15].

Lemma 2.1 ([16, Corollary 2.5, p. 735], [8]) *Let 0 be a singular point of a front $f: (R^2, 0) \rightarrow (R^3, 0)$. Then 0 is a cuspidal edge if and only if $d\lambda(\eta) \neq 0$ at 0. In particular, at a cuspidal edge, the null direction and the singular direction are transversal.*

In this paper we shall use the first and second fundamental forms defined in [9] for surfaces in R^3 with corank 1 singularities and given as follows. Let $q \in R^2$ be a corank 1 singular point of $f: R^2 \rightarrow R^3$ and $p = f(q)$. The Euclidean metric $\langle \cdot, \cdot \rangle$ of R^3 induces a pseudometric on T_qR^2 given by the *first fundamental form* $I: T_qR^2 \times$

$T_qR^2 \rightarrow R$ defined by $I(X, Y) = \langle df_q(X), df_q(Y) \rangle$, where df_q is the differential map of f at q . The coefficients of I at q are

$$E(q) = \langle f_u, f_u \rangle (q), \quad F(q) = \langle f_u, f_v \rangle (q), \quad \text{and} \quad G(q) = \langle f_v, f_v \rangle (q),$$

and, given $X = x\partial_u + y\partial_v \in T_qR^2$, then $I(X, X) = x^2E(q) + 2xyF(q) + y^2G(q)$, where (u, v) is a coordinate system on the source and $\partial_u = (\partial/\partial u)_q$ and $\partial_v = (\partial/\partial v)_q$.

Let us denote the image of f by M and the tangent line to M at p by $T_pM = \text{Im } df_q$. So there is a plane N_pM satisfying $T_pR^3 = T_pM \oplus N_pM$. Consider the orthogonal projection

$$\begin{aligned} \perp : T_pR^3 &\longrightarrow N_pM \\ w &\longrightarrow w^\perp. \end{aligned}$$

The *second fundamental form* $II : T_qR^2 \times T_qR^2 \rightarrow N_qM$ is defined by $II(\partial_u, \partial_u) = f_{uu}^\perp(q)$, $II(\partial_u, \partial_v) = f_{uv}^\perp(q)$ and $II(\partial_v, \partial_v) = f_{vv}^\perp(q)$, and we extend it in the unique way as a symmetric bilinear map. The *second fundamental form along a normal vector* $\nu \in N_pM$ is the function $II_\nu : T_qR^2 \times T_qR^2 \rightarrow R$ defined by $II_\nu(X, Y) = \langle II(X, Y), \nu \rangle$, and its coefficients at q are

$$l_\nu(q) = \langle f_{uu}^\perp(q), \nu \rangle, \quad m_\nu(q) = \langle f_{uv}^\perp(q), \nu \rangle, \quad n_\nu(q) = \langle f_{vv}^\perp(q), \nu \rangle.$$

It was shown in [9] that $\Delta_p = \{II(X, X) | I(X, X)^{1/2} = 1\}$ is a parabola in N_pM , which can degenerate in a half-line, line or a point. (Δ_p is called the *curvature parabola* of M at p .) If p is a cuspidal edge, Δ_p is a half-line in N_pM ; for details, see [9].

3 Normal Form of Cuspidal Edges

In this section we give a normal form of cuspidal edges by using only coordinate transformations on the source and isometries on the target. These changes of coordinates preserve the geometry of the image.

Let $f : (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ and let 0 be a cuspidal edge with $f = (f_1, f_2, f_3)$. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a unit normal vector field along f and let (u, v) be the usual Cartesian coordinate system of R^2 . So $\text{rank } df_0 = 1$ and then we may assume that $f_u(0) = ((f_1)_u(0), 0, 0)$, where $(f_1)_u(0) \neq 0$, by a rotation of R^3 if necessary. The map on the source $(\tilde{u}, \tilde{v}) = (f_1(u, v), v)$ is a coordinate transformation. In fact,

$$\det \begin{pmatrix} \tilde{u}_u & \tilde{u}_v \\ \tilde{v}_u & \tilde{v}_v \end{pmatrix} = \det \begin{pmatrix} \tilde{u}_u & \tilde{u}_v \\ 0 & 1 \end{pmatrix} = \tilde{u}_u = (f_1)_u \neq 0 \quad \text{at } 0.$$

By coordinates (\tilde{u}, \tilde{v}) , f is written $f(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{f}_2(\tilde{u}, \tilde{v}), \tilde{f}_3(\tilde{u}, \tilde{v}))$ for some functions \tilde{f}_2, \tilde{f}_3 . Needless to say, $(\tilde{f}_2)_u = (\tilde{f}_3)_u = (\tilde{f}_2)_v = (\tilde{f}_3)_v = 0$ at 0.

Since 0 is a cuspidal edge and $\eta = \partial_v$ at 0 is a null vector field, $\lambda_\nu \neq 0$. Rewriting $f(u, v) = (u, f_2(u, v), f_3(u, v))$, we have

$$\lambda = \det \begin{pmatrix} 1 & (f_2)_u & (f_3)_u \\ 0 & (f_2)_v & (f_3)_v \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix},$$

and therefore $(0, (f_2)_{vv}, (f_3)_{vv}) \neq 0$. Since $S(f)$ is a regular curve, $S(f)$ is transverse to the v -axis. Thus, $S(f)$ can be parametrized by $(u, g(u))$. Considering the coordinate transformation on the source

$$\tilde{u} = u, \quad \tilde{v} = v - g(u),$$

we may assume that $f(u, v) = (u, f_2(u, v), f_3(u, v))$ and $S(f) = \{v = 0\}$.

On the other hand, there exist functions a_2, a_3, b_2, b_3 such that

$$f_i(u, v) = a_i(u) + vb_i(u, v), \quad i = 2, 3.$$

Since $f_v = 0$ on $\{v = 0\}$, it holds that $b_i(u, v) = 0$ on $\{v = 0\}$, $i = 1, 2$. Then by the Malgrange preparation theorem, there exist functions \bar{b}_2, \bar{b}_3 such that $b_i(u, v) = v\bar{b}_i(u, v)$, $i = 1, 2$.

Rewriting \bar{b} as b , we may assume that f is of the form

$$f(u, v) = (u, a_2(u) + v^2b_2(u, v), a_3(u) + v^2b_3(u, v)).$$

By the above arguments, $f_{vv}(0) \neq 0$; that is, $(b_2, b_3) = ((f_2)_{vv}, (f_3)_{vv}) \neq (0, 0)$ at 0.

Now, using the rotation of R^3 given by the matrix

$$(3.1) \quad A_\theta = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A}_\theta \end{pmatrix}, \quad \tilde{A}_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

we get

$$A_\theta f = (u, \cos \theta a_2(u) - \sin \theta a_3(u) + v^2[\cos \theta b_2(u, v) - \sin \theta b_3(u, v)], \\ \sin \theta a_2(u) + \cos \theta a_3(u) + v^2[\sin \theta b_2(u, v) + \cos \theta b_3(u, v)]).$$

Since $(b_2, b_3) \neq (0, 0)$ at 0, there exists some number θ such that

$$(3.2) \quad \cos \theta b_2(0) - \sin \theta b_3(0) > 0 \quad \text{and} \quad \sin \theta b_2(0) + \cos \theta b_3(0) = 0.$$

Setting

$$\bar{a}_2(u) = \cos \theta a_2(u) - \sin \theta a_3(u), \quad \bar{b}_2(u, v) = \cos \theta b_2(u, v) - \sin \theta b_3(u, v), \\ \bar{a}_3(u) = \sin \theta a_2(u) + \cos \theta a_3(u), \quad \bar{b}_3(u, v) = \sin \theta b_2(u, v) + \cos \theta b_3(u, v),$$

f is rewritten as

$$f(u, v) = (u, \bar{a}_2(u) + v^2\bar{b}_2(u, v), \bar{a}_3(u) + v^2\bar{b}_3(u, v)),$$

with $\bar{a}_2(0) = \bar{a}'_2(0) = \bar{a}_3(0) = \bar{a}'_3(0) = 0$, $\bar{b}_2(0) \neq 0$ and $\bar{b}_3(0) = 0$, where $\bar{a}'_2 = d\bar{a}_2/du$, for example. We remark that $\bar{b}_2(0) > 0$ holds.

Next, using the coordinate transformation on the source

$$\tilde{u} = u, \quad \tilde{v} = v\sqrt{2\bar{b}_2(u, v)},$$

one can rewrite f as

$$f(\tilde{u}, \tilde{v}) = \left(\tilde{u}, \bar{a}_2(\tilde{u}) + \frac{v^2}{2}, \bar{a}_3(\tilde{u}) + \tilde{v}^2\tilde{b}_3(\tilde{u}, \tilde{v}) \right),$$

for some function \tilde{b}_3 satisfying $\tilde{b}_3(0) = 0$.

Rewriting \tilde{u} as u , \tilde{a} as a and \tilde{b} as b , we may assume that f is of the form

$$f(u, v) = \left(u, a_2(u) + \frac{v^2}{2}, a_3(u) + v^2 b_3(u, v) \right).$$

Since $b_3(0) = 0$, there exist functions $a_4(u)$ and $b_4(u, v)$ such that $b_3(u, v) = a_4(u) + vb_4(u, v)$, with $a_4(0) = 0$. We remark that, as 0 is a cuspidal edge, $d\lambda(\eta) \neq 0$. This is equivalent to $(b_3)_v(0) \neq 0$. Thus, $b_4(0) \neq 0$.

Hence, f can be written as

$$f(u, v) = \left(u, a_2(u) + \frac{v^2}{2}, a_3(u) + v^2 a_4(u) + v^3 b_4(u, v) \right).$$

Changing the numbering, we get

$$(3.3) \quad f(u, v) = \left(u, a_1(u) + \frac{v^2}{2}, b_2(u) + v^2 b_3(u) + v^3 b_4(u, v) \right),$$

where $a_1(0) = a'_1(0) = b_2(0) = b'_2(0) = b_3(0) = 0$, $b_4(0) \neq 0$. By rotations $(u, v) \mapsto (-u, -v)$ on R^2 and $(x, y, z) \mapsto (-x, y, -z)$ on R^3 , we may assume that $b'_2(0) \geq 0$. Summarizing the above arguments, we have the following theorem.

Theorem 3.1 *Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ and let 0 be a cuspidal edge. Then there exist a diffeomorphism-germ $\varphi: (R^2, 0) \rightarrow (R^2, 0)$ and an isometry-germ $\Phi: (R^3, 0) \rightarrow (R^3, 0)$ satisfying*

$$(3.4) \quad \Phi \circ f \circ \varphi(u, v) =$$

$$\left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{1}{2}v^2, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3 \right) + h(u, v),$$

($b_{03} \neq 0$, $b_{20} \geq 0$), where

$$h(u, v) = \left(0, u^4 h_1(u), u^4 h_2(u) + u^2 v^2 h_3(u) + uv^3 h_4(u) + v^4 h_5(u, v) \right),$$

with $h_1(u), h_2(u), h_3(u), h_4(u), h_5(u, v)$ smooth functions.

We call this parametrization the *normal form* of cuspidal edges. One can easily verify that all coefficients of (3.4) are uniquely determined, since the rotation (3.2) means that $\eta\eta f(0) = (0, 1, 0)$, where ηf means the directional derivative $df(\eta)$. This unique expansion of a cuspidal edge implies that the above coefficients can be considered as geometric invariants of the cuspidal edge. It means that any cuspidal edge has this form using only coordinate changes on the source and isometries of R^3 .

We shall deal with the six geometric invariants of cuspidal edges given by the formula (3.4) in the following sections.

4 Singular Curvature, Normal Curvature, and Umbilic Curvature

In this section, we review the singular curvature κ_s , the limiting normal curvature κ_n ([15]), and the umbilic curvature κ_u ([9]). We show $\kappa_n = \kappa_u$ and compute the curvature κ and the torsion τ , as well as κ_s and κ_n , of the cuspidal edge given by (3.4). As an immediate consequence we obtain an expression relating the singular and limiting normal curvatures with the curvature of the cuspidal curve as a space curve.

Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ and let 0 be a cuspidal edge. Let $\gamma(t)$ be the singular curve, $\hat{\gamma} = f \circ \gamma$, and choose the null vector $\eta(t)$ such that $(\gamma'(t), \eta(t))$ is a positively oriented frame field along γ . The *singular curvature* κ_s and the *limiting normal curvature* κ_ν at t are the functions ([15])

$$(4.1) \quad \kappa_s(t) = \operatorname{sgn}(d\lambda(\eta)) \frac{\det(\hat{\gamma}'(t), \hat{\gamma}''(t), \nu(\gamma(t)))}{|\hat{\gamma}'(t)|^3} = \operatorname{sgn}(d\lambda(\eta)) \frac{\langle \hat{\gamma}''(t), n(t) \rangle}{|\hat{\gamma}'(t)|^2},$$

$$\kappa_\nu(t) = \frac{\langle \hat{\gamma}''(t), \nu(\gamma(t)) \rangle}{|\hat{\gamma}'(t)|^2}, \quad n(t) = \nu(\gamma(t)) \times \frac{\hat{\gamma}'(t)}{|\hat{\gamma}'(t)|},$$

where \times denotes the vector product in R^3 . Then $\kappa_s(t)$ can be considered as the limiting geodesic curvature of curves with the singular curve on their right-hand sides. The definitions given in (4.1) do not depend on the parametrization for the singular curve, nor the orientation of R^2 . Furthermore, κ_s does not depend on the choice of ν , and κ_ν depends on the choice of ν . For more details see [15]. We consider the absolute value of κ_ν and set $\kappa_n = |\kappa_\nu|$. We call κ_n the *absolute normal curvature* or just the *normal curvature* of the cuspidal edge.

The umbilic curvature κ_u is a function defined in [9] for corank 1 singular points of surfaces in R^3 , unless for Whitney umbrellas (i.e., surfaces image of any map germ $(R^2, 0) \rightarrow (R^3, 0)$, which is \mathcal{A} -equivalent to (x, y^2, xy)), and so κ_u is well defined for the cuspidal edge f at $\hat{\gamma}(t)$. Its definition is given in terms of the first and second fundamental forms of $M = \operatorname{Im} f$ defined in Section 2.

Under the above setting, let $\alpha: R \rightarrow N_p M$ be a parametrization for Δ_p , where $p = \hat{\gamma}(t)$. Since Δ_p is a half-line, $|\alpha(s) \times \alpha'(s)|/|\alpha'(s)|$ does not depend on the parametrization $\alpha(s)$ for Δ_p , nor on the value s satisfying $\alpha'(s) \neq 0$. Set $\kappa_u(t) = |\alpha(s) \times \alpha'(s)|/|\alpha'(s)|$. Since $N_p M$ is a normal plane of $\hat{\gamma}'(t)$,

$$(4.2) \quad \kappa_u(t) = \frac{|\alpha(s) \times \alpha'(s)|}{|\alpha'(s)|} = \left| \left\langle \frac{|\alpha(s) \times \alpha'(s)|}{|\alpha'(s)|}, \frac{\hat{\gamma}'(t)}{|\hat{\gamma}'(t)|} \right\rangle \right|$$

$$= \frac{|\det(\alpha(s), \alpha'(s), \hat{\gamma}'(t))|}{|\alpha'(s) \times \hat{\gamma}'(t)|}$$

holds for any s such that $\alpha'(s) \neq 0$. Notice that $\kappa_u(t)$ is the distance between p and the line ℓ containing Δ_p .

For later computation, it is convenient to take an adapted pair of vector fields and an adapted coordinate system. If a singular point of the map-germ f is a cuspidal edge, then $S(f)$ is a regular curve on the source and the null vector field is transverse to $S(f)$. Thus, we can take a pair of vector fields and a coordinate system as follows: A pair of vector field (ξ, η) on $(R^2, 0)$ is called *adapted* if it satisfies the following:

- (a) ξ is tangent to $S(f)$ on $S(f)$;
- (b) η is a null vector on $S(f)$;
- (c) (ξ, η) is positively oriented.

A coordinate system (u, v) on $(R^2, 0)$ is called *adapted* if it satisfies the following:

- (a) the u -axis is the singular curve,
- (b) ∂_v gives a null vector field on the u -axis, and
- (c) there are no singular points except the u -axis.

We remark that the coordinate system (u, v) in the formula (3.4) is adapted. Condition (a) for adapted vector field (ξ, η) is characterized by $\xi\lambda = 0$ on $S(f)$, where λ is the signed area density, and Condition (b) is characterized by $\eta f = 0$ on $S(f)$. Formulas for coefficients in (3.3) by using adapted coordinate systems are stated in the sequel. In [10], we also define adapted coordinate system. In that definition, a condition $|f_u| = 1$ is imposed in addition to the above, but we do not assume it here.

Remark 4.1 If (ξ, η) is an adapted pair of vector fields, then $\xi\eta f = 0$ holds on $S(f)$, since $\eta f = 0$ on $S(f)$. Furthermore, $\{\xi f, \eta\eta f, \nu\}$ is linearly independent, since $\det(\xi f, \eta\eta f, \nu) = \eta\lambda \neq 0$ at 0. For the same reason, if (u, v) is an adapted coordinate system, then $f_{uv} = 0$ holds on $S(f)$ and $\{f_u, f_{vv}, \nu\}$ is linearly independent.

Taking an adapted coordinate system, it holds that

$$(4.3) \quad \kappa_s(u, 0) = \text{sgn}(\lambda_\nu) \frac{\det(f_u, f_{uu}, \nu)}{|f_u|^3}(u, 0).$$

See [15] for details.

For an adapted pair of vector fields (ξ, η) on $(R^2, 0)$, it can be easily seen that

$$\kappa_s(u, v) = \text{sgn}(\eta\lambda) \frac{\det(\xi f, \xi\xi f, \nu)}{|\xi f|^3}(u, v),$$

and

$$(4.4) \quad \begin{aligned} \kappa_n(u, v) &= \frac{|\det(\xi f, \eta\eta f, \xi\xi f)|}{|\xi f|^2 |\xi f \times \eta\eta f|}(u, v) \\ &= \frac{\text{sgn}(\eta\lambda \langle \nu, \xi\xi f \rangle) \det(\xi f, \eta\eta f, \xi\xi f)}{|\xi f|^2 |\xi f \times \eta\eta f|}(u, v), \end{aligned}$$

where $(u, v) \in S(f)$.

Lemma 4.2 Formula (4.4) of $\kappa_n(u, v)$ does not depend on the choice of pairs of adapted vector fields.

Proof Let us take another pair of adapted vector fields $(\tilde{\xi}, \tilde{\eta})$ such that

$$(4.5) \quad \tilde{\xi} = a\xi + b\eta, \quad \tilde{\eta} = c\xi + d\eta,$$

where a, b, c, d are smooth functions of (u, v) satisfying $ad - bc \neq 0$, and on $S(f)$, satisfying $b = c = 0$. Moreover, as $(\tilde{\xi}, \tilde{\eta})$ is positively oriented on $S(f)$, $ad > 0$ holds on $S(f)$. Then we have

$$(4.6) \quad \begin{aligned} \tilde{\xi} f &= a\xi f + b\eta f = a\xi f && (\text{on } S(f)), \\ \tilde{\xi}\tilde{\xi} f &= a(\xi a\xi f + a\xi\xi f + \xi b\eta f + b\xi\eta f) + b(\eta a\xi f + a\eta\xi f + \eta b\eta f + b\eta\eta f) \\ &= a\xi a\xi f + a^2\xi\xi f && (\text{on } S(f)), \\ \tilde{\eta} f &= c\xi f + d\eta f \\ \tilde{\eta}\tilde{\eta} f &= c(\xi c\xi f + c\xi\xi f + \xi d\eta f + d\xi\eta f) + d(\eta c\xi f + c\eta\xi f + \eta d\eta f + d\eta\eta f) \\ &= d\eta c\xi f + d^2\eta\eta f && (\text{on } S(f)). \end{aligned}$$

Then it holds that

$$\frac{\det(\tilde{\xi}f, \tilde{\eta}\tilde{\eta}f, \tilde{\xi}\tilde{\xi}f)}{|\tilde{\xi}f|^2|\tilde{\xi}f \times \tilde{\eta}\tilde{\eta}f|} = \frac{a^3 d^2 \det(\xi f, \eta\eta f, \xi\xi f)}{|a^3||d^2||\xi f|^2|\xi f \times \eta\eta f|} = \text{sgn}(a) \frac{\det(\xi f, \eta\eta f, \xi\xi f)}{|\xi f|^2|\xi f \times \eta\eta f|},$$

and the lemma follows. ■

Since formula (4.4) does not depend on the choice of pairs of adapted vector fields, one can choose $\xi = \partial_u, \eta = \partial_v$ and an adapted coordinate system, getting

$$(4.7) \quad \kappa_n(u, 0) = \frac{1}{E} \frac{|\det(f_u, f_{vv}, f_{uu})|}{|f_u \times f_{vv}|}(u, 0).$$

We can state properties of the umbilic curvature and singular curvature in terms of the second fundamental form and also relate the umbilic and normal curvatures.

Theorem 4.3 *Let $f: (R^2, q) \rightarrow (R^3, p)$ be a map-germ, q a cuspidal edge, and ν a unit normal vector field along f . Then the following hold:*

- (i) $\nu(q)$ is orthogonal to the line ℓ which contains Δ_p . Therefore, $\kappa_u(q) = \frac{|II_\nu(X, X)|}{I(X, X)}$, for any $X \in T_qR^2$.
- (ii) $\kappa_u(q) = \kappa_n(q)$.
- (iii) $\kappa_s(q) = 0$ if and only if $II(X, X)$ is parallel to ν at p , where X is a non-zero tangent vector to $S(f)$ at q .
- (iv) $\kappa_u(q) = \kappa_s(q) = 0$ if and only if $II(X, X) = 0$, where X is a non-zero tangent vector to $S(f)$ at q .

Proof Let M be the image of f . It was shown in [9, Lemma 1.1] that the second fundamental form does not depend on the choice of the local system of coordinates on $(R^2, 0)$. So we can take an adapted coordinate system (u, v) .

Writing $\gamma(t) = (u(t), 0)$ and $\hat{\gamma}(t) = f \circ \gamma(t)$, we have $\hat{\gamma}'(t) = u'(t)f_u(u(t), 0)$. Since $f_v(u(t), 0) = 0, X = x\partial_u + y\partial_v$ is a unit vector in $T_{\hat{\gamma}(t)}R^2$ (with relation the pseudometric given by I) if and only if $x = \pm 1/\sqrt{E(u(t), 0)}$. As $f_{uv}(u(t), 0) = 0$, a parametrization for $\Delta_{\hat{\gamma}(t)}$ at $\hat{\gamma}(t)$ is

$$\alpha(s) = \frac{1}{E(u(t), 0)} f_{uu}^\perp(u(t), 0) + s^2 f_{vv}^\perp(u(t), 0),$$

and so $\alpha'(s) = 2s f_{vv}^\perp(u(t), 0)$.

Then, to conclude (i), it is enough to verify that $\nu(q)$ is orthogonal to $f_{vv}^\perp(q)$. On $S(f) = \{(u, v); v = 0\}$ it also holds that $f_v = 0$, and by Remark 4.1, $f_{vv} \neq 0$ holds. Therefore, we can write $f_v = vh$, where $h(u, v) \neq 0$ on $(R^2, 0)$, which implies that

$$\nu = \varepsilon \frac{f_u \times h}{|f_u \times h|} = \varepsilon \frac{f_u \times f_{vv}}{|f_u \times f_{vv}|} = \varepsilon \frac{f_u \times f_{vv}^\perp}{|f_u \times f_{vv}|}$$

on the singular set, where $\varepsilon = 1$ or -1 , and therefore ν is orthogonal to f_{vv}^\perp on $S(f)$, as we claimed. So, it follows from [9, Remark 2.10(3)] that

$$\kappa_u(q) = \frac{|II_\nu(X, X)|}{I(X, X)}$$

holds for any $X \in T_qR^2$.

Denoting $\kappa_u(t)$ by $\kappa_u(u, 0)$, it holds from (4.2) that

$$\kappa_u(u, 0) = \frac{|\det(\frac{1}{E}f_{uu}^\perp + s^2 f_{vv}^\perp, 2s f_{vv}^\perp, f_u)|}{|2s f_u^\perp \times f_{vv}|}(u, 0) = \frac{|\det(f_{uu}, 2s f_{vv}, f_u)|}{E|2s f_u \times f_{vv}|}(u, 0).$$

Therefore, from (4.7), we get that $\kappa_u(u, 0) = \kappa_n(u, 0)$, concluding (ii).

Hence,

$$(4.8) \quad \begin{aligned} \kappa_u(u, 0) &= \frac{|\det(f_u, f_{vv}, f_{uu})|}{E|f_u \times f_{vv}|} = \frac{1}{E} \left| \left\langle \frac{f_u \times f_{vv}}{|f_u \times f_{vv}|}, f_{uu} \right\rangle \right| \\ &= \frac{|\langle \nu, f_{uu} \rangle|}{E} = \frac{|\langle \nu, f_{uu}^\perp \rangle|}{E}, \end{aligned}$$

at $q = (u, 0)$.

Now, consider the orthonormal frame $\{\nu(q), \nu(q) \times f_u(q)/|f_u(q)|\}$ for $N_p M$. Noticing that $\det(f_u, f_{uu}, \nu) = \det(f_u, f_{uu}^\perp, \nu)$ at q , then, from (4.3) and (4.8), it holds that

$$\frac{1}{E} II(\partial_u, \partial_u) = \frac{1}{E} f_{uu}^\perp = \kappa_u \nu + \operatorname{sgn}(\lambda_\nu) \kappa_s \nu \times \frac{f_u}{|f_u|},$$

at q , which implies that $|II(\partial_u, \partial_u)/E(q)|^2 = \kappa_u^2(q) + \kappa_s^2(q)$, and consequently we conclude (iii) and (iv) of the theorem. ■

A standard approach to getting information about the geometry of surfaces is analyzing their generic contacts with planes and spheres. Such contacts are measured by composing the implicit equation of the plane or sphere with the parametrisation of the surface and seeing what types of singularities arise. Then we label the contact according with the type of singularity. In [9] Nuño-Ballesteros and the first author deal with such study for surfaces in R^3 with corank 1 singularities. We recall that a singular point of a function is said to be of type $\Sigma^{2,2}$ if all of the partial derivatives of the function up to order 2 at the singular point are equal to zero. With the conditions of Theorem 4.3 and by [9, Theorems 2.11 and 2.15], it follows that (a) if $\kappa_n(q) = 0$, then the plane at p orthogonal to $\nu(q)$ is the only plane in R^3 having contact of type $\Sigma^{2,2}$ with f ; (b) if $\kappa_n(q) \neq 0$, then the sphere with center at

$$u = p + \varepsilon \frac{1}{\kappa_n(q)} \nu(q)$$

is the only sphere in R^3 having contact of type $\Sigma^{2,2}$ with f , where $\varepsilon = \operatorname{sgn}(II_\nu(X, X))$, for any unit vector $X \in T_q R^2$.

When f is of normal form, the relations between the singular curvature, the limiting normal curvature (so, the umbilic curvature), and curvature and torsion of the space curve $f|_{S(f)}$ are given in the following result.

Theorem 4.4 *Let $f(u, v)$ be a map-germ of the form (3.4). For the space curve $f|_{S(f)}$ at the origin, it holds that*

$$\begin{aligned} \kappa_s &= a_{20}, & \kappa'_s &= a_{30} + b_{12}b_{20}, & \kappa_n &= \kappa_u = b_{20}, & \kappa'_n &= b_{30} - a_{20}b_{12}, \\ \kappa &= \sqrt{a_{20}^2 + b_{20}^2}, & \kappa' &= \frac{a_{20}a_{30} + b_{20}b_{30}}{\sqrt{a_{20}^2 + b_{20}^2}}, & \tau &= \frac{a_{20}a_{30} - b_{20}a_{30}}{a_{20}^2 + b_{20}^2}. \end{aligned}$$

Proof The curvature and the torsion of the curve $\widehat{\gamma} = f \circ \gamma$ are calculated as usual and we shall omit that here. Using the parametrization $f(u, v)$ given by (3.3), we have:

$$f_u = (1, a'_1, b'_2 + v^2 b'_3 + v^3(b_4)_u) \quad \text{and} \quad f_v = (0, v, 2vb_3 + 3v^2 b_4 + v^3(b_4)_v),$$

where $a'_i = da_i/du$ and $b'_i = db_i/du$ for $i = 2, 3$. This implies that the u -axis is the singular curve and the v -direction is the null direction. So (u, v) is an adapted system of coordinates. Since $f_u(0) = (1, 0, 0)$, $f_{uu}(0) = (0, a''_1(0), b''_2(0))$, $f_{vv}(0) = (0, 1, 0)$, a unit normal vector at 0 is $\nu = (0, 0, 1)$ and the signed area density satisfies $\lambda_\nu(0) = 1$. Then, using (4.3) and (4.8), we get

$$\kappa_s(0) = a''_1(0) = a_{20} \quad \text{and} \quad \kappa_n(0) = b''_2(0) = b_{20}. \quad \blacksquare$$

As a consequence, we have the following corollary.

Corollary 4.5 *Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ, let 0 be a cuspidal edge, and let $\gamma(t)$ be a parametrization of $S(f)$, and $\widehat{\gamma}(t) = f \circ \gamma(t)$. Let $\kappa(t)$ be the curvature of $\widehat{\gamma}(t)$ as a curve in R^3 , $\kappa_s(t)$ its singular curvature and $\kappa_n(t)$ its limiting normal curvature. Then*

$$\kappa(t)^2 = \kappa_s(t)^2 + \kappa_n(t)^2.$$

5 Other Geometric Invariants up to Order Three

Comparing (3.4) and Theorem 4.4, there are three other independent invariants of cuspidal edges up to order three.

5.1 Cuspidal Curvature

The *cuspidal curvature* κ_c for cuspidal edges is defined in [10] as

$$\kappa_c(u, v) = \frac{|\xi f|^{3/2} \det(\xi f, \eta \eta f, \eta \eta \eta f)}{|\xi f \times \eta \eta f|^{5/2}}(u, v), \quad (u, v) \in S(f),$$

where (ξ, η) is an adapted pair of vector fields on $(R^2, 0)$. If $f(u, v)$ is a map-germ of the form (3.4), then it holds that $\kappa_c(0, 0) = b_{03}$. See [10] for detailed description and geometric meanings.

5.2 Cusp-directional Torsion

Let $f = (f_1, f_2, f_3): (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ, 0 a cuspidal edge, and $\gamma(t)$ a parametrization of $S(f)$. Take a pair of adapted vector fields (ξ, η) on $(R^2, 0)$. We define the *cusp-directional torsion* (or *cuspidal torsion* for short) on singular points consisting of cuspidal edges as follows:

$$(5.1) \quad \kappa_t(u, v) = \frac{\det(\xi f, \eta \eta f, \xi \eta \eta f)}{|\xi f \times \eta \eta f|^2}(u, v) - \frac{\det(\xi f, \eta \eta f, \xi \xi f) \langle \xi f, \eta \eta f \rangle}{|\xi f|^2 |\xi f \times \eta \eta f|^2}(u, v), \quad (u, v) \in S(f).$$

By Remark 4.1, the denominator of $\kappa_t(u, v)$ does not vanish, and therefore κ_t is a bounded function on cuspidal edges. The following proposition shows that the cusp-directional torsion is well defined.

Proposition 5.1 *The definition of cusp-directional torsion does not depend on the choice of the pair (ξ, η) of adapted vector fields on $(R^2, 0)$.*

Proof Define a new pair $(\tilde{\xi}, \tilde{\eta})$ of adapted vector fields as in (4.5). By (4.6), we have

$$\tilde{\xi}\tilde{\eta}f = x_1\xi f + x_2\eta f + ad\eta c\xi\xi f + ad^2\xi\eta\eta f$$

holds on $S(f)$, where x_1, x_2 are some functions. Thus, again by (4.6), it holds that

$$\begin{aligned} & \frac{\det(\tilde{\xi}f, \tilde{\eta}\tilde{\eta}f, \tilde{\xi}\tilde{\eta}\tilde{\eta}f)}{|\tilde{\xi}f \times \tilde{\eta}\tilde{\eta}f|^2} - \frac{\det(\tilde{\xi}f, \tilde{\eta}\tilde{\eta}f, \tilde{\xi}\tilde{\xi}f) \langle \tilde{\xi}f, \tilde{\eta}\tilde{\eta}f \rangle}{|\tilde{\xi}f|^2 |\tilde{\xi}f \times \tilde{\eta}\tilde{\eta}f|^2} \\ &= \frac{\det(\xi f, \eta\eta f, \eta c\xi\xi f + d\xi\eta\eta f)}{d|\xi f \times \eta\eta f|^2} - \frac{\langle \xi f, \eta c\xi\xi f + d\eta\eta f \rangle |\xi f, \eta\eta f, \xi\xi f|}{d|\xi f|^2 |\xi f \times \eta\eta f|^2} \\ &= \frac{\det(\xi f, \eta\eta f, \xi\eta\eta f)}{|\xi f \times \eta\eta f|^2} - \frac{\det(\xi f, \eta\eta f, \xi\xi f) \langle \xi f, \eta\eta f \rangle}{|\xi f|^2 |\xi f \times \eta\eta f|^2}. \end{aligned}$$

Thus, the proposition follows. ■

In the following equation we give the expression of the cusp-directional torsion for an adapted coordinate system:

$$\kappa_t(u, 0) = \frac{\det(f_u, f_{vv}, f_{uvv})}{|f_u \times f_{vv}|^2}(u, v) - \frac{\det(f_u, f_{vv}, f_{uu}) \langle f_u, f_{vv} \rangle}{|f_u|^2 |f_u \times f_v|^2}(u, 0).$$

Moreover, if (u, v) satisfies $\langle f_u, f_{vv} \rangle(u, 0) = 0$, then we have the following simple expression:

$$\kappa_t(u, 0) = \frac{\det(f_u, f_{vv}, f_{uvv})}{|f_u \times f_{vv}|^2}(u, 0).$$

The next result gives $\kappa_t(0)$ for the normal form of cuspidal edges, and we omit its proof, as it is a straightforward calculation.

Proposition 5.2 *Let f be a map-germ of the form (3.4). Then $\kappa_t(0) = b_{12}$ holds.*

Let us state the geometric meaning of the invariant κ_t .

Proposition 5.3 *Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ, 0 a cuspidal edge, ν a unit normal vector field along f , $\gamma(t)$ a parametrization of $S(f)$, and $q \in S(f)$. If $\text{pr}_\nu \circ f$ is locally a bijection, then $\kappa_t = 0$ on $S(f)$ near q . Here, pr_ν is the orthogonal projection to the orthogonal plane to $\nu = \text{Im } df_0(T_0R^2) \times \nu(0)$.*

Proof We may assume that f is given by (3.3). Then

$$\text{pr}_{\nu(0)} \circ f(u, v) = (u, a_2(u) + v^2a_3(u) + v^3b(u, v)).$$

For a sufficiently small u_0 , $(u_0, a_2(u_0) + v^2 + v^3b(u_0, v))$ is located on the line $u = u_0$. If $u_0 \neq u_1$, then these lines do not have a crossing. Hence, $\text{pr}_\nu \circ f$ is locally

bijjective if and only if $g_u(v) = a_2(u) + a_3(u)v^2 + v^3b(u, v)$ is monotone, for any sufficiently small u . Since

$$g'_u(0) = 0, g''_u(0) = 2a_3(u) \text{ and } g'''_u(0) = 6b(u, 0),$$

a necessary condition that $\text{pr}_{v(0)} \circ f$ is locally bijective is $a_3(u) = 0$, for any sufficiently small u . Since $a_3(0) = 0$ and $a'_3(0) = \kappa_t(0)$, this proves the assertion. ■

Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a map-germ, 0 a cuspidal edge, and M the image of f . Then the slice locus $M \cap N_0M$ is a cusp. When a pair of adapted vector field (ξ, η) satisfies $\langle \xi f, \eta \eta f \rangle \equiv 0$ on $S(f)$, then $\eta \eta f \in N_0M$ points to the direction where the cusp comes in N_0M . We call this direction of the *cuspidal-direction*. Proposition 5.3 implies that the cusp-directional torsion measures the rotation of cusp-direction along the singular curve of the cuspidal edge. This is a reason that we call κ_t cusp-directional torsion. Remark that (5.1) is well defined for non-degenerate singularities whose null direction is transverse to the singular direction (for example, the cuspidal cross cap $(u, v) \mapsto (u, v^2, uv^3)$). In the appendix, a global property of κ_t is discussed.

5.3 Edge Inflectional Curvature

The invariants introduced in the previous sections for a map-germ given by (3.4) suggest that there is one more geometric invariant that should tell us about a_{30} or b_{30} . Let (ξ, η) be a pair of adapted vector fields on $(R^2, 0)$. We define the *edge inflectional curvature* as follows:

$$\kappa_i(u, v) = \frac{\det(\xi f, \eta \eta f, \xi \xi \xi f)}{|\xi f|^3 |\xi f \times \eta \eta f|}(u, v) - 3 \frac{\langle \xi f, \xi \xi f \rangle \det(\xi f, \eta \eta f, \xi \xi f)}{|\xi f|^5 |\xi f \times \eta \eta f|}(u, v),$$

$(u, v) \in S(f)$. If where ξ is chosen satisfying $|\xi f| = 1$ on $S(f)$, then we have

$$\kappa_i(u, v) = \frac{\det(\xi f, \eta \eta f, \xi \xi \xi f)}{|\xi f \times \eta \eta f|}(u, v).$$

Proposition 5.4 *The function κ_i does not depend on the choice of the pair (ξ, η) of adapted vector fields.*

Proof Define a new pair $(\tilde{\xi}, \tilde{\eta})$ of adapted vector fields as in (4.5). By (4.6), we have

$$\tilde{\xi} \tilde{\xi} \tilde{\xi} f = a((\xi a)^2 + a \xi \xi a) \xi f + 3a^2 \xi a \xi \xi f + a^3 \xi \xi \xi f$$

holds on $S(f)$. Thus, again by (4.6), we see that

$$\begin{aligned} & \frac{\det(\tilde{\xi} f, \tilde{\eta} \tilde{\eta} f, \tilde{\xi} \tilde{\xi} \tilde{\xi} f)}{|\tilde{\xi} f|^3 |\tilde{\xi} f \times \tilde{\eta} \tilde{\eta} f|} - 3 \frac{\langle \tilde{\xi} f, \tilde{\xi} \tilde{\xi} f \rangle \det(\tilde{\xi} f, \tilde{\eta} \tilde{\eta} f, \tilde{\xi} \tilde{\xi} f)}{|\tilde{\xi} f|^5 |\tilde{\xi} f \times \tilde{\eta} \tilde{\eta} f|} \\ &= \frac{3 \xi a \det(\xi f, \eta \eta f, \xi \xi f)}{a |\xi f|^3 |\xi f \times \eta \eta f|} + \frac{\det(\xi f, \eta \eta f, \xi \xi \xi f)}{|\xi f|^3 |\xi f \times \eta \eta f|} \\ & \quad - \frac{3(\xi a \langle \xi f, \xi f \rangle + a \langle \xi f, \xi \xi f \rangle) \det(\xi f, \eta \eta f, \xi \xi f)}{a |\xi f|^5 |\xi f \times \eta \eta f|} \\ &= \frac{\det(\xi f, \eta \eta f, \xi \xi \xi f)}{|\xi f|^3 |\xi f \times \eta \eta f|} - 3 \frac{\langle \xi f, \xi \xi f \rangle \det(\xi f, \eta \eta f, \xi \xi f)}{|\xi f|^5 |\xi f \times \eta \eta f|} \end{aligned}$$

holds on $S(f)$ as we claimed. ■

The expression for κ_i at an adapted coordinate system is the following one:

$$\kappa_i(u, 0) = \frac{\det(f_u, f_{vv}, f_{uuu})}{|f_u|^3 |f_u \times f_{vv}|}(u, 0) - 3 \frac{\langle f_u, f_{uu} \rangle \det(f_u, f_{vv}, f_{uu})}{|f_u|^5 |f_u \times f_{vv}|}(u, 0).$$

The next result gives the edge inflectional curvature at 0 for the normal form of cuspidal edges. Its proof is a straightforward calculation, and so we omit it here.

Proposition 5.5 *Let f be a map-germ given by (3.4). Then $\kappa_i(0) = b_{30}$ holds.*

Let us consider the geometric meaning of the invariant κ_i . Since $\eta f = 0$, $\eta\eta f$ points in the direction in which the cusp comes, and

$$\det(\xi f, \xi\xi\xi f, \eta\eta f) = \langle \xi f \times \xi\xi\xi f, \eta\eta f \rangle.$$

So, the invariant κ_i measures the difference between the vector $\xi f \times \xi\xi\xi f$ and the cusp direction $\eta\eta f$. Here, $\xi f \times \xi\xi\xi f = \hat{\gamma}'(s) \times \hat{\gamma}'''(s)$, where s is the arc-length of $\hat{\gamma}$. Let κ and τ be the curvature and torsion of $\hat{\gamma}$, and assume $\kappa > 0$. Consider the Frenet frame $\{t, n, b\}$ and assume that $\hat{\gamma}' \times \hat{\gamma}'''$ is constant. Then, by the Frenet-Serret formulas, it holds that

$$\kappa^2 \tau t + (-2\kappa' \tau - \kappa \tau')n + (\kappa'' - \kappa \tau^2)b \equiv 0.$$

Since $\kappa > 0$, we have $\tau \equiv 0$ and $\kappa'' \equiv 0$. This means that $\hat{\gamma}$ is a plane curve and a clothoid, or a circle.

6 Geometric Invariants up to Order Three

Let f be a map-germ given by (3.4). Then

$$\begin{aligned} \kappa &= \sqrt{a_{20}^2 + b_{20}^2}, & \tau &= \frac{a_{20}b_{30} - b_{20}a_{30}}{a_{20}^2 + b_{20}^2}, & \kappa_s &= a_{20}, \\ \kappa_n &= b_{20}, & \kappa_c &= b_{03}, & \kappa_t &= b_{12}, & \kappa_i &= b_{30} \end{aligned}$$

at 0. We see that κ is written in terms of a_{20} and b_{20} . However, the other six invariants are independent of each other. Moreover, they determine all the third-order coefficients of the normal form (3.4). Therefore, we have the following theorem.

Theorem 6.1 *Let $f, g: (R^2, 0) \rightarrow (R^3, 0)$ be map-germs, and let 0 be cuspidal edges that have the same invariants $\tau, \kappa_s, \kappa_n, \kappa_c, \kappa_t$ and κ_i at 0 respectively, and let $\kappa_n(0) \neq 0$. Then there exists a diffeomorphism-germ $\varphi: (R^2, 0) \rightarrow (R^2, 0)$ and an isometry-germ $\Phi: (R^3, 0) \rightarrow (R^3, 0)$ that satisfies*

$$f(u, v) - \Phi(g(\varphi(u, v))) = O(4),$$

where $O(4) = \{h(u, v): (R^2, 0) \rightarrow (R^3, 0) | (\partial^{i+j} / \partial u^i \partial v^j) h(0) = 0, i + j \leq 3\}$. Using the differential of invariants, if f and g have the same invariants $\kappa_s, \kappa_n, \kappa_c, \kappa_t, \kappa'_s$, and κ'_n at 0, then the same assertion holds.

Proof By the formula for $\tau, \kappa_s, \kappa_n, \kappa_c, \kappa_t,$ and $\kappa_i,$ if $\kappa_n \neq 0,$ these six values determine all the coefficients $a_{20}, a_{30}, b_{20}, b_{30}, b_{12},$ and b_{03} in (3.4). Thus, we have the result. By the same arguments, the second claim is proven by Theorem 4.4. ■

We remark that for given real numbers $\tau, \kappa_s, \kappa_n, \kappa_c, \kappa_t,$ and $\kappa_i,$ there exists a map-germ f at 0 such that 0 is a cuspidal edge, and its six invariants at 0 are $\tau, \kappa_s, \kappa_n, \kappa_c, \kappa_t$ and $\kappa_i,$ respectively, just by substituting these real numbers into (3.4) and applying $h(u, v) = 0.$ For global realization of fronts, see [18]. In Figure 1 we have drawn surfaces that are images of map-germs given by (3.4). Invariants not specified are zero.

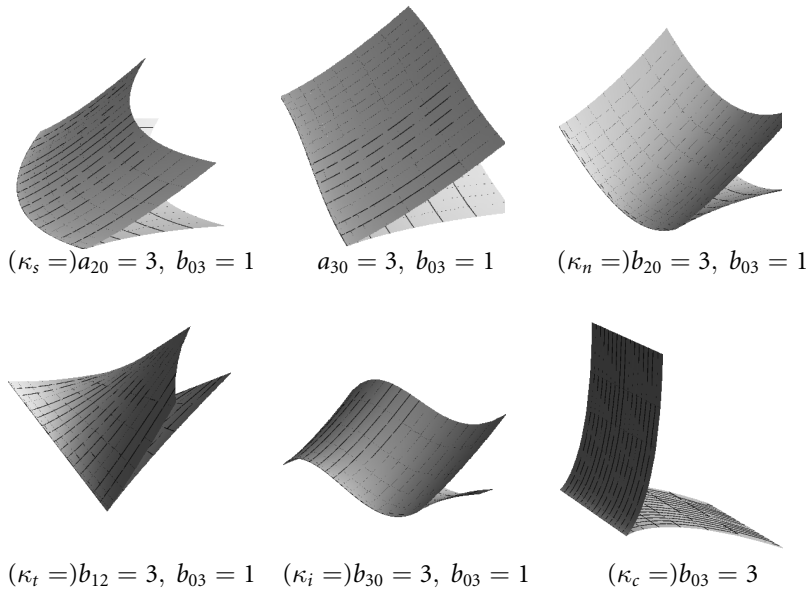


Figure 1: Invariants of cuspidal edges

6.1 Example: Tangent Developable

Let $\hat{\gamma}: R \rightarrow R^3$ be a unit speed space curve that has curvature $\kappa(u) > 0$ and torsion $\tau(u) \neq 0,$ for all $u \in I,$ and let $\{t, n, b\}$ be the Frenet frame. Let $f: I \times R \rightarrow R^3$ be given by $f(u, v) = \hat{\gamma}(u) + v\hat{\gamma}'(u).$ Then f is called a *tangent developable surface* (see Figure 2). Then $S(f) = \{(u, 0)\}$ and $(u, 0)$ is a cuspidal edge. The unit normal vector field is $b(u),$ and the area density function is proportional to $v.$ Therefore, taking $\xi = \partial_u$ and $\eta = -\partial_u + \partial_v,$ (ξ, η) is an adapted pair of vector fields. So, by the Frenet formulas, we have

$$\xi f = t + v\kappa n, \quad \xi\xi f = \kappa n + v(\kappa n)', \quad \text{and} \quad \xi\xi\xi f = -\kappa^2 t + \kappa' n + \kappa\tau b + v(\kappa n)''.$$

Thus,

$$\xi f = t, \quad \xi\xi f = \kappa n, \quad \xi\xi\xi f = -\kappa^2 t + \kappa' n + \kappa\tau b \quad (\text{on } S(f)).$$

Furthermore, we have

$$\eta f = -\nu\kappa n, \quad \eta\eta f = \nu(\kappa n)' - \kappa n, \quad \xi\eta\eta f = \nu(\kappa n)'' + \kappa^2 t - \kappa' n - \kappa\tau b$$

and

$$\eta\eta\eta f = -\nu(\kappa n)''' + 2(-\kappa^2 t + \kappa' n + \kappa\tau b).$$

Thus, it holds that

$$\eta\eta f = -\kappa n, \quad \xi\eta\eta f = \kappa^2 t - \kappa' n - \kappa\tau b, \quad \eta\eta\eta f = 2(-\kappa^2 t + \kappa' n + \kappa\tau b) \quad (\text{on } S(f)).$$

Finally, since $\eta\lambda = \nu\kappa' - \kappa$, it holds that $\text{sgn}(\eta\lambda) = -1$. Therefore, we obtain

$$\kappa_s = -\kappa(u), \quad \kappa_n = 0, \quad \kappa_c = -\frac{2\tau(u)}{\sqrt{\kappa(u)}}, \quad \kappa_t = \tau(u), \quad \kappa_i = -\kappa(u)\tau(u) \quad \text{at } (u, 0).$$

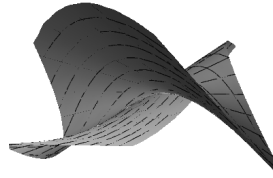


Figure 2: Tangent developable surface.

Appendix A Global Property of Cusp Directional Torsion

In this subsection, we consider a global property of the function κ_t .

Lemma A.1 *Let $f: (R^2, 0) \rightarrow (R^3, 0)$ be a frontal and 0 a non-degenerate singularity whose singular direction and null direction are transversal. Let M be the image of f . Then a slice locus $M \cap N_0M$ is a curve $\hat{\sigma}$ with $\hat{\sigma}'(0) = 0$ and $\hat{\sigma}''(0) \neq 0$.*

Proof Let (u, ν) be an adapted coordinate system, since we can take it by the assumption of f . Since $\langle f(u, \nu), f_u(0) \rangle_u \neq 0$ at 0, there exists a function $u(\nu)$ ($u(0) = 0$) such that $\langle f(u(\nu), \nu), f_u(0) \rangle_u \equiv 0$. We set $\sigma(\nu) = (u(\nu), \nu)$. Then the image $\hat{\sigma}(\nu) = f \circ \sigma(\nu)$ coincides with the slice locus $M \cap N_0M$. Remark that since ∂_ν is a null direction, so $\langle f(u, \nu), f_u(0) \rangle_\nu(0) = 0$, and thus $u'(0) = 0$ holds. By a calculation, we have $\hat{\sigma}'(0) = 0$, $\hat{\sigma}''(0) = f_u(0)u''(0) + f_{\nu\nu}(0)$. Since 0 is a non-degenerate singularity, $\det(f_u, f_{\nu\nu}, \nu)(0) \neq 0$ holds. Therefore, we have $\hat{\sigma}''(0) \neq 0$. ■

Lemma A.2 *Let $\sigma: (R, 0) \rightarrow (R^2, 0)$ be a curve with $\sigma'(0) = 0$ and $\sigma''(0) \neq 0$. Then there exist an orthonormal basis $\{x, y\}$ of R^2 and a parameter ν of $(R, 0)$ such that $j^3\sigma(\nu) = \nu^2x/2 + \alpha\nu^3y/6$, $\alpha \in R$ holds.*

Proof Since $\sigma'(0) = 0$ and $\sigma''(0) \neq 0$, we can set $j^3\sigma(v) = (a_2v^2 + a_3v^3, b_2v^2 + b_3v^3)$, where $a_2, a_3, b_2, b_3 \in R$ and $(a_2, b_2) \neq 0$. We can assume that $a_2 \neq 0$. Set θ satisfying $\sin \theta a_2 + \cos \theta b_2 = 0$. Then we have

$$\tilde{A}_\theta(j^3\sigma(v)) = ((\cos \theta a_2 - \sin \theta b_2)v^2 + (\cos \theta a_3 - \sin \theta b_3)v^3, (\sin \theta a_3 + \cos \theta b_3)v^3)$$

(See (3.1) for \tilde{A}_θ). Set

$$\tilde{v} = \sqrt{2}v((\cos \theta a_2 - \sin \theta b_2) + v(\cos \theta a_3 - \sin \theta b_3))^{1/2}.$$

Then $\tilde{A}_\theta(j^3\sigma(\tilde{v})) = (\tilde{v}^2/2, \alpha\tilde{v}^3)$, $\alpha \in R$, holds. Setting ${}^t x = \tilde{A}_\theta^{-1}({}^t(1, 0))$ and ${}^t y = \tilde{A}_\theta^{-1}({}^t(0, 1))$, we have the result. ■

Let Σ be a two dimensional manifold and $f: \Sigma \rightarrow R^3$ a frontal. Let $\gamma: S^1 \rightarrow \Sigma$ be a simple closed curve consists only of non-degenerate singularities whose singular direction and null direction are transversal, namely $df(\gamma') \neq 0$. In [10], this type of singularities are called non-degenerate singular points of the *second kind*. Denote $\hat{\gamma} = f \circ \gamma$. Let u be an arclength parameter of $\hat{\gamma}$ and d_1, d_2 an orthonormal frame along $\hat{\gamma}$, namely an orthonormal frame field of the normal plane $(\hat{\gamma}')^\perp$ of $\hat{\gamma}$. Then we have

$$\begin{pmatrix} e' \\ d'_1 \\ d'_2 \end{pmatrix} = \begin{pmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{pmatrix} \begin{pmatrix} e \\ d_1 \\ d_2 \end{pmatrix},$$

where, $e = \hat{\gamma}'$. If the curvature of $\hat{\gamma}$ does not vanish and e, d_1, d_2 is the Frenet frame, then $c_1 = \kappa, c_2 = 0, c_3 = \tau$.

For a sufficiently small ε , a map $(t_1, t_2, t_3) \mapsto \hat{\gamma}(t_1) + t_2d_1 + t_3d_2, (-\varepsilon < t_2, t_3 < \varepsilon)$ is diffeomorphic. Thus by Lemmas A.1 and A.2, f can be represented as $\hat{\gamma}(u) + v^2x(u)/2 + \alpha v^3y(u)/6 + v^4z(u, v)$, where $x(u), y(u), z(u, v)$ are vector fields along $\hat{\gamma}(u)$ and $\{x(u), y(u)\}$ is an orthonormal basis of $(\hat{\gamma}(u))^\perp$. Remark that $\eta\eta f$ is proportional to x . Then there exists a function $\theta(u)$ such that

$$\begin{aligned} x(u) &= \cos \theta(u)d_1(u) - \sin \theta(u)d_2(u), \\ y(u) &= \sin \theta(u)d_1(u) + \cos \theta(u)d_2(u) \end{aligned}$$

hold. By a direct calculation, $\kappa_t(u) = c_3(u) - \theta'(u)$ holds. Hence, $\frac{1}{2\pi} \int_\gamma (c_3(u) - \kappa_t(u))du \in Z$ holds. This integer n is called the *intersection number of the frame $(\hat{\gamma}, x)$ with respect to the frame $(\hat{\gamma}, d_1)$* . Thus, we have $\int_\gamma \kappa_t(u)du = \int_\gamma c_3(u)du - 2\pi n$. On the other hand, if the curvature of $\hat{\gamma}$ never vanish, then one can take d_1 as the principal normal vector. Then c_3 is the torsion of $\hat{\gamma}$. Hence, $\int_\gamma \kappa_t(u)du$ is equal to the difference of the total torsion of $\hat{\gamma}$ between intersection number of $(\hat{\gamma}, x)$ with respect to $(\hat{\gamma}, d_1)$.

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