On the Theory of Continued Fractions.

(Second Paper).

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The present paper is a further contribution towards the object defined in my former paper,* namely, to derive the principal known results regarding Continued Fractions, and some new theorems, by transforming the functions considered from infinite series to Continued Fractions, by use of the theory of determinants.

Part III.

In Part III. we shall show that the continued fractions given by Gauss † and Heine ‡ for the quotient of two hypergeometric or generalised hypergeometric functions may be obtained by a direct use of Heilermann's § transformation.

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* *Proc. Edin. Math. Soc.* **34** (1916), pp. 109–132.

† *Disquisitiones generales circa seriem infinitam* $1+\frac{\alpha\beta}{1\,.\,\gamma}\,x+\,\ldots\ldots\ldots$ *Gesammelte Werke* **3**.

‡ *Untersuchungen über die Reihe* $1+\frac{(1-q^{\alpha})\,(1-q^{\beta})}{(1-q)\,(1-q^{\gamma})}\,x+\,\ldots$. *Journal für Math.* **34**.

§ *Journal für Math.* **33** (1845), p. 174.

Heilermann's formula may be stated thus:

"*If the quotient* $\dfrac{A_0+A_1x+A_2x^2+\ldots}{B_0+B_1x+B_2x^2+\ldots}$ *be converted into a continued fraction of the form* $\dfrac{b_0}{1+}\,\dfrac{b_1x}{1+}\,\dfrac{b_2x}{1+}\,\ldots\ldots\ldots$, *then the elements of the continued fraction are*

$$b_0=A_0/B_0$$

$$b_{2n}=-\frac{\Delta_{2n-3}}{\Delta_{2n-2}}\,\frac{\Delta_{2n}}{\Delta_{2n-1}}$$

$$b_{2n+1}=-\frac{\Delta_{2n-2}}{\Delta_{2n-1}}\,\frac{\Delta_{2n+1}}{\Delta_{2n}}$$

where e.g. $\Delta_{2\times 2-1}=\begin{vmatrix} A_3 & A_2 & B_2 & B_3 \\ A_2 & A_1 & B_1 & B_2 \\ A_1 & A_0 & B_0 & B_1 \\ A_0 & 0 & 0 & B_0 \end{vmatrix}$

and $\Delta_{2\times 2}=\begin{vmatrix} A_4 & A_3 & A_2 & B_3 & B_4 \\ A_3 & A_2 & A_1 & B_2 & B_3 \\ A_2 & A_1 & A_0 & B_1 & B_2 \\ A_1 & A_0 & 0 & B_0 & B_1 \\ A_0 & 0 & 0 & 0 & B_0 \end{vmatrix}$."

These determinants are of the kind known as *bigradients.* It will appear that in the present case they are all factorisable, a circumstance to which the success of the method is largely due.

By use of this transformation we shall now convert into continued fractions the following expressions

$(O)\quad F(\alpha,\ \beta+d_2,\ \gamma+d_3,\ x)\div F(\alpha,\ \beta,\ \gamma,\ x)$

$(P)\quad F(\beta+d_2,\ \gamma+d_3,\ x)\div F(\beta,\ \gamma,\ x)$

$(Q)\quad F(\alpha,\ \gamma+d_3,\ x)\div F(\alpha,\ \gamma,\ x)$

$(R)\quad F(\gamma+d_3,\ x)\div F(\gamma,\ x)$

$(S)\quad F(\alpha,\ \beta+d_2,\ \gamma,\ x)\div F(\alpha,\ \beta,\ x)$

$(T)\quad F(\beta+d_2,\ \gamma,\ x)\div F(\beta,\ \gamma,\ x)$

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where

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1!\gamma} c x + \frac{\alpha(\alpha+d_1)\beta(\beta+d_2)}{2!\gamma(\gamma+d_3)} c^2 x^2 + \dots$$

$$F(\alpha, \beta+d_2, \gamma+d_3, x) = 1 + \frac{\alpha(\beta+d_2)}{1!(\gamma+d_3)} c x + \dots$$

$$(U) \quad \phi(\alpha, \beta+1, \gamma+1, q, x) \div \phi(\alpha, \beta, \gamma, q, x)$$

$$(V) \quad \phi(\beta+1, \gamma+1, q, x) \div \phi(\beta, \gamma, q, x)$$

$$(W) \quad \phi(\alpha, \gamma+1, q, x) \div \phi(\alpha, \gamma, q, x)$$

$$(X) \quad \phi(\gamma+1, q, x) \div \phi(\gamma, q, x)$$

$$(Y) \quad \phi(\alpha, \beta+1, q, x) \div \phi(\alpha, \beta, q, x)$$

$$(Z) \quad \phi(\beta+1, \gamma, q, x) \div \phi(\beta, \gamma, q, x)$$

where $$\phi(\alpha, \beta, \gamma, q, x) = 1 + \frac{(a+q^\alpha)(b+q^\beta)}{(1-q)(c+q^\gamma)} x + \dots$$

In the above series some additional quantities such as the a's, b's, c's and d's have been introduced in order that we may pass readily from one determinant to another; they are not introduced, as might at first sight be supposed, for the purpose of generalising the series or the determinants. The quantity c in the hypergeometric series is introduced in order to effect simul taneous changes, if necessary, in the series and in the continued fractions without altering the values of x.

The determinants will be denoted according to the titles of the sections to which they belong, so that O_n denotes the bigradient of the $(n+1)^{\text{th}}$ order occurring in section (O)

(*O*).

On $O_{2\times3-1}$ performing the operations

$$(1) \quad \text{col}_6 - \text{col}_1 = c\,\frac{\alpha(d_3\beta - d_2\gamma)}{\gamma(\gamma+d_3)}\,\text{col}_6$$

$$\text{col}_5 - \text{col}_2 = c\,\frac{\alpha(d_3\beta - d_2\gamma)}{\gamma(\gamma+d_3)}\,\text{col}_5'$$

$$\text{col}_4 - \text{col}_3 = c\,\frac{\alpha(d_3\beta - d_2\gamma)}{\gamma(\gamma+d_3)}\,\text{col}_4'$$

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$$(2) \quad \text{col}_2 - \text{col}_6' = c\,\frac{(\beta+d_2)\,(d_3\,\alpha - d_1\,\gamma - d_1\,d_3)}{(\gamma+d_3)\,(\gamma+2d_3)}\,\text{col}_2'$$

$$\text{col}_3 - \text{col}_5' = c\,\frac{(\beta+d_2)\,(d_3\,\alpha - d_1\,\gamma - d_1\,d_3)}{(\gamma+d_3)\,(\gamma+2d_3)}\,\text{col}_3'$$

we obtain as a co-factor the determinant

$$|\ \text{col}_2'\ \ \text{col}_3'\ \ \text{col}_4'\ \ \text{col}_5'\ |\,.$$

If in this determinant we put $\alpha - d_1$, $\beta - d_2$, $\gamma - 2d_3$ for α, β, γ respectively, we obtain the determinant $O_{2\times 2-1}$

$$O_5 = (-c)^{5^2}\,[\alpha^3\,(\alpha+d_1)^2\,(\alpha+2d_2)]$$
$$[(d_3\,\beta - d_2\,\gamma)^3\,(d_3\,\beta - d_2\,\gamma - d_2\,d_3)^2\,(d_3\,\beta - d_2\,\gamma - 2d_2\,d_3)]$$
$$\times\,\frac{[(\beta+d_2)^2\,(\beta+2d_2)]\ [d_3\,\alpha - d_1\,\gamma - d_1\,d_3)^2\,(d_3\,\alpha - d_1\,\gamma - 2d_2\,d_3)]}{\gamma^3\,(\gamma+d_3)^5\,(\gamma+2d_3)^4\,(\gamma+3d_3)^5\,(\gamma+4d_3)^2\,(\gamma+5d_2)}.$$

Similarly,

$$O_{2n-1} = (-c)^{n^2}\,[\alpha^n\,(\alpha+d_1)^{n-1} \ldots \{\alpha+(n-1)\,d_1\}]$$
$$[(d_3\,\beta - d_2\,\gamma)^n \ldots \{d_3\,\beta - d_2\,\gamma - (n-1)\,d_2\,d_3\}]$$
$$[(\beta+d_2)^{n-1}\,(\beta+2d_2)^{n-2} \ldots \{\beta+(n-1)\,d_2\}]$$
$$\times\,\frac{[(d_3\,\alpha - d_1\,\gamma - d_1\,d_3)^{n-1} \ldots \{d_3\,\alpha - d_1\,\gamma - (n-1)\,d_2\,d_3\}]}{\gamma^n\,(\gamma+d_3)^{2n-1}\,(\gamma+2d_3)^{2n-2} \ldots \{\gamma+(2n-1)\,d_3\}}$$

$$O_{2n} = c^{n\,(n+1)}\,[\alpha^n \ldots \{\alpha+(n-1)\,d_1\}]\,[(d_3\,\beta - d_2\gamma)^n \ldots (d_3\,\beta - d_2\gamma - nd_2\,d_3)]$$
$$\times\,\frac{[(\beta+d_2)^n \ldots (\beta+nd_2)]\ [(d_3\,\alpha - d_1\,\gamma - d_2\,d_3)^n \ldots (d_3\,\alpha - d_1\,\gamma - nd_1\,d_3)]}{\gamma^n\,(\gamma+d_3)^{2n}\,(\gamma+2d_3)^{2n-1} \ldots (\gamma+2nd_3)}.$$

Hence *the elements of the continued fraction are determined by the equations*

$$b_{2n} = c\,\frac{(\beta+nd_2)\,(d_3\,\alpha - d_1\,\gamma - nd_1\,d_3)}{(\gamma+2nd_3)\{\gamma+(2n-1)\,d_3\}}$$

$$b_{2n+1} = c\,\frac{(\alpha+nd_1)\,(d_3\,\beta - d_2\,\gamma - nd_2\,d_3)}{(\gamma+2nd_3)\{\gamma+(2n+1)\,d_3\}}$$

(*P*). Here the results can be obtained from the corresponding results of (*O*) by making $\alpha = 1$, $d_1 = 0$.

(*Q*). Here $\beta = 1$, $d_2 = 0$.

(*R*). Here $\alpha = \beta = 1$, $d_1 = d_2 = 0$.

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Ex. If $d_3=1$, $\gamma=\frac{1}{2}$, and $c=\frac{1}{4}$, we see that

$$\begin{vmatrix} \frac{1}{7!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{6!} \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{4!} \\ \frac{1}{3!} & 1 & 1 & \frac{1}{2!} \\ 1 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{3^3 . 5^2 . 7}.$$

In this particular case

$$R_{2n-1}=(-1)^n \frac{1}{3^{2n-1} . 5^{2n-2} \ldots (4n-3)^2 . (4n-1)}$$

$$R_{2n}=\frac{1}{3^{2n}\, 5^{2n-2} \ldots (4n+1)}.$$

(*S*). Here $\gamma=1$, $d_3=0$.

(*T*).

It may easily be shown that

$$T_{2n-1}=\left(-\frac{cd_2}{\gamma}\right)^n Q'_{2n-2} \text{ or } = \left(\frac{cd_2}{\gamma}\right)^n P_{2n-2}$$

and
$$T_{2n}=\left(\frac{cd_2}{\gamma}\right)^n Q'_{2n-1}.$$

If in Q'_m we put α for $\beta+d_2$ and d_1 for d_2, we obtain Q_m. Thus we have

$$b_{2n}=cd_3 \frac{\beta+nd_2}{\{\gamma+(2n-2)\,d_3\}\ \{\gamma+(2n-1)\,d_3\}}$$

$$b_{2n+1}=c\,\frac{d_3\beta-d_2\gamma-(n-1)\,d_2 d_3}{\{\gamma+(2n-1)\,d_3\}\,(\gamma+2nd_3)}.$$

(*U*).

On $U_{2\times 2-1}$ performing the operations

$$(1)\quad \text{col}_4-\text{col}_1=\frac{(a+q^{\alpha})\,(cq^{\beta}-bq^{\gamma})}{(c+q^{\gamma})\,(c\,+\,q^{\gamma+1})}\,\text{col}_4'$$

$$\text{col}_3-\text{col}_2=\frac{(a+q^{\alpha})\,(cq^{\beta}-bq^{\gamma})}{(c+q^{\gamma})\,(c\,+\,q^{\gamma+1})}\,\text{col}_3'$$

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$$(2) \quad \text{col}_2 - \text{col}_4' = \frac{(b+q^{\beta+1})\,(cq^{\alpha} - aq^{\gamma+1})}{(c+q^{\gamma+1})\,(c+q^{\gamma+2})},$$

we obtain as a co-factor the determinant

$$|\ \text{col}_2'\ \text{col}_3'\ |.$$

In this last determinant, if we put $\alpha-1$, $\beta-1$, $\gamma-2$ for α, β, γ respectively, we obtain $\Delta_{2\times 2-1}$.

$$U_{2n-1} = (-1)^n[(a+q^{\alpha})^n \ldots (a+q^{\alpha+n-1})]$$

$$[(cq^{\beta} - bq^{\gamma})^n\,(cq^{\beta+1} - bq^{\gamma+2})^{n-1} \ldots (cq^{\beta+n-1} - bq^{\gamma+2n-2})]$$

$$\times \frac{[(b+q^{\beta+1})^{n-1} \ldots (b+q^{\beta+n-1})]\,[(cq^{\alpha} - aq^{\gamma+1})^{n-1} \ldots (cq^{\alpha+n-2} - aq^{\gamma+2n-3})]}{(c+q^{\gamma})^n\,(c+q^{\gamma+1})^{2n-1}\,(c+q^{\gamma+2})^{2n-2} \ldots (c+q^{\gamma+2n-1})}$$

$$U_{2n} = [(a+q^{\alpha})^n \ldots (a+q^{\alpha+n-1})]\,[(cq^{\beta} - bq^{\gamma})^n \ldots (cq^{\beta+n-1} - bq^{\gamma+2n-2})]$$

$$\times \frac{[(b+q^{\beta+1})^n \ldots (b+q^{\beta+n})]\,[(cq^{\alpha} - aq^{\gamma+1})^n \ldots (cq^{\alpha+n-1} - aq^{\gamma+2n-1})]}{(c+q^{\gamma})^n\,(c+q^{\gamma+1})^{2n}\,(c+q^{\gamma+2})^{2n-1} \ldots (c+q^{\gamma+2n})}$$

giving

$$b_{2n} = \frac{(b+q^{\beta+n})\,(cq^{\alpha+n-1} - aq^{\gamma+2n-1})}{(c+q^{\gamma+2n-1})\,(c+q^{\gamma+2n})}$$

$$b_{2n+1} = \frac{(a+q^{\alpha+n})\,(cq^{\beta+n} - bq^{\gamma+2n})}{(c+q^{\gamma+2n})\,c+q^{\gamma+2n+1})}.$$

(V). Here the results can be obtained from the corresponding results of (U) by making

$$a=1 \text{ and } q^{\alpha}=0.$$

(W). Here $b=1$, $q^{\beta}=0$.

(X). Here $a=b=1$, $q^{\alpha}=q^{\beta}=0$.

(Y). Here $c=1$, $q^{\gamma}=0$.

(Z).

Here

$$Z_{2n-1}=(-1)^{\frac{1}{2}n(n-1)}\left[(b+q^{\beta+1})^{n-1}\ \dots\ (b+q^{\beta+n-1})\right]$$

$$\left[q^{n\beta}(cq^{\beta+1}-bq^{\gamma+1})^{n-1}\ \dots\ (cq^{\beta+n-1}-q^{\gamma+2n-3})\right]$$

$$\times\frac{q^{\{(n-1)\gamma+(n-2)(\gamma+2)+\dots+1.(\gamma+2n-4)\}}}{(c+q^{\gamma})^{2n-1}(c+q^{\gamma+1})^{2n-2}\ \dots\ (c+q^{\gamma+2n-2})}$$

$$Z_{2n}=(-1)^{\frac{1}{2}n(n+1)}\left[(b+q^{\beta+1})^{n}\dots(b+q^{\beta+n})\right]$$

$$\left[q^{n\beta}(cq^{\beta+1}-bq^{\gamma+1})^{n-1}\dots(cq^{\beta+n-1}-bq^{\gamma+2n-3})\right]$$

$$\times\frac{q^{\{n\gamma(n-1)(\gamma+2)\ \dots\ +\ 1.(\gamma+2n-2)\}}}{(c+q^{\gamma})^{2n}(c+q^{\gamma+1})^{2n-1}\dots(c+q^{\gamma+2n-1})},$$

and so

$$b_{2n}=-q^{\gamma+2n-2}\ \frac{b+q^{\beta+n}}{(c+q^{\gamma+2n-2})}$$

$$b^{2n+1}=\frac{cq^{\beta+n}-bq^{\gamma+2n-1}}{(c+q^{\gamma+2n-1})(c+q^{\gamma+2n})}.$$

Part IV.

In Part III. we have been dealing with the conversion of a quotient

$$\frac{A_0+A_1x+A_2x^2+\dots}{B_0+B_1x+B_2x^2+\dots}$$

into a continued fraction. *We shall now examine the relation which this bears to the continued fraction obtained from the quotient*

$$\frac{(B_0+B_1x^2+B_2x^4+\dots)+x(A_0+A_1x^2+A_2x^4+\dots)}{(B_0+B_1x^2+B_2x^4+\dots)-x(A_0+A_1x^2+A_2x^4+\dots)}$$

when we take the continued fraction for this quotient in the form

$$\frac{1}{a_0+}\frac{x}{a_1+}\frac{x}{a_2+}\frac{x}{a_3+}\dots$$

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It will be shown that the two continued fractions are closely onnected : in fact, *the elements of the latter continued fractions are*

$$a_0 = 1\ ; \quad a_1 = -B_0/2A_0\ ; \quad a_2 = -2\ ; \quad a_3 = -A_0^2/2\Delta_1\ ; \quad a_4 = 2$$

$$a_{4n} = -a_{4n+2} = 2$$

$$a_{4n+1} = -\tfrac{1}{2}\,\frac{\Delta^2_{2n-1}}{\Delta_{2n-1}\,\Delta_{2n}}; \quad a_{4n+3} = -\tfrac{1}{2}\,\frac{\Delta^2_{2n}}{\Delta_{2n-1}\,\Delta_{2n+1}}$$

where Δ's are the same bigradients as those of Part III.

To establish this result, let the bigradients be denoted by Δ''s, so

$$\Delta_7' = \begin{vmatrix} A_3 & B_3 & A_2 & B_2 & B_2 & -A_2 & B_3 & -A_3 \\ B_3 & A_2 & B_2 & A_1 & -A_1 & B_2 & -A_2 & B_3 \\ A_2 & B_2 & A_1 & B_1 & B_1 & -A_1 & B_2 & -A_2 \\ B_2 & A_1 & B_1 & A_0 & -A_0 & B_1 & -A_1 & A_2 \\ A_1 & B_1 & A_0 & B_0 & B_0 & -A_0 & B_1 & -A_1 \\ B_1 & A_0 & B_0 & 0 & 0 & B_0 & -A_0 & B_1 \\ A_0 & B_0 & 0 & 0 & 0 & 0 & B_0 & -A_0 \\ B_0 & 0 & 0 & 0 & 0 & 0 & 0 & B_0 \end{vmatrix}.$$

If on this determinant we perform the operations

$$\begin{aligned} (1) \quad & \text{col}_8 + \text{col}_1 = 2\,\text{col}_8' \\ & \text{col}_7 + \text{col}_2 = 2\,\text{col}_7' \\ & \text{col}_6 + \text{col}_3 = 2\,\text{col}_6' \\ & \text{col}_5 + \text{col}_4 = 2\,\text{col}_5' \end{aligned}$$

$$(2) \quad \text{col}_1 - \text{col}_8'\ ;\ \text{col}_2 - \text{col}_7'\ ;\ \text{col}_3 - \text{col}_6'\ ;\ \text{col}_4 - \text{col}_5',$$

we obtain another determinant whose columns with columns and rows with rows can be interchanged in such a manner as to make all the elements of the upper right hand quarter of the determinant zeros.

$$\Delta_7' = B_0\,2^4 \begin{vmatrix} A_2 & A_1 & B_2 \\ A_1 & A_0 & B_1 \\ A_0 & 0 & B_0 \end{vmatrix} \begin{vmatrix} A_3 & A_2 & B_2 & B_3 \\ A_2 & A_1 & B_1 & B_2 \\ A_1 & A_0 & B_0 & B_1 \\ A_0 & 0 & 0 & B_0 \end{vmatrix} = B_0\,2^4\,\Delta_2\,\Delta_3$$

$$\Delta'_{4n} = (-1)^n B_0 \, 2^{2n} \, \Delta^2_{2n-1}$$

$$\Delta'_{4n+1} = B_0 \, 2^{2n+1} \, \Delta_{2n-1} \, \Delta_{2n}$$

$$\Delta'_{4n+2} = (-1)^{n+1} B_0 \, 2^{2n+1} \, \Delta^2_{2n}$$

$$\Delta'_{4n+3} = B_0 \, 2^{2n+2} \, \Delta_{2n} \, \Delta_{2n+1} \, .$$

We know from Heilermann's transformation that

$$a_{2n} = \frac{\Delta'^2_{2n-1}}{\Delta'_{2n-2} \, \Delta'_{2n}} \; ; \quad a_{2n+1} = - \frac{\Delta'^2_{2n}}{\Delta'_{2n-1} \, \Delta'_{2n+1}} \, .$$

The rest follows easily.

If we make use of the results of section (R), the quotient becomes

$$\frac{F(\gamma, cx^2) + xF(\gamma+1, cx^2)}{F(\gamma, cx^2) - xF(\gamma+1, cx^2)}$$

and $$a_{4n+1} = -\tfrac{1}{2}\,\frac{\gamma+2n}{\gamma} \; ; \quad a_{4n+3} = \frac{1}{2c}\,\gamma\,(\gamma+2n+1).$$

Putting $\frac{1}{2}$ for γ and $-\frac{1}{4}$ for c, we obtain

$$\frac{\cos x + \sin x}{\cos x - \sin x} = \frac{1}{1-}\,\frac{2x}{1+}\,\frac{x}{1+}\,\frac{x}{3-}\,\frac{x}{1-}\,\frac{x}{5+}\,\frac{x}{1+}\,\frac{x}{7-}\dots .$$

$$\therefore \quad \tan x = \frac{x}{1-x+}\,\frac{x}{1+}\,\frac{x}{3-}\,\frac{x}{1-}\,\frac{x}{5+}\,\frac{x}{1-}\,\frac{x}{7-}\,\frac{x}{1-}\,\frac{x}{9+}\,\frac{x}{1+}\,\frac{x}{11-}\,\frac{x}{1-}\dots$$

In connexion with the above result, we can establish *a simple rule for the extension of a continued fraction of the form*

$$\frac{1}{\alpha_0+}\,\frac{y}{\alpha_1+}\,\frac{y}{\alpha_2+}\,\frac{y}{\alpha_3+}\dots ,$$

namely, its extended form is

$$\frac{1}{\alpha_0 - x -}\,\frac{x}{a_2+}\,\frac{x}{a_3+}\,\frac{x}{a_4+}\dots ,$$

where $$x = \sqrt{y} \; ; \quad a_{4n} = -a_{4n+2} = 1$$

$$a_{4n+1} = -\alpha_{2n} \; ; \; a_{4n+3} = \alpha_{2n+1} \, .$$

The $2n^{\text{th}}$ convergent of the latter is the same as the n^{th} convergent of the former.

II. We shall next consider the fractions which are reciprocal to those of Part III.

The quotient, when inverted, is

$$\frac{B_0 + B_1 x + B_2 x^2 + \dots}{A_0 + A_1 x + A_2 x_2 + \dots};$$

we shall suppose that it is converted into a continued fraction of the same form as in Part III., viz.

$$\frac{k_0}{1} + \frac{k_1 x}{1} + \frac{k_2 x}{1} + \dots .$$

The elements of the continued fraction are

$$k_{2n} = -\frac{\Delta''_{2n-3}}{\Delta''_{2n-2}} \frac{\Delta''_{2n}}{\Delta''_{2n-1}}$$

$$k_{2n+1} = -\frac{\Delta''_{2n-2}}{\Delta''_{2n-1}} \frac{\Delta''_{2n+1}}{\Delta''_{2n}}$$

where $\Delta''_{2n-1} = (-1)^n \Delta_{2n-1}$.

And if in Δ''_{2n} the A's and B's are interchanged, we obtain Δ_{2n}.

We shall now consider some special cases.

(f) First, let us take for the A's and B's the following values

$$A_0 = B_0 = 1$$

$$A_n = \frac{1}{n!\,(\gamma+1)\dots(\gamma+n)}$$

$$B_n = \frac{1}{n!\,(\gamma+1)\dots(\gamma+n-1)}.$$

Then we can show that

$$k_{2n} = -\frac{(n+1)(\gamma+n)}{n(\gamma+n-1)} \frac{1}{(\gamma+2n-1)(\gamma+2n+1)}$$

$$k_{2n+1} = \frac{n(\gamma+n-1)}{(n+1)(\gamma+n)} \frac{1}{(\gamma+2n)(\gamma+2n+1)}$$

$$k_0 = 1; \quad k_1 = -\frac{1}{\gamma(\gamma+1)}.$$

If $\gamma=\frac{1}{2}$ and $x=-\frac{x^2}{4}$, then we have

$$x\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{1}{1}-\frac{x^2}{3(1.1)}+\frac{2.3x^2}{5}+\frac{1.1x^2}{7(2.3)}+\frac{3.5x^2}{9}+\frac{2.3x^2}{11(3.5)}+\frac{4.7x^2}{13}+\dots.$$

Hence

$$\cot x=\frac{1}{1}+\frac{x^2}{1.3}-\frac{6x^2}{5}-\frac{x^2}{6.7}-\frac{15x^2}{9}-\frac{6x^2}{15.11}-\frac{28x^2}{13}-\frac{15x^2}{28.15}-\dots.$$

(*g*) Next we shall take for the A's and B's the special values

$$B_0=A_0=1$$

$$B_n=\frac{\beta(\beta+1)\dots(\beta+n-1)}{n!\,\gamma\dots(\gamma+n-1)}$$

$$A_n=\frac{(\beta+1)\dots(\beta+n)}{n!\,\gamma\dots(\gamma+n-1)}.$$

Then it can be shown that

$$k_{2n}=\frac{\beta-\gamma-n+1}{(\gamma+2n-2)(\gamma+2n-1)}$$

$$k_{2n+1}=\frac{\beta+n}{(\gamma+2n-1)(\gamma+2n)}$$

$$k_1=1/\gamma;\quad k_0=1.$$

In these two cases, viz. (*f*) and (*g*), as well as in many others, it may be shown that if a function is converted into a continued fraction of the form

$$\frac{1}{1}+\frac{a_1x}{1}+\frac{a_2x}{1}+\dots\dots\quad(1)$$

and also of the form

$$1+\frac{k_1x}{1}+\frac{k_2x}{1}+\dots\quad(2),$$

and if $\frac{1}{1}$ is regarded as the first convergent of (1) and 1 as that of (2), then the odd convergents of the one are equal to the corresponding odd convergents of the other, while the even convergents cannot be equal. This may conveniently be proved by expressing the denominators and numerators of the convergents as continuants.

III. Finally, if the quotient

$$\frac{B_0 + A_0 + (B_1 + A_1)\,x + (B_2 + A_2)\,x^2 + \ldots}{B_0 - A_0 + (B_1 - A_1)\,x + (B_2 - A_2)\,x^2 + \ldots}$$

be converted into a continned fraction of the form

$$\frac{1}{b_0 +}\;\frac{b_1\,x}{1}\;+\;\frac{b_2\,x}{1}\;+\;\frac{b_3\,x}{1}\;+\ldots,$$

it is readily shown that

$$b_0 = \frac{B_0 - A_0}{B_1 + A_1}\,; \quad b_1 = \frac{(B_1 - A_1)\,(B_0 + A_0) - (B_1 + A_1)\,(B_0 - A_0)}{(B_0 + A_0)^2}$$

and generally

$$b_{2n} = -\frac{\Delta'''_{2n-3}}{\Delta'''_{2n-2}}\;\frac{\Delta'''_{2n}}{\Delta'''_{2n-1}}$$

$$b_{2n+1} = -\frac{\Delta'''_{2n-2}}{\Delta'''_{2n-1}}\;\frac{\Delta'''_{2n+1}}{\Delta'''_{2n}}$$

where $\Delta'''_{2n-1} = 2^n\,\Delta_{2n-1}\,; \quad \Delta'''_{2n} = 2^n\{\Delta_{2n} + (-1)^n\,\Delta''_{2n}\}.$

If we consider the special case in which

$$A_0 = B_0 = 1$$

$$A_n = \frac{1}{n!\;(\gamma+1)\,\ldots\,(\gamma+n)}$$

$$B_n = \frac{1}{n!\,\gamma\,(\gamma+1)\,\ldots\,(\gamma+n-1)},$$

we can show that

$$b_{2n} = \frac{\gamma + (n+1)(\gamma+n)}{\gamma + n(\gamma+n-1)}\;\frac{1}{(\gamma+2n-1)\,(\gamma+2n)}$$

$$b_{2n+1} = \frac{\gamma + n\,(\gamma+n-1)}{\gamma + (n+1)\,(\gamma+n)}\;\frac{1}{(\gamma+2n)\,(\gamma+2n+1)}$$

$$b_0 = 0\,; \quad \text{and } b_1 = \frac{1}{2\gamma\,(\gamma+1)}\,.$$

Putting $-x^2/4$ for x, $\frac{1}{2}$ for γ, we get

$$b_1 = -\frac{1}{2\times 3}$$

$$b_{2n} = -\frac{1+(n+1)(2n+1)}{1+n(2n-1)} \quad \frac{1}{(4n-1)(4n+1)}$$

$$b_{2n+1} = -\frac{1+n(2n-1)}{1+(n+1)(2n+1)} \quad \frac{1}{(4n+1)(4n+3)}.$$

Hence $\dfrac{\sin x - x\cos x}{\sin x + x\cos x}$ * [* *Cf.* II. (*f*)].

$$= \frac{x^2}{3(1+1.1)-} \; \frac{(1+2.3)x^2}{5\ -} \; \frac{(1+1.1)x^2}{7(1+2.3)\ -} \; \frac{(1+3.5)x^2}{9\ -} \; \frac{(1+2.3)x^2}{11(1+3.5)-\dots}.$$