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TENSOR PRODUCTS OF JORDAN ALGEBRAS

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Let J_1 and J_2 be two Jordan algebras with unit elements. We define various tensor products of J_1 and J_2. The first, which we call the Kronecker product, is the most obvious and is based on the tensor product of the vector spaces. We find conditions sufficient for its existence and for its non-existence. Motivated by the universal mapping property for the tensor product of associative algebras we define, in Section 2, tensor products of J_1 and J_2 by means of a universal mapping property. The tensor products always exist for special Jordan algebras and need not coincide with the Kronecker product when the latter exists. In Section 3 we construct a more concrete tensor product for special Jordan algebras. Here the tensor product of a special Jordan algebra and an associative Jordan algebra coincides with the Kronecker product of these algebras. We show that this "special" tensor product is the natural tensor product for some Jordan matrix algebras.

In the final section we deal with Jordan algebras of operators on a Hilbert space and with JW-algebras, abstract analogues of weakly closed Jordan algebras of operators on a Hilbert space. We generalise the tensor product of Section 3 to define a tensor product of JW-algebras. This is shown to exist when the JW-algebras are hermitian parts of von Neumann factors.

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Preliminaries. The definitions and results of this section can all be found in [**6**].

An *algebra* is taken to be a not necessarily associative algebra, with a unit element, over the field of real numbers. All homomorphisms are assumed to preserve unit elements. For an algebra A we denote by dim A the dimension of the vector space underlying A. We denote the composition $(a, b) \rightarrow (ab\text{-}ba)$ by $[ab]$. We denote the *associator* of the elements a, b and c by $[a, b, c]$, i.e., $[a, b, c] = (ab)c - a(bc)$. The *nucleus* of an algebra A is the set of elements which associate with every pair of elements of the algebra. The *centre* of A is the set of elements of the nucleus which commute with all elements of A. For A_1 and A_2 associative algebras with involutions, we denote their algebraic tensor product with the natural product involution by $A_1 \otimes A_2$. For an associative algebra A we write $\{a_1, a_2, \ldots, a_r\}$ to denote

$$\tfrac{1}{2}(a_1a_2 \ldots a_r + a_r \ldots a_2a_1)$$

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A *Jordan algebra* is a commutative algebra satisfying the identity $(a^2x)a = a^2(xa)$ for arbitrary x and a in the algebra. The linear mapping $x \to xa = ax$ will be denoted by R_a. For an associative algebra A we denote by A^+ the Jordan algebra consisting of the vector space underlying A provided with a product composition $a \cdot b = \frac{1}{2}(ab + ba)$. We denote $a \cdot a$ by $a^{\cdot 2}$. A Jordan algebra J is called *special* if, for some associative algebra A, there is a monomorphism of J into A^+. If A is an associative algebra with involution over a field D we denote by $\mathfrak{h}_D(A)$ the Jordan subalgebra of A^+ made up of the symmetric elements of A. Where there is no possibility of confusion we shall drop the suffix D. We denote the set of antisymmetric elements by $\mathfrak{a}(A)$.

We call a homomorphism of a Jordan algebra J into A^+, where A is an associative algebra, an *associative specialisation* of J in A. A pair $(\mathfrak{S}(J), \pi)$, consisting of an associative algebra $\mathfrak{S}(J)$ and an associative specialisation π of J in $\mathfrak{S}(J)$, is called a *special envelope* for J if, for any associative specialisation μ of J in a associative algebra A, there exists a unique homomorphism ν of $\mathfrak{S}(J)$ in A such that $\nu \circ \pi = \mu$. The special envelope always exists and can be constructed as in [**6**]; it is defined by a universal mapping so it is unique. The associative specialisation π is 1-1 if and only if J is special. There exists a unique involution in $\mathfrak{S}(J)$ leaving $\pi(x)$ invariant, for $x \in J$; we shall always consider $\mathfrak{S}(J)$ as an algebra with this involution.

Remark. The notation of the preceding paragraph is not exactly the same as that in [**6**] since we have assumed that all algebras have unit elements and all homomorphisms are unit-preserving.

We denote the *free associative algebra* with generators $x_1, \ldots, x_r$ by $\Phi\{x_1, \ldots, x_r\}$ or $\Phi^{(r)}\{\quad\}$, the *free nonassociative algebra* by $\Phi\{\{x_1, \ldots, x_r\}\}$. If A is an algebra we say that $f \in \Phi\{\{x_1, \ldots, x_r\}\}$ is an *identity* for A if f is mapped into 0 under every homomorphism of $\Phi\{\{x_1, \ldots, x_r\}\}$ into A. An identity is said to be *multilinear* if it is homogeneous of degree $\leqq 1$ in all the x's. The *free Jordan algebra* $FJ(x_1, \ldots, x_r)$, or $FJ^{(r)}$ is determined from $\Phi\{\{x_1, \ldots, x_r\}\}$ by the identities $\{f, g\}, f = [x_ix_j], g = (x_i^2x_j)x_i - x_i^2(x_jx_i), i, j = 1, \ldots, r$; the *free special Jordan algebra* $FSJ(x_1, \ldots, x_r)$, or $FSJ^{(r)}$, is defined to be the subalgebra of $\Phi\{x_1, \ldots, x_r\}^+$ generated by 1 and the x_i. The elements of $FSJ^{(r)}$ are called *Jordan elements* of $\Phi\{x_1, \ldots, x_r\}$. It is known that

$$\mathfrak{S}(FJ^{(r)}) = \mathfrak{S}(FSJ^{(r)}) = \Phi^{(r)}\{\quad\},$$

where the involution in $\Phi\{x_1, \ldots, x_r\}$ is the reversal operation, i.e., $1^* = 1$ and $(x_{i_1} \ldots x_{i_k})^* = x_{i_k} \ldots x_{i_1}$.

We denote by $\mathscr{M}_n(D)$ the algebra of $n \times n$ matrices with entries in an algebra D with involution, and assume that $\mathscr{M}_n(D)$ is provided with the standard involution [**6**, Chapter I.5]. We are concerned with the cases where D is the field **R**, **C** or **H** of reals, complex numbers or quaternions respectively, or the algebra **O** of octonions. We write $\mathscr{H}_n(D)$ instead of $\mathfrak{h}(\mathscr{M}_n(D))$. We know that $\mathfrak{S}(\mathscr{H}_n(\mathbf{R})) = \mathscr{M}_n(\mathbf{R})$ and $\mathfrak{S}(\mathscr{H}_n(\mathbf{C})) = \mathscr{M}_n(\mathbf{C})$ for $n \geqq 2$. For $n > 2$, one has $\mathfrak{S}(\mathscr{H}_n(\mathbf{H})) = \mathscr{M}_n(\mathbf{H})$ (see [**6**, Chapter 3]).

We denote by $\mathfrak{J}(V, f)$ the Jordan algebra of the form $\Phi \oplus V$, where V is a vector space over a field Φ, with symmetric bilinear form f. It is known that $\mathfrak{S}(\mathfrak{J}(V, f))$ is the *Clifford algebra* $\mathfrak{C}(V, f)$, i.e. the tensor algebra based on V modulo the ideal generated by elements $x \otimes x - f(x, x)1$. The Jordan algebra $\mathfrak{J}(V, f)$ is special and can be realised in $\mathfrak{C}(V, f)^+$; it is generated by 1 and the basis elements (e_i) of V. If one can choose a basis such that $f(e_i, e_j) = \delta_{ij}$, the set of elements e_i in $\mathfrak{C}(V, f)$ is called a *spin system* and $\mathfrak{J}(V, f)$ is called a *spin algebra*; the e_i satisfy the anticommutation relations $e_i \cdot e_j = \frac{1}{2}(e_i e_j + e_j e_i) = 0$, $i \neq j$.

1. The Kronecker product. Let J_1 and J_2 be two Jordan algebras, a_ν, $x \in J_1$, b_ν, $y \in J_2$. Write $((a_i a_j)x)a_k = e_{ijk}$, $(a_i a_j)(x a_k) = \bar{e}_{ijk}$ and a like notation with f's for b_ν, $y \in J_2$. Thus $[a_i a_j, x, a_k] = e_{ijk} - \bar{e}_{ijk}$.

LEMMA 1. *Let J be a free Jordan algebra with generators x, a_ν. The identities, modulo commutation, in J, homogeneous of degree 2 in a_i, 1 in x and 1 in a_j may be described by the equations*

$$\begin{aligned} a_i^2(x a_j) + 2(a_i a_j)(x a_i) &= (a_i^2 x)a_j + 2((a_i a_j)x)a_i \\ &= (a_i(a_i x))a_j + (a_i(a_i a_j))x + (a_i(a_j x))a_i. \end{aligned}$$

In particular $2(e_{iji} - \bar{e}_{iji}) + e_{iij} - \bar{e}_{iij}$ *is an identity.*

Proof. From the linearised Jordan identity, denoted $h(x_1, x_2, x_3, x_4)$ in [**6**, Chapter I (51)], we deduce the lemma by selecting the x_i in all ways compatible with the degree requirements.

We denote the vector space $J_1 \otimes J_2$, provided with the product composition $(x_1 \otimes x_2, y_1 \otimes y_2) \to x_1 y_1 \otimes x_2 y_2$, by $J_1 \otimes^k J_2$. When $J_1 \otimes^k J_2$ is a Jordan algebra we call it the *Kronecker product* of the Jordan algebras J_1 and J_2.

LEMMA 2. *Let J_1 and J_2 be Jordan algebras, a_ν, $x \in J_1$, b_ν, $y \in J_2$. Then*

$$\left[\left(\sum_\nu a_\nu \otimes b_\nu\right)^2, x \otimes y, \sum_\nu a_\nu \otimes b_\nu\right] = \sum_{\substack{i,j,k \\ j \neq k}} e_{kkj} \otimes f_{kkj} - \bar{e}_{kkj} \otimes \bar{f}_{kkj} + 2\, e_{jki} \otimes f_{jki} - 2\, \bar{e}_{jki} \otimes \bar{f}_{jki}.$$

Proof. Since $[a_i \otimes b_i, x \otimes y, a_j \otimes b_j] = (a_i x)a_j \otimes [b_i, y, b_j] + [a_i, x, a_j] \otimes b_i(y b_j)$, one has $[(a_i \otimes b_i)^2, x \otimes y, a_i \otimes b_i] = 0$ in $J_1 \otimes^k J_2$. Thus

$$\begin{aligned} &\left[\left(\sum_\nu a_\nu \otimes b_\nu\right)^2, x \otimes y, \sum_\nu a_\nu \otimes b_\nu\right] = \sum_{j \neq k} [a_k^2 \otimes b_k^2, x \otimes y, a_j \otimes b_j] \\ &\quad + 2 \sum_{i, j \neq k} [a_j a_k \otimes b_j b_k, x \otimes y, a_i \otimes b_i] = \sum_{\substack{i,j,k \\ j \neq k}} \{(a_k^2 x)a_j \otimes [b_k^2, y, b_j] \\ &\quad + [a_k^2, x, a_j] \otimes b_k^2(y b_j) + 2\, ((a_j a_k)x)a_i \otimes [b_j b_k, y, b_i] \\ &\quad + 2[a_j a_k, x, a_i] \otimes (b_j b_k)(y b_i)\} = \sum_{\substack{i,j,k \\ j \neq k}} \{e_{kkj} \otimes (f_{kkj} - \bar{f}_{kkj}) \\ &\quad + (e_{kkj} - \bar{e}_{kkj}) \otimes \bar{f}_{kkj} + 2e_{jki} \otimes (f_{jki} - \bar{f}_{jki}) + 2(e_{jki} - \bar{e}_{jki}) \otimes \bar{f}_{jki}\}. \end{aligned}$$

THEOREM 1. *$FJ^{(2)} \otimes^k FJ^{(3)}$ is not a Jordan algebra.*

Proof. Using Lemma 2 we show that the element $[(a_1 \otimes b_1 + a_2 \otimes b_2)^2, x \otimes y, a_1 \otimes b_1 + a_2 \otimes b_2]$ is non-null. Choosing $x = 1$, so that $e_{ijk} = \bar{e}_{ijk}$, the above element can be written as

$$2(a_1a_2)a_1 \otimes (f_{121} - \bar{f}_{121}) + a_1^2a_2 \otimes (f_{112} - \bar{f}_{112}) \\ + 2(a_1a_2)a_2 \otimes (f_{122} - \bar{f}_{122}) + a_1a_2^2 \otimes (f_{122} - \bar{f}_{122}).$$

Identities which involve the 64 terms $e_{ijk} \otimes f_{ijk}$, $e_{ijk} \otimes \bar{f}_{ijk}$, $\bar{e}_{ijk} \otimes f_{ijk}$, $\bar{e}_{ijk} \otimes \bar{f}_{ijk}$, $i, j, k = 1$ or 2, are linear combinations of eight identities, a typical one being

$$(a_1a_2)a_1 \otimes (2(f_{121} - \bar{f}_{121}) + f_{112} - \bar{f}_{112}).$$

Writing these down as an 8×64 matrix, it is obvious that the element considered cannot be written as a linear combination of identities.

COROLLARY 1. *Given Jordan algebras J_1 and J_2, one of which has at least two free elements, the other at least three, then $J_1 \otimes^k J_2$ is not a Jordan algebra.*

Proof. Denote two of the free elements, in J_1 say, by a_1, a_2, and three in J_2 by b_1, b_2, y. The proof of the theorem then applies to this case.

THEOREM 2. *The Kronecker product of Jordan algebras J_1 and J_2 exists if J_1 or J_2 is associative, or if J_1 or J_2 is a spin algebra.*

Proof. Let $a_\nu, x \in J_1$, $b_\nu, y \in J_2$. Assume J_1 associative. By Lemmas 1 and 2,

$$\left[\left(\sum a_\nu \otimes b_\nu\right)^2, x \otimes y, \sum a_\nu \otimes b_\nu\right] = 2 \sum_{i \neq j \neq k} a_j a_k a_i x \otimes (f_{jki} - \bar{f}_{jki}).$$

This is null since $f_{jki} - \bar{f}_{jki}$ is the linearised Jordan identity in b_j, b_k, y, b_i. For the second proof we assume J_1 a spin algebra. The $e_{ijk} = 0$ unless $i = j = k$, so

$$\begin{aligned}\left[\left(\sum_i a_i \otimes b_i\right)^2, x \otimes y, \sum_i a_i \otimes b_i\right] &= 2 \sum_i e_i \otimes (f_{iii} - \bar{f}_{iii} \\ &\qquad + 2f_{iii} - 2\bar{f}_{iii}) \\ &= 6 \sum_i e_i \otimes (f_{iii} - \bar{f}_{iii}) \\ &= 6 \sum_i e_i \otimes [a_i^2, x, a_i] = 0.\end{aligned}$$

PROPOSITION 1. *Let $J_1 = \mathscr{H}_n(D)$, $J_2 = \mathscr{H}_m(D)$, where D is* **R**, **C** *or* **H**. *When $n = m = 3$ we may also let D be* **O**. *The Kronecker product of $\mathbf{J}_1$ and J_2 exists if and only if n or m is less than three.*

Proof. If n or m is less than three, J_1 or J_2 will be associative or a spin algebra.

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If n and m are greater than two it is sufficient to consider $n = m = 3$. Let

$$a_1 = b_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a_2 = b_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad x = y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $e_{121} = \frac{1}{8}a_2$, $\bar{e}_{121} = \frac{1}{4}a_2$, $e_{112} = \frac{1}{2}a_2$, $\bar{e}_{112} = \frac{1}{4}a_2$, $e_{122} = \bar{e}_{122}$ and $e_{221} = \bar{e}_{221}$. Using Lemma 2 it is a simple calculation to show

$$[(a_1 \otimes b_1 + a_2 \otimes b_2)^2, x \otimes y, a_1 \otimes b_1 + a_2 \otimes b_2] = \tfrac{1}{3}a_2 \otimes b_2 \neq 0.$$

PROPOSITION 2. *Let J_1 and J_2 be special Jordan algebras. When J_1 or J_2 is associative then $J_1 \otimes^k J_2$ is special and is the Jordan subalgebra of $(\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2))^+$ generated by J_1 and J_2. If neither J_1 nor J_2 is associative, $J_1 \otimes^k J_2$ is not a subalgebra of $(\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2))^+$.*

Proof. We embed $J_1 \otimes J_2$ in $\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2)$. Calculating in $\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2)$, for $x_1, y_1 \in J_1$, $x_2, y_2 \in J_2$ we have

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) - (x_1 \cdot y_1) \otimes (x_2 \cdot y_2) = \tfrac{1}{4}(x_1y_1 - y_1x_1) \otimes (x_2y_2 - y_2x_2).$$

When J_1 or J_2 is associative this is zero, so the multiplication in $J_1 \otimes^k J_2$ coincides with the multiplication in $(\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2))^+$. It is obvious that $J_1 \otimes^k J_2$ is generated by $J_1 \otimes 1$ and $1 \otimes J_2$. The last statement of the proposition follows from the equation above.

PROPOSITION 3. *Let F be an alternative algebra with involution such that all symmetric elements are in the nucleus, and let C be a commutative associative algebra. Then $\mathscr{H}_3(F) \otimes^k C$ is the Jordan algebra $\mathscr{H}_3(F \otimes C)$, where $F \otimes C$ has the involution defined by $(x \otimes y)^* = x^* \otimes y$.*

Proof. By identifying $\mathscr{M}_3(F) \otimes C$ with $\mathscr{M}_3(F \otimes C)$ we also identify the subspaces $\mathscr{H}_3(F) \otimes C$ and $\mathscr{H}_3(F \otimes C)$. If the symmetric elements of $F \otimes C$ are in the nucleus, $\mathscr{H}_3(F \otimes C)$ will be a Jordan algebra [**6**, Chapter 1, Theorem 4]. It is thus sufficient to show that

$$[x \otimes y, a \otimes b, u \otimes v] = [a \otimes b, x \otimes y, u \otimes v] = [a \otimes b, u \otimes v, x \otimes y] = 0$$

for x symmetric in F, $a, u \in F$, and $y, b, v \in C$. We have relations of the type $[x \otimes y, a \otimes b, u \otimes v] = (xa)u \otimes [y, b, v] + [x, a, u] \otimes y(bv) = 0$ since x is in the nucleus of F, and C is associative; so the proposition is proved.

LEMMA 3. *Let J_1 and J_2 be Jordan algebras. Assume J_2 associative. Then any multilinear identity for J_1 is also a multilinear identity for $J_1 \otimes^k J_2$.*

Proof. Let f be a multilinear identity for J_1. Since f is multilinear, to prove that f is an identity for $J_1 \otimes^k J_2$ it is sufficient to verify that

$$f(x_1 \otimes y_1, x_2 \otimes y_2, \ldots, x_n \otimes y_n) = 0$$

whenever $f(x_1, x_2, \ldots, x_n) = 0$, where $x_1, x_2, \ldots, x_n \in J_1, y_1, y_2, \ldots, y_n \in J_2$. Since $f(x_1, \otimes y_1, x_2 \otimes y_2, \ldots, x_n \otimes y_n) = f(x_1, x_2, \ldots, x_n) \otimes (y_1 y_2 \ldots y_n)$, this is evident. The lemma follows since an identity homogeneous in elements of J_1 will be homogeneous, of the same order, in elements of $J_1 \otimes^k J_2$.

PROPOSITION 4. *Let J_1 and J_2 be Jordan algebras* (over any field of characteristic 0). *If J_2 is associative then any identity for J_1 is also an identity for $J_1 \otimes^k J_2$.*

Proof. Suppose f is an identity for J_1; it can be linearised to a multilinear identity, g say. By Lemma 3, g is an identity for $J_1 \otimes^k J_2$. By [**6**, Ch. I, Theorem 7], the T-ideal generated by f is identical to the T-ideal generated by g. Thus f also is an identity for $J_1 \otimes^k J_2$.

2. The general tensor product. Let J_1 and J_2 denote Jordan algebras. We say that a Jordan algebra J, with monomorphisms u_1 and u_2 of J_1 and J_2 respectively into J, is the $\mathfrak{J}$*-tensor product* of J_1 and J_2 if it has the universal mapping property that, whenever v_1 and v_2 are homomorphisms of J_1 and J_2 respectively in a Jordan algebra F such that $[R_{v_1(J_1)} R_{v_2(J_2)}] = 0$, there exists a unique homomorphism w of J into F such that $v_i = w \circ u_i$, $i = 1, 2$. The $\mathfrak{J}$-tensor product is unique when it exists and we denote it by $J_1 \otimes^j J_2$. We may identify J_1 and J_2 with their images in $J_1 \otimes^j J_2$, and their unit elements with the unit element in $J_1 \otimes^j J_2$.

The $\mathfrak{J}$-tensor product does not always exist. The following counterexample is essentially the same as that for the non existence of a free product of the Jordan algebras involved (cf. [**6**, p. 424]).

LEMMA 4. *When $J_1 = \mathscr{H}_3(\mathbf{O})$, $J_2 = FJ^{(2)}$, the element*

$$[[p_3, p_1, p_1], [p_2, p_1, p_1], [p_2, p_1, p_1]],$$

where $p_k = [x^k, y, y]$, is an identity for J_1, but is not an identity for J_2.

Proof. The subalgebra of J_1 generated by 1, x and y is of the form $\mathscr{H}_3(D)$, where D is a subalgebra of $\mathbf{O}$ [**6**, Chapter IX. 1], so x satisfies an equation $x^3 + \alpha x^2 + \beta x + \gamma = 0$. We take the associator with y and y, obtaining $p_3 + \alpha p_2 + \beta p_1 = 0$. Then we take the associator with p_1 and p_1, and finally with $[p_2, p_1, p_1]$ and itself, to obtain the first result. As in [**6**, Chapter I. 1], one has $[a, b, b] \neq 0$ in $FJ^{(2)}$. The element considered is this with $a = [p_3, p_1, p_1]$, $b = [p_2, p_1, p_1]$, neither of which are identities, for the same reason.

THEOREM 3. *When $J_1 = \mathscr{H}_3(\mathbf{O})$, $J_2 = FJ^{(2)}$, the J-tensor product of J_1 and J_2 cannot exist.*

Proof. Suppose the $\mathfrak{J}$-tensor product exists. Since it contains J_1 as a subalgebra with the same unit element, by [**5**, Theorem 3], $J_1 \otimes^j J_2$ is isomorphic to $J_1 \otimes^k A$ for some associative Jordan algebra A. By Lemma 4 and Proposition 4 the element of Lemma 4 is an identity for J_1, $\otimes^k A$; being non-zero in J_2 it can not be an identity for $J_1 \otimes^j J_2$.

PROPOSITION 5. *Whenever J_1 or J_2 is associative there is a unique epimorphism w_{jk} of $J_1 \otimes^j J_2$ onto $J_1 \otimes^k J_2$.*

Proof. For any $x_i \in J_i$, $z \in J_1 \otimes^k J_2$, one has

$$z[R_{x_1 \otimes 1} R_{1 \otimes x_2}] = [(x_1 \otimes 1), z, (1 \otimes x_2)] = 0.$$

The existence and uniqueness of the morphism follows from the universal mapping property. The morphism is surjective since $J_1 \otimes^k J_2$ is generated by J_1 and J_2.

We denote the *free product* of two algebras A and B, *in the class of associative algebras*, by $A * B$. This free product is known to exist in general. We denote the *free product* of J_1 and J_2, *in the class of Jordan algebras*, by $J_1 *^j J_2$, when it exists. We may write the Jordan algebras J_1 and J_2 as quotient algebras $FJ^{(r)}/\mathfrak{a}_1$ and $FJ^{(s)}/\mathfrak{a}_2$ where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are ideals in $FJ^{(r)}$ and $FJ^{(s)}$ respectively. When the free product exists it is isomorphic to the free composition; so $J_1 \otimes^j J_2 \cong FJ^{(r+s)}/I(\mathfrak{a}_1, \mathfrak{a}_2)$ where $I(\mathfrak{a}_1, \mathfrak{a}_2)$ denotes the ideal in $FJ^{(r+s)}$ generated by $\mathfrak{a}_1$ and $\mathfrak{a}_2$.

PROPOSITION 6. *If J_1 and J_2 are special Jordan algebras, then $J_1 *^j J_2$ exists.*

Proof. The special Jordan algebra $(\mathfrak{S}(J_1) * \mathfrak{S}(J_2))^+$ contains J_1 and J_2 as subalgebras, so $J_1 *^j J_2$ exists [**7**, §4].

COROLLARY 2. *There is an epimorphism of $J_1 *^j J_2$ onto the subalgebra of $(\mathfrak{S}(J_1) * \mathfrak{S}(J_2))^+$ generated by J_1 and J_2.*

Proof. This follows from the universality of the free product.

The free product of special Jordan algebras need not be special; for example $FSJ(x, y) *^j FSJ(z) = FJ(x, y, z)$. Also, the free product of associative Jordan algebras need not be associative; for example, $FJ(x) *^j FJ(y) = FJ(x,y)$.

PROPOSITION 7. *The $\mathfrak{J}$-tensor product of Jordan algebras exists whenever the free product exists.*

Proof. As for associative algebras, the intersection of ideals of a Jordan algebra is an ideal and one can construct the smallest ideal generated by a set of elements. Let $\mathfrak{K}$ be the ideal of $J_1 *^j J_2$ generated by $(J_1 *^j J_2)[R_{v_1(J_1)} R_{v_2(J_2)}]$. Then $(J_1 *^j J_2)/\mathfrak{K}$ satisfies the universal mapping property.

PROPOSITION 8. *The $\mathfrak{J}$-tensor product of associative Jordan algebras is not necessarily associative; the epimorphism w_{jk} is not necessarily an isomorphism, even when both algebras are associative.*

Proof. Let $J_1 = FJ(x)$ and $J_2 = FJ(y)$. Since $J_1 \otimes^k J_2$ is associative, the second part of the proposition will follow from the first. Now $J_1 \otimes^j J_2 = (J_1 *^j J_2)/\mathfrak{K}$, where $\mathfrak{K}$ is the ideal in $FJ(x, y)$ generated by elements of the form $(PX)Y - (PY)X$, for $P \in FJ(x, y)$, $X \in FJ(x)$, $Y \in FJ(y)$. We shall show that the element $[xy, x, y]$ of $FJ(x, y)$ is nonzero in $J_1 \otimes^j J_2$, to prove non-associativity. Let $FC(X)$ denote $\Phi\{\{X\}\}/\mathfrak{T}$ where $\mathfrak{T}$ is the T-ideal generated by the identities $[x_i x_j]$, for $x_i, x_j \in X$ (see [**6**, I §6]). We may write $J_1 \otimes^j J_2$ as $FC(x, y)/\mathfrak{M}$, where $\mathfrak{M}$ is the ideal in $FC(x, y)$ generated by elements of the form $[x^2, y, x]$, $[y^2, x, y]$ and $(QX)Y - (QY)X$, $Q \in FC(x, y)$. The element $[xy, x, y]$ is of degree 4, 2 in x and 2 in y; so it is sufficient to examine those elements in $\mathfrak{M}$ of like degree and verify whether $[xy, x, y]$ is a combination of these. Linearising the Jordan identities to the form $h(x_1, x_2, x_3, x_4)$ [**6**, I (51)], and selecting the x_i in all posible ways compatible with the degree 4, 2 in x and 2 in y, we obtain only two identities, the first two below. The third identity is the only possible identity of degree 4, 2 in x and 2 in y, of the form $(QX)Y - (QY)X$.

$$-x^2y^2 + (y^2x)x + 2((xy)x)y - 2(xy)^2$$
$$-x^2y^2 + (x^2y)y + 2((xy)y)x - 2(xy)^2$$
$$((xy)x)y - ((xy)y)x$$

The element $[xy, x, y] = ((xy)x)y - (xy)^2$ is not a consequence of these identities, so is not in the ideal $\mathfrak{M}$.

The same procedure could be used to prove that the J-tensor product of $FJ^{(2)}$ and $FJ^{(1)}$ is not a special Jordan algebra, if this is true. However, the smallest degree of an identity true for all special Jordan algebras but not for all Jordan algebras is 8 and without a major simplification the procedure is not practical.

In the same way as for general Jordan algebras we may define an $\mathfrak{H}$-tensor product of Jordan algebras in the variety $\mathfrak{H}$ of special Jordan algebras and their homomorphic images. We can prove, in the same way as for the $\mathfrak{J}$-tensor product, that the $\mathfrak{H}$-tensor product always exists for special Jordan algebras. Also, since for J_1 and J_2 as used in Proposition 8, the $\mathfrak{H}$ and $\mathfrak{H}$ tensor products coincide, the $\mathfrak{H}$-tensor product of associative Jordan algebras is not necessarily associative. For other results on the $\mathfrak{H}$-tensor product we shall use the following theorem of Cohn as applied to the variety $\mathfrak{H}$.

THEOREM (Cohn [**2**, Chapter IV, Theorem 4.5]). *Let $\mathfrak{a}$ be an ideal of $FSJ^{(n)}$. Then $\mathfrak{S}(FSJ^{(n)}/\mathfrak{a}) = \Phi^{(n)}\{\quad\}/\bar{\mathfrak{a}}$, where $\bar{\mathfrak{a}}$ is the ideal of $\Phi^{(n)}\{\quad\}$ generated by the set $\mathfrak{a}$. Also $FSJ^{(n)}/\mathfrak{a}$ is special if and only if $\bar{\mathfrak{a}} \cap FSJ^{(n)} = \mathfrak{a}$.*

PROPOSITION 9. *The $\mathfrak{H}$-tensor product of the special Jordan algebras $FSJ^{(2)}$ and $FSJ^{(\mu)}$ is not a special Jordan algebra.*

Proof. The $\mathfrak{H}$-free product of $FSJ^{(2)}$ and $FSJ^{(1)}$ is $FSJ^{(3)}$. Thus $FSJ^{(2)} \otimes^t FSJ^{(1)}$ is $FSJ^{(3)}/\mathfrak{K}$ where $\mathfrak{K}$ is the ideal as in Proposition 7. The ideal $\mathfrak{K}$ is generated by elements $p[R_Z R_X]$ where $p \in FSJ^{(3)}$, $X \in FSJ^{(2)}$ and $Z \in FSJ^{(1)}$. These elements can be expressed in terms of the associative multiplication as $\frac{1}{4}[p[ZX]]$. The $\mathfrak{H}$-tensor product is thus isomorphic to $FSJ^{(3)}/\mathfrak{b}$ where $\mathfrak{b}$ is the ideal of $FSJ^{(3)}$ generated by elements $[p[ZX]]$. By the theorem of Cohn mentioned above, a quotient algebra $FSJ^{(3)}/\mathfrak{a}$ is special if and only if $\{uxyz\} \in \mathfrak{a}$ for every generator u of the ideal $\mathfrak{a}$ (see [**1**]). We suppose that $\{uxyz\} \in \mathfrak{b}$, where $u = 2xzx - zx^2 - x^2z$, i.e. where we have chosen $p = x$, $X = x$ and $Z = z$. Then, as in [**1**], there is a $\phi(k, x, y, z) \in FSJ(x, y, z, k)$ such that $\phi(u, x, y, z) = \{uxyz\}$, and we may write

$$\phi(k, x, y, z) = \alpha_1\{kxyz\} + \alpha_2\{xkyz\} + \alpha_3\{kyxz\} + \alpha_4\{ykxz\} + \alpha_5\{xykz\} + \alpha_6\{yxkz\}.$$

Putting $k = 2xzx - zx^2 - x^2z$, a simple calculation shows that $\alpha_1 = 1$ is the only non-zero coefficient. Thus the Jordan element $\phi(k, x, y, z)$ equals $\{kxyz\}$ and the latter is known not to be a Jordan element [**2**, p. 305].

PROPOSITION 10. *Let J_1 and J_2 be special Jordan algebras. Then* $\mathfrak{S}(J_1 \otimes^i J_2) = \mathfrak{S}(J_1) * \mathfrak{S}(J_2)$.

Proof. We may write $J_1 = FSJ(X)/\mathfrak{a}_1$ and $J_2 = FSJ(Y)/\mathfrak{a}_2$. Let $\mathfrak{K}$ be the ideal such that $(J_1 *^i J_2)/\mathfrak{K} = J_1 \otimes^i J_2$. Then

$$\mathfrak{S}(J_1 *^i J_2) = \mathfrak{S}((FSJ(X)/\mathfrak{a}_1 *^i FSJ(Y)/\mathfrak{a}_2)/\mathfrak{K}$$
$$= \mathfrak{S}(FSJ(X, Y)/I(\mathfrak{a}_1, a_2, \mathfrak{K})).$$

Since $\mathfrak{S}(FSJ(X, Y)) = \Phi\{X, Y\}$ and the ideal of $\Phi\{X, Y\}$ generated by $\mathfrak{K}$ is null, we apply Cohn's theorem above to get $S(J_1 \otimes^i J_2) = \Phi\{X, Y\}/\bar{I}(\mathfrak{a}_1, \mathfrak{a}_2)$ where $\bar{I}(\mathfrak{a}_1, \mathfrak{a}_2)$ is the ideal of $\Phi\{X, Y\}$ generated by $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Since $\mathfrak{a}_1$ is generated by elements in X and $\mathfrak{a}_2$ is generated by elements in Y, one has

$$\Phi\{X, Y\}/\bar{I}(\mathfrak{a}_1, \mathfrak{a}_2) = \Phi\{X\}/\bar{\mathfrak{a}}_1 * \Phi\{Y\}/\bar{\mathfrak{a}}_2.$$

Applying Cohn's theorem once again, the result follows.

3. The special tensor product. We are not yet able to give a concrete construction of the tensor products of the previous section. We have seen that the $\mathfrak{H}$-tensor product of special Jordan algebras is not necessarily special. The $\mathfrak{J}$-tensor product of special Jordan algebras is unlikely to be special. In this section we define another tensor product. We are able to construct it in a concrete way. Also it remains special when its component algebras are special Jordan algebras.

We say that a Jordan algebra J, with monomorphisms s_1 and s_2 of J_1 and J_2 respectively in J, is the $\mathfrak{S}$-*tensor product of* J_1 and J_2 if, for associative specialisations v_1 and v_2 of J_1 and J_2 respectively in an associative algebra A such that $[v_1(J_1), v_2(J_2)] = 0$, there exists a unique associative specialisation w of J in A such that $w \circ s_k = v_k$, $k = 1, 2$. The $\mathfrak{S}$-tensor product is unique when it exists.

We denote by $J_1 \otimes^s J_2$ the Jordan subalgebra of $(\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2))^+$ generated by the images of J_1 and J_2. It follows from Proposition 2 that, when J_1 and J_2 are special, $J_1 \otimes^s J_2$ coincides with $J_1 \otimes^k J_2$ if and only if J_1 or J_2 is associative.

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PROPOSITION 11. *When J_1 and J_2 are special Jordan algebras, then $J_1 \otimes^s J_2$ is the $\mathfrak{S}$-tensor product of J_1 and J_2.*

Proof. We identify J_i with its image in $(\mathfrak{S}(J_i))^+$, $i = 1, 2$. The homomorphisms s_i are defined by $s_1(x_1) = x_1 \otimes 1$ and $s_2(x_2) = 1 \otimes x_2$ for $x_i \in J_i$. Let the v_i be associative specialisations of the J_i in A, whose images commute. These extend to homomorphisms of the $\mathfrak{S}(J_i)$ into A. From the universal mapping property of $\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2)$, there is a unique homomorphism of $\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2)$ into A taking $x_1 \otimes x_2$ to $v_1(x_1)v_2(x_2)$, for $x_i \in J_i$. Thus there is an associative specialisation $\bar{w}$ of $(\mathfrak{S}(J_1) \otimes \mathfrak{S}(J_2))^+$ into A taking $x_1 \otimes x_2$ to $v_1(x_1)v_2(x_2)$. Since $J_1 \otimes^s J_2$ is generated by J_1 and J_2, the restriction of $\bar{w}$ to $J_1 \otimes^s J_2$ is the required w.

LEMMA 4. *Let $a, b \in A$, an associative algebra. The condition $[R_aR_b] = 0$ as an operator on A is equivalent to the condition that $[ab]$ be in the center of A.*

Proof. One has $x[R_aR_b] = (a \cdot x) \cdot b - (b \cdot x) \cdot a$. By an easy calculation it follows that this equals $\frac{1}{4}(x[ab] - [ab]x)$. Thus $[R_aR_b] = 0$ if and only if $x[ab] = [ab]x$ for all $x \in A$, i.e. if and only if $[ab]$ is in the centre of A.

PROPOSITION 12. *When J_1 and J_2 are special there is a unique morphism w_{us} of $J_1 \otimes^u J_2$ onto $J_1 \otimes^s J_2$, where u denotes i or j.*

Proof. Since $[ab] = 0$ implies that $[R_aR_b] = 0$, the proposition follows from the universal mapping property for $J_1 \otimes^u J_2$.

THEOREM (N. Jacobson [**6**, Chapter II, Theorem 19]). *If J is a special Jordan algebra such that $J \cong \mathfrak{h}(\mathfrak{S}(J))$ and $\mathfrak{a}(\mathfrak{S}(J)) = [\mathfrak{h}(\mathfrak{S}(J)), \mathfrak{h}(\mathfrak{S}(J))]$, or if J is generated by 1 and two generators, then the subalgebra of $\mathfrak{S}(J) \otimes \mathfrak{S}(J)$ generated by the elements $a \otimes 1 + 1 \otimes a$, $a \in J$, coincides with the subalgebra of elements invariant under the exchange atomorphism $a \otimes b \to b \otimes a$.*

COROLLARY 3. *If J is a Jordan algebra as above, then $J \otimes^s J$ contains the symmetric elements of $\mathfrak{S}(J) \otimes \mathfrak{S}(J)$ invariant under the exchange automorphism.*

We know that for special Jordan algebras J_1 and J_2, in general $J_1 \otimes^i J_2$ and $J_1 \otimes^s J_2$ are not isomorphic since the former need not be special. The question arises whether they are isomorphic when $J_1 \otimes^i J_2$ is a special Jordan algebra.

THEOREM 4. *Let J_1 and J_2 be special Jordan algebras such that their special envelopes are involutive algebras over* **C**. *Then*

$$J_1 \otimes J_2 \subset J_1 \otimes^s J_2 \subset \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2)).$$

When $J_j = \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_j))$, $j = 1, 2$, then $J_1 \otimes J_2 = \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$ and the vector space underlying $J_1 \otimes^s J_2$ is $J_1 \otimes J_2$.

Proof. We identify J_j with the corresponding subalgebra of $\mathfrak{S}(J_j)^+$. If

$z \in \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$ then $z = \frac{1}{4}(z + z^*)$, so we may write z as

$$\tfrac{1}{2} \sum_j (x_j \underset{\mathbf{C}}{\otimes} y_j + x_j^* \underset{\mathbf{C}}{\otimes} y_j^*) = \tfrac{1}{4} \sum_j \{(x_j + x_j^*) \otimes (y_j + y_j^*)\}$$
$$- \tfrac{1}{4} \sum_j \{(i(x_j - x_j^*)) \otimes (i(y_j - y_j^*))\}.$$

Thus $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2)) \subset \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1)) \otimes \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2))$. The mapping $(x_1, x_2) \mapsto x_1 \otimes_{\mathbf{C}} x_2$ is bilinear from $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1)) \times \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2))$ to $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$ so, by the universal mapping property for tensor products of vector spaces, $x_1 \otimes x_2 \mapsto x_1 \otimes_{\mathbf{C}} x_2$ defines a linear homomorphism of $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2))$ onto $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$. We prove that this homomorphism is an injection so that $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1)) \otimes \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2)) \subset \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$ and, a fortiori, that $J_1 \otimes J_2 \subset \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$. Let M denote the submodule of $\mathfrak{S}(J_1) \times \mathfrak{S}(J_2)$ generated by elements $(\lambda z_1, z_2) - (z_1, \lambda z_2)$, $\lambda \in \mathbf{C}$, $z_j \in \mathfrak{S}(J_j)$. Since any $z \in \mathfrak{S}(J_j)$ can be expressed uniquely as $x + iy$, where $x, y \in \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_j))$, the natural mappings projecting $\mathfrak{S}(J_1) \times \mathfrak{S}(J_2)$ onto $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1)) \times \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2))$ will map an element of M to an element of the corresponding submodule defining $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1)) \otimes \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2))$. The other submodule defining the tensor products is unaffected by the change of field. Thus the homomorphism is an injection. Identifying $x_1 \otimes x_2$ of $J_1 \otimes J_2$ with

$$(x_1 \otimes 1) \cdot (1 \otimes x_2) \text{ of } (\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))^+$$

we may consider $J_1 \otimes J_2$ as a subspace of $J_1 \otimes^s J_2$ and the elements of $J_1 \otimes^s J_2$ are obtained from $J_1 \otimes J_2$ by multiplication and linear combinations in $(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))^+$. Thus $J_1 \otimes^s J_2$ remains in $\mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2))$. The last part of the theorem follows since

$$J_1 \otimes J_2 \subset J_1 \otimes^s J_2 \subset \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1) \otimes_{\mathbf{C}} \mathfrak{S}(J_2)) \subset \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_1)) \otimes \mathfrak{h}_{\mathbf{C}}(\mathfrak{S}(J_2)).$$

Remark. When the $\mathfrak{S}(J_i)$ are algebras with involution over $\mathbf{H}$, the mapping $x \otimes y \mapsto x \otimes_{\mathbf{H}} y$ of $J_1 \otimes J_2$ to $\mathfrak{h}_{\mathbf{H}}(\mathfrak{S}(J_1) \otimes_{\mathbf{H}} \mathfrak{S}(J_2))$ is not injective except for trivial cases. For example

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \otimes_{\mathbf{H}} \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \otimes_{\mathbf{H}} \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$$

while the tensor products in $\mathscr{H}_2(\mathbf{H}) \otimes \mathscr{H}_2(\mathbf{H})$ are not equal.

THEOREM 5. *$\mathscr{H}_n(D) \otimes^s \mathscr{H}_m(D) = \mathscr{H}_{nm}(D)$ whenever $D = \mathbf{R}$ or $\mathbf{C}$.*

Proof. By Theorem 4 we need consider only $D = \mathbf{R}$. It is trivial that $\mathscr{H}_n(\mathbf{R}) \otimes \mathscr{H}_m(\mathbf{R}) \subset \mathscr{H}_{nm}(\mathbf{R})$ and the same reasoning as in Theorem 4 shows that $\mathscr{H}_n(\mathbf{R}) \otimes^s \mathscr{H}_m(\mathbf{R}) \subset \mathscr{H}_{nm}(\mathbf{R})$. To prove the theorem it is thus sufficient to verify that every element of $\mathscr{H}_{nm}(\mathbf{R})$ is a sum of Jordan products of elements in $\mathscr{H}_n(\mathbf{R}) \otimes \mathscr{H}_m(\mathbf{R})$. Consider $x \in \mathscr{H}_n(D)$ and $y \in \mathscr{H}_m(D)$ as operators on vector spaces V_n and V_m with bases (e_α) and (f_β) respectively. Denote the inner product in $V_n \otimes V_m$ by $(\cdot\,|\,\cdot)$ and the matrix of $z \in \mathscr{H}_{nm}(\mathbf{R})$ by (z_{ij}); for $z = x \otimes y$, $((x \otimes y)(e_\alpha \otimes f_\gamma)|e_\beta \otimes f_\delta)$ is thus written as z_{ij} where $i = i(\alpha, \gamma)$ and $j = j(\beta, \delta)$. Since

$$((x \otimes y)(e_\alpha \otimes f_\gamma)|e_\beta \otimes f_\delta) = ((x \otimes y)(e_\beta \otimes f_\gamma)|e_\alpha \otimes f_\delta),$$

we see that any $z \in \mathscr{H}_n(\mathbf{R}) \otimes \mathscr{H}_m(\mathbf{R})$ is such that $z_{ij} = z_{k\nu}$ for some i, j, k, ν with $(i) \cap (j) \cap (k) \cap (\nu) = \emptyset$. However, using Jordan multiplication for elements of $\mathscr{H}_n(\mathbf{R}) \otimes \mathscr{H}_m(\mathbf{R})$ we can generate all $\mathscr{H}_{nm}(\mathbf{R})$. Indeed, choosing z in $\mathscr{H}_n(\mathbf{R}) \otimes \mathscr{H}_m(\mathbf{R})$ such that $z_{is} = 0$ for all $s \neq j$ and $z_{ks} = 0$ for all $s \neq \nu$, we may use formulae 18′-23′ of [**6**, Chapter III] to show that $z^{\cdot 2}$ is realised as $(p_{\alpha\beta})$ where $p_{ij} \neq p_{k\nu}$; we do the same for other identified matrix entries, and by summing we generate all $\mathscr{H}_{nm}(\mathbf{R})$.

Corollary 5. *If $D = \mathbf{R}$ or $\mathbf{C}$, then*

$$\dim\,(\mathscr{H}_n(D) \otimes_D \mathscr{H}_m(D)) \geqq \dim \mathscr{H}_n(D) \times \dim \mathscr{H}_m(D).$$

It might be of use for quantum physics to consider tensor products of spin algebras. Denote the spin algebra $\mathfrak{J}(V, f)$ by $\mathfrak{J}_n$ when V has dimension n, and denote the enveloping Clifford algebra by $\mathfrak{C}_n$. We denote the canonical basis of $\mathbf{H}$ by $(1, i, j, k)$ and identify i with $\sqrt{-1}$. The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$$

are denoted by σ_1, σ_2, σ_3, σ_4 and σ_5 respectively.

We tabulate some results on spin algebras of low dimension. In the second row for $\mathfrak{J}_4$ one may use the tensor products $1 \otimes \sigma_i$ and/or the canonical mapping $\tau : C \to \mathscr{M}_2(\mathbf{R})$ to embed the σ_i, $i = 1, 2, 3$, in $\mathscr{M}_4(\mathbf{R})$. An elementary calculation then gives σ_4 in $\mathscr{M}_4(\mathbf{R})$. The procedure can be repeated to embed the other $\mathfrak{J}_n$ in real matrix algebras. For instance we can embed $\mathfrak{J}_6$ in $\mathscr{M}_8(\mathbf{R})$ in the same way, using the canonical mapping of $\mathbf{H}$ into $\mathscr{M}_4(\mathbf{R})$.

$\mathfrak{J}$	embedded in	with generators 1 and	$\mathfrak{S}(J_n) = \mathfrak{C}_n$
$\mathfrak{J}_1 = \mathbf{R} \oplus \mathbf{R}$	$\mathscr{M}_2(\mathbf{R})$	σ_1	$\mathbf{R} \oplus \mathbf{R}$
$\mathfrak{J}_2 = \mathscr{H}_2(\mathbf{R})$	$\mathscr{M}_2(\mathbf{R})$	σ_1, σ_2	$\mathscr{M}_2(\mathbf{R})$
$\mathfrak{J}_3 = \mathscr{H}_2(\mathbf{C})$	$\mathscr{M}_2(\mathbf{C})$	$\sigma_1, \sigma_2, \sigma_3$	$\mathscr{M}_2(\mathbf{C})$
$\mathfrak{J}_4$	$\mathscr{M}_2(\mathbf{H})$	$\sigma_1, \sigma_2, \sigma_3, \sigma_4$	$\mathscr{M}_2(\mathbf{H})$
	$\mathscr{M}_4(\mathbf{R})$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	
$\mathfrak{J}_5 = \mathscr{H}_2(\mathbf{H})$	$\mathscr{M}_2(\mathbf{H})$	$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$	$\mathscr{M}_2(\mathbf{H}) \oplus \mathscr{M}_2(\mathbf{H})$ $\cong \mathscr{M}_2(\mathbf{H}) \otimes (\mathbf{R} \oplus \mathbf{R})$

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The $\mathfrak{S}$-tensor product does not seem suitable for spin-algebras. It cannot be a spin algebra itself since, due to the matrix structure of the tensor product, the identity will be decomposable into more than two orthogonal equivalent idempotents. Investigation of spin algebras of low dimension shows that there is no natural mapping from the tensor product of spin algebras to another spin algebra and that the Clifford algebras do not have a suitable regularity for tensor products.

Also, the Kronecker product of spin algebras, which exists by Theorem 2, will not be a spin algebra. This comes about since the Kronecker product destroys the matrix structure. A more specific proof is that it follows from Proposition 14 below. It is clear that the Kronecker product of a spin algebra with a Jordan algebra not of the form $\mathfrak{J}(V, f)$ is not a spin algebra since the Kronecker product will not be based on a vector space of the form $\mathbf{R} \oplus V$.

PROPOSITION 14. *The mapping* $(\alpha + x) \otimes (\beta + y) \rightarrow \alpha\beta + \beta x + \alpha y + x \otimes y$ *is a linear bijection of* $\mathfrak{J}_m \otimes^k \mathfrak{J}_n$ *onto* $\mathfrak{J}_{n+m+nm}$, *but is not multiplicative.*

Proof. The mapping is obviously linear injective. Since $\dim \mathfrak{J}_m \otimes^k \mathfrak{J}_n = m + n + mn + 1 = \dim \mathfrak{J}_{m+n+mn}$, it is a bijection. However $((1 + x) \otimes (1 + y))^2 = 4(1 + x) \otimes (1 + y)$ and

$$(1 + x + y + x \otimes y)^2 = 4 + 2(x + y + x \otimes y),$$

so it is not a homomorphism.

4. Tensor product of JW-algebras. We shall use the term von Neumann algebra to mean abstract von Neumann algebra, and morphism of von Neumann algebras to mean normal homomorphism. The algebra of all bounded operators on a Hilbert space H is denoted by $\mathscr{L}(H)$; the Jordan algebra of all self-adjoint operators on H is denoted by $\mathscr{H}(H)$. The Hilbert space tensor product of Hilbert spaces H and K is denoted $H \otimes^h K$. The concrete von Neumann tensor product of concrete von Neumann algebras A and B is denoted by $A \otimes^c B$; as in [**4**], the (large) abstract tensor product of von Neumann algebras A and B is denoted by $A \otimes^\mu B$.

Let A be a von Neumann algebra and define A^+ to be the real vector space underlying A, provided with the weakest operator topology (known to be independent of the realisation of A) and with Jordan multiplication $a \cdot b$. A closed Jordan subalgebra of self-adjoint elements of A^+ is called a *JW-algebra*; so $\mathfrak{h}(A)$, the hermitian part of A, is a JW-algebra. A weakly closed Jordan algebra of bounded self-adjoint operators on a Hilbert space is called a *concrete JW-algebra*. A continuous homomorphism of JW-algebras will be called a *JW-morphism*. A JW-morphism from a JW-algebra J to $\mathfrak{h}(A)$ is called a *W*-specialisation* of J in A. A pair (M, π) consisting of a von Neumann algebra M and an associative specialisation π of J in M is called a *W*-envelope* of J if, for any W^*-specialisation μ of J in a von Neumann algebra A, there exists a unique morphism ν of M in A such that $\nu \circ \pi = \mu$; the W^*-specialisation is uniquely defined if it exists. We write $M = \overline{\mathfrak{S}}(J)$. We know that if M is a factor, then $M = \overline{\mathfrak{S}}(\mathfrak{h}(M))$ [**8**].

We say that a JW-algebra J, with monomorphisms s_1 and s_2 of J_1 and J_2, respectively, in J, is the *JW-tensor product* of the JW-algebras J_1 and J_2 if, for W^*-specialisations v_1 and v_2 of J_1 and J_2 respectively in a von Neumann algebra A, such that $[v_1(J_1)\, v_2(J_2)] = 0$, there exists a unique W^*-specialisation of J in A such that $w \circ s_k = v_k$. Thus the JW-tensor product is uniquely defined, if it exists. We denote by $J_1 \otimes^{ws} J_2$ the JW-subalgebra of $(\overline{\mathfrak{S}}(J_1) \otimes^{\mu} \overline{\mathfrak{S}}(J_2))^+$ generated by J_1 and J_2.

PROPOSITION 15. *Let J_1 and J_2 be two JW-algebras having W^*-envelopes. Then $J_1 \otimes^{ws} J_2$ is the JW-tensor product of J_1 and J_2.*

Proof. Denote the canonical associative specialisation of J_i in $\mathfrak{S}(J_i)$ by $\pi_i(J_i)$, $i = 1, 2$. Let $x_i \in J_i$. The morphisms s_i are defined by $s_1(x_1) = \pi_1(x_1) \otimes 1$ and $s_2(x_2) = 1 \otimes \pi_2(x_2)$, and $s_1(x_1)s_2(x_2) = s_1(x_1) \cdot s_2(x_2)$. Let v_i be a W^*-specialisation of J_i in a von Neumann algebra A such that the images of v_1 and v_2 commute in A. The v_i extend to morphisms of the $\overline{\mathfrak{S}}(J_i)$ in A. From the universal mapping property [**4**, Proposition 8.2], there is a unique morphism of $\overline{\mathfrak{S}}(J_1) \otimes^{\mu} \overline{\mathfrak{S}}(J_2)$ in A taking $s_1(x_1)s_2(x_2)$ to $v_1(x_1)v_2(x_2)$. Thus there is a W^*-specialisation $\bar{w}$ of $(\overline{\mathfrak{S}}(J_1) \otimes^{\mu} \overline{\mathfrak{S}}(J_2))^+$ in A taking $s_1(x_1)s_2(x_2)$ to $v_1(x_1)v_2(x_2)$. Since $J_1 \otimes^{ws} J_2$ is generated by the $s_i(J_i)$, the restriction of $\bar{w}$ to $J_1 \otimes^{ws} J_2$ is the required w.

THEOREM 6. *When M and N are factors, the JW-tensor product of $\mathfrak{h}(M)$ and $\mathfrak{h}(N)$ can be identified with $\mathfrak{h}(M \otimes^{\mu} N)$.*

Proof. The method of proof of Theorem 4 can be applied here. In fact, convergence is not affected by considering the algebras as real or complex and one proves that

$$\mathfrak{h}(M) \otimes^{\mu} \mathfrak{h}(N) \subset \mathfrak{h}(M) \otimes^{ws} \mathfrak{h}(N) \subset \mathfrak{h}(M \otimes^{\mu} N) \subset \mathfrak{h}(M) \otimes^{\mu} \mathfrak{h}(N)$$

where $\mathfrak{h}(M) \otimes^{\mu} \mathfrak{h}(N)$ denotes the closure of $\mathfrak{h}(M) \otimes \mathfrak{h}(N)$ in $\mathfrak{h}(M \otimes^{\mu} N) \subset M \otimes^{\mu} N$.

COROLLARY 6. *For M and N discrete factors, $\mathfrak{h}(M) \otimes^{ws} \mathfrak{h}(N) = \mathfrak{h}(M \otimes^{c} N)$. Also $\mathscr{H}(H) \otimes^{ws} \mathscr{H}(K) = \mathscr{H}(H \otimes^{h} K)$ for Hilbert spaces H and K.*

Proof. For M and N discrete, $M \otimes^{\mu} N = M \otimes^{c} N$ so the first part of the corollary is evident. The second part of the corollary is also evident since $\mathscr{L}(H) \otimes^{c} \mathscr{L}(K) = \mathscr{L}(H \otimes^{h} K)$.

Remark. Let V be the prehilbert space formed from a II_1 factor A whose inner product is defined by $(a|b) = \text{trace}\, a \cdot b$, for $a, b \in A$. The space $\mathfrak{J}(V, f)$, where f is the orthonormal bilinear form defined by the inner product, is called the *infinite dimensional spin factor* and is of interest in the theory of JW-algebras. We investigate the structure of its W^*-envelope by considering sub-

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objects of (V, f). Considering the elements of $\mathfrak{C}_{2n+1}$ as matrices, a complex structure can be given to $\mathfrak{C}_{2n+1}$ by means of a mapping Γ such that $i\Gamma = \Gamma F$; here F is a matrix such that $F^2 = -I$ and $F^t = -F$. For Fermi systems in quantum mechanics the complexification corresponds physically to the consideration of momenta as well as of coordinates. The mapping

$$e_j \mapsto \bar{e}_j, j = 1, \ldots, n, \qquad e_{2n+1} \mapsto i \prod_{j=1}^{2n} \bar{e}_j$$

is easily seen to define an isomorphism of $\Gamma(\mathfrak{C}_{2n+1})$ onto the complex Clifford algebra for a $2n$-dimensional space with basis $(\bar{e}_j)$. Using the universal mapping property for Clifford algebras and the fact that the complex even-dimensional Clifford algebras are tensor products of copies of $\mathscr{M}_2(\mathbf{C})$, it is easily seen that the W^*-envelope for $\mathfrak{J}(V, f)$ will be set-theoretically isomorphic to the large abstract von Neumann tensor product of an infinite number of copies of $\mathscr{M}_2(\mathfrak{C})$. This space is excessively large and exists only by virtue of Zorn's lemma. Thus besides the fact, as seen in § 3, that the $\mathfrak{S}$-tensor product is not particularly suitable for spin algebras, for the infinite-dimensional spin factor the W^*-envelope is too large to be of any use.

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