Canad. Math. Bull. Vol. **38** (2), 1995 pp. 223–229

SHUFFLING OF LINEAR ORDERS

JOHN LINDSAY ORR

ABSTRACT. A linearly ordered set A is said to *shuffle* into another linearly ordered set B if there is an order preserving surjection $A \longrightarrow B$ such that the preimage of each member of a cofinite subset of B has an arbitrary pre-defined finite cardinality. We show that every countable linearly ordered set shuffles into itself. This leads to consequences on transformations of subsets of the real numbers by order preserving maps.

The purpose of this note is to present some new results in the study of ordered sets, in particular Theorem 1 and Proposition 2 below. These ideas arose in work on the structure of certain operator algebras [6]. However, the techniques of this note are entirely order-theoretic and combinatoric. In fact the main results seem quite natural questions to consider in their own right and it was quite surprising to discover that they rested on rather deep combinatoric theory [3, 2]. Because of the general nature of the present results and their proven value in [6], one hopes they will spark interest and find other applications in a wider field.

This work was conducted while the author held an SERC(UK) Research Assistantship at the University of Lancaster and while visiting the University of Waterloo. In particular, I would like to thank Professor K. R. Davidson for his hospitality during that time.

The following theorem is the main result of this paper:

THEOREM 1. *Let $K \subseteq [0,1]$ be a compact set. If $S \subseteq [0,1]$ meets each component of the relative complement, K^c, of K in a finite number of points then there is an increasing bijection f of the unit interval to itself such that $f(K)$ contains all but finitely many points of S.*

We can express the order theoretic ideas of Theorem 1 without reference to the real number system. The following proposition, which is a simple consequence of Theorem 7, achieves this.

PROPOSITION 2. *Let A be a countable linearly ordered set and let L_a $(a \in A)$ be finite linearly ordered sets. Then there is an increasing map*

$$\sigma: A \longrightarrow L = \sum_{a \in A} L_a$$

which maps onto all but finitely many points of L, and, in any event, onto at least one point in every L_a.

Received by the editors August 30, 1993; revised March 22, 1994.
AMS subject classification: 06A05.

The simplest examples show how one is led to the conditions of the theorem. If K is empty and S is not, then the requirement that $f(K)$ may miss finitely many points of S becomes clear. For simple infinite sets the result is straightforward. For example if $K = \{0\} \cup \{1/2^n : n \geq 1\}$ and $S = \{1/n : n \geq 1\}$ then a piecewise linear map taking $1/2^n$ to $1/n$ achieves the result. If K is ordered as $\mathbf{Z}$ or other simple infinite order types, the appropriate generalizations are easily made.

Proposition 2 does not hold in general when A may be uncountable order type. For example if A were the real line and each L_a a two point set then σ would have to map a rational into all but finitely many of the sets L_a. Indeed, if $L_a = \{\ell_1, \ell_2\}$ and $\sigma(x) = \ell_1 < \ell_2 = \sigma(y)$ then $x < y$ and there is a rational number q between x and y which is mapped to one of ℓ_1, ℓ_2. But this corresponds countably many rationals with uncountably many reals, which is impossible.

Theorem 1 is proved by considering mappings of the components of K^c into themselves. These components are pairwise disjoint open intervals which are linearly ordered by comparison of endpoints. The number of components of K^c which must be mapped to a single component of K^c is equal to the number of components of $(K \cup S)^c$ which lie in that component. This need to map linearly ordered sets into themselves in such a way that almost every point is hit a specified number of times motivated the following definition.

DEFINITION. Let A be a countable linearly ordered set. A function $f\colon A \longrightarrow \mathbf{Z}^+$ is called an *order of shuffling* on A. Given an order of shuffling f on A, we shall say that a linearly ordered set B *shuffles into* (A,f) if there is a surjective order homomorphism σ of B onto A such that the cardinality of $\sigma^{-1}\{a\}$ is at least $f(a)$ for all but finitely many $a \in A$. If this holds for all $a \in A$ we say that B shuffles into (A,f) *exactly*.

The main technical result of the paper, Theorem 7, will be that every countable linearly ordered set shuffles into itself with any order of shuffling.

We must now recall some definitions and terminology from set and order theory. An *order type* is an equivalence class of linearly ordered sets, under the relation of order isomorphism. The advantage of considering order types is that we can speak of the *set* of all countable order types whereas we can only talk of the *class* of all countable linearly ordered sets. (Since every countable order type has a representative which is a subset of the rationals, this set is well defined.)

Conventionally, the symbols $\mathbf{0}, \mathbf{1}, \boldsymbol{\omega}, \boldsymbol{\omega}^*$ and $\boldsymbol{\eta}$ denote respectively the order types of the empty set, the one-point set, the positive integers, the negative integers and the rational numbers. The symbol $\boldsymbol{\omega}_1$ denotes the first uncountable ordinal. We shall sometimes blur the distinction between order types and linearly ordered sets, for example letting the same symbol denote a relation between order types and the corresponding relation between their representatives.

A linearly ordered set A is dense if for any $a < b$ in A there is c in A with $a < c < b$. On the other hand, A is said to be *scattered* if it does not contain any subset of relative order type $\boldsymbol{\eta}$.

Finally, if $(I, \leq)$ is a linearly ordered set and for each $i \in I$, $(L_i, \leq_i)$ is a linearly ordered set then the order sum $\sum_{i \in I} L_i$ is defined to be the set $\{(i, a) : i \in I, a \in L_i\}$ with the lexicographic ordering. If I is a finite set the sum is generally written

$$L_1 + \cdots + L_n.$$

We shall be working with the set $\mathcal{C}$ of countable order types with the relation $A \preceq B$ meaning "A is a homomorphic image of B." This relation is a transitive, reflexive binary relation on $\mathcal{C}$. Such relations are called *quasi-orderings*. Building on deep results of Nash-Williams [4, 5] and Laver [3], Landraitis in [2] proved a powerful combinatorial result for the quasi-ordering $(\mathcal{C}, \preceq)$. In order to discuss this result we must consider some properties of quasi-ordered sets.

A quasi-ordered set $(S, \preceq_S)$ is said to be *well quasi-ordered* (wqo) if whenever s_n is an infinite sequence of elements of S then there are indices $n < m$ such that $s_n \preceq_S s_m$. By Ramsey's Theorem, every such sequence contains an increasing subsequence. From this it is clear that if $(R, \preceq_R)$ and $(S, \preceq_S)$ are wqo then $(R \times S, \preceq_R \times \preceq_S)$ is wqo.

Landraitis' theorem (Theorem 3 below) is that $(\mathcal{C}, \preceq)$ is wqo. This is proved indirectly, using a stronger condition, *better quasi-ordering* (bqo), which was introduced by Nash-Williams in [5]. The property bqo is more technical to define, but is better poised for use in the transfinite induction arguments which are needed in this area. In fact Landraitis proves that $(\mathcal{C}, \preceq)$ is bqo and then makes use of the elementary fact that bqo $\Rightarrow$ wqo. However, bqo does not play any direct part in this work and so we shall not discuss it further, but refer the interested reader to [5] or Rosenstein's excellent description in [8, Chapter 10].

THEOREM 3 (LANDRAITIS, [2] COROLLARY 3.4). *The set $\mathcal{C}$ of countable linear orderings, with the quasi-ordering of "is a homomorphic image of", is well quasi-ordered.*

We can now begin the proof of Theorem 7 by proving two lemmas.

LEMMA 4. *Given a linearly ordered set A with an order of shuffling f, suppose that A shuffles into (A,f). Then there is an n such that whenever*

$$B = \sum_{1 \leq i \leq m} B_i$$

and at least n of the B_i, including B_1 and B_m, satisfy $A \preceq B_i$ then B shuffles into (A,f) exactly.

REMARK. We call the least such n the degree of exactness for (A,f).

PROOF. Let σ be an order homomorphism implementing the shuffling of A into (A,f). Let $a_1, \ldots, a_k$ be those a in A for which the cardinality of $\sigma^{-1}(a)$ is less than $f(a)$, enumerated in increasing order.

The range of σ covers each of the intervals

$$(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty)$$

(where $(a, +\infty)$ has the conventional meaning $\{x \in A : x > a\}$, and $(-\infty, a)$ likewise) and σ maps onto each element of these intervals with the appropriate multiplicity. It is conceivable that A does not map onto any of the a_i, so we must include in B at least $f(a_i)$ copies of A to map onto each a_i. Thus we take $n = k + 1 + \sum_{i=1}^{k} f(a_i)$. Given a set $B = \sum_{1 \leq i \leq m} B_i$ satisfying the hypotheses, identify a sequence $B_{i_1}, \ldots, B_{i_n}$ such that $A \preceq B_i$ and

$$1 = i_1 < i_2 < \cdots < i_n = m.$$

Let τ_r map $B_{i_r} \subseteq B$ onto A. Construct an exact shuffling of B onto A as follows: First map the initial segment $\tau_1^{-1}\sigma^{-1}(-\infty, a_1)$ of B onto $(-\infty, a_1)$ by $\sigma\tau_1$. We shall map the segment $\tau_{f(a_1)+2}^{-1}\sigma^{-1}(a_1, a_2)$ onto (a_1, a_2) with $\sigma\tau_{f(a_1)+2}$. Meanwhile, map everything which lies between these two segments, in particular, all of $B_{i_2}, \ldots, B_{i_{f(a_1)+1}}$, onto a_1. Continue in this way, letting $g(i) = f(a_1)+\cdots+f(a_i)+i+1$ and mapping each segment $\tau_{g(i)}^{-1}\sigma^{-1}(a_i, a_{i+1})$ onto (a_i, a_{i+1}) with $\sigma\tau_{g(i)}$. We have required that exactly enough of the B_{i_r} map onto A to be able to continue in this way, finally mapping a final segment of B_{i_n} onto $(a_k, +\infty)$. ■

We shall need Hausdoff's canonical form for countable scattered order types [1]. A proof of this well-known fact can also be found in [8].

DEFINITION. Let $S_0 = \{\mathbf{0}, \mathbf{1}\}$ and inductively, for each countable ordinal α, let S_α be the set of all order types which can be written in the form $\sum_{i \in I} L_i$ where I is either $\boldsymbol{\omega}$ or $\boldsymbol{\omega}^*$ and the L_i all belong to $\bigcup_{\beta<\alpha} S_\beta$.

THEOREM 5 (HAUSDORFF). *The set of all countable scattered order types is equal to* $\bigcup_{\alpha<\boldsymbol{\omega}_1} S_\alpha$.

LEMMA 6. *Let A be a countable scattered linear ordering and let f be an order of shuffling on A. Then A shuffles into (A,f).*

PROOF. We proceed by induction on S_α. The assertion is certainly true for S_0 so suppose that it also holds for all S_β with $\beta < \alpha$. Let A belong to S_α. Now $A = \sum_{i \in I} L_i$ where each $L_i \in \bigcup_{\beta<\alpha} S_\beta$ and I is ordered as one of $\boldsymbol{\omega}$ or $\boldsymbol{\omega}^*$. For definiteness we shall assume I is ordered as $\boldsymbol{\omega}$ since the proof in the other case is directly analogous.

As was observed with the definition of wqo, the product of wqo sets is wqo. Thus, by Landraitis' Theorem, the set

$$S = \{i : L_i \preceq L_j \text{ and } L_{i+1} \preceq L_{j+1} \text{ for some } j > i\}$$

is cofinite. Let $m = \max S^c$. Then for each $i > m$ there are infinitely many $j > i$ with $L_i \preceq L_j$ and $L_{i+1} \preceq L_{j+1}$.

Regard the sets L_i as subsets of A and, by restriction, obtain orderings of shuffling $f_i = f|_{L_i}$ on L_i. By hypothesis each L_i shuffles into (L_i, f_i) and so clearly $A' = L_1 + \cdots + L_m$ shuffles into $(A', f|_{A'})$. Thus, it will suffice to show that $A'' = \sum_{m<i<\boldsymbol{\omega}} L_i$ shuffles into (A'', f'') exactly, where $f'' = f|_{A''}$.

Consider first L_{m+1}. Now L_{m+1} shuffles into (L_{m+1}, f_{m+1}) by hypothesis, so let n_1 be the degree of exactness for (L_{m+1}, f_{m+1}) from Lemma 4. By the last sentence of the second

https://doi.org/10.4153/CMB-1995-032-1 Published online by Cambridge University Press

paragraph, we can pick $k_2 > k_1$ (setting $k_1 = m+1$) so that $L_{m+1} \preceq L_{k_2-1}$ and $L_{m+2} \preceq L_{k_2}$, and so that $L_{m+1} \preceq L_i$ for at least n_1 values of $k_1 \leq i < k_2$. Then $\sum_{i=k_1}^{k_2-1} L_i$ shuffles into (L_{m+1}, f_{m+1}) exactly.

Moreover, L_{m+2} shuffles into (L_{m+2}, f_{m+2}) by hypothesis, so we take n_2 to be its degree of exactness. We can repeat the argument of the last paragraph, finding $k_3 > k_2$ such that $L_{m+2} \preceq L_{k_3-1}$, $L_{m+3} \preceq L_{k_3}$ and $L_{m+2} \preceq L_i$ for at least n_2 values of $k_2 \leq i < k_3$. This procedure can now be iterated, and the homomorphisms so obtained can be patched together to give an exact shuffling of A'' into (A'', f''). ∎

THEOREM 7. *Let A be a countable linearly ordered set and let f be an order of shuffling on A. Then A shuffles into (A, f).*

PROOF. By another theorem of Hausdorff [1] (see also [8, Theorem 4.9]) every linearly ordered set can be written in the form

$$\sum_{i \in I} L_i$$

where I is a dense linear ordering and the L_i are scattered linear orderings. Since A is assumed to be countable we may write A in this form where both I and all the L_i are countable. By a well-known theorem of Cantor, I must have order type

$$\mathbf{1}, \quad \boldsymbol{\eta}, \quad \mathbf{1}+\boldsymbol{\eta}, \quad \boldsymbol{\eta}+\mathbf{1} \quad \text{or} \quad \mathbf{1}+\boldsymbol{\eta}+\mathbf{1}.$$

If I has a greatest or a least element then by the last lemma the L_i corresponding to those elements shuffle into themselves with their restricted order of shuffling. Thus, it only remains to show that $A' = \sum_{i \in \boldsymbol{\eta}} L_i$ shuffles into itself with the restricted order of shuffling.

First we remark that if S is a countable scattered linearly ordered set then there is a homomorphism $\tau : \boldsymbol{\eta} \longrightarrow S$ such that $\tau^{-1}\{s\}$ is infinite for each $s \in S$. This is easily verified by transfinite induction using Hausdorff's theorem (Theorem 5.) Thus if η_i is a copy of $\boldsymbol{\eta}$ there are order homomorphisms $\tau_i : \eta_i \longrightarrow L_i$ with this property for each $i \in \boldsymbol{\eta}$. Thus there is an order homomorphism

$$\tau : \boldsymbol{\eta} = \sum_{i \in \boldsymbol{\eta}} \eta_i \longrightarrow \sum_{i \in \boldsymbol{\eta}} L_i = A'$$

such that $\tau^{-1}\{a\}$ is infinite for each $a \in A'$. Hence τ induces a homomorphism with the same property from A' to itself by the formula $\hat{\tau}(a) = \tau(i)$ for $a \in L_i$. ∎

We are now in a position to prove our main result, Theorem 1:

PROOF. Without loss we can replace S with $S \cup K$, with the consequence that S is now closed. Suppose K has no interior. Let U_a $(a \in A)$ be the components of $[0, 1] \setminus K$ indexed in order by a countable linearly ordered set A. By the conditions on S, the components of $[0, 1] \setminus S$ are ordered as $L = \sum_{a \in A} L_a$ where L_a are all finite. Index these components as $V_{a,i}$ $(a \in A, i \in L_a)$. By Proposition 2, let σ map A onto all but finitely many points

https://doi.org/10.4153/CMB-1995-032-1 Published online by Cambridge University Press

of L and, in any event, onto at least one point of every L_a. We can ensure that σ maps onto the greatest and least elements of L, if such exist. For simply delete the greatest or least elements of A to get A' and obtain a map σ' of A' onto all but finitely many points of $\sum_{a\in A'} L_a$. Then extend σ' to a map σ which maps the extreme points of A to the corresponding extremal points of L.

Now begin to define the map f on $[0,1]$ as follows: If $t \in U_a$, let x_1 and x_2 be respectively the infimum and supremum of the union of those intervals U_b with $\sigma(b) = \sigma(a)$ and write $V_{\sigma(a)} = (y_1, y_2)$. On (x_1, x_2) let f be the linear map taking x_i to y_i $(i = 1, 2)$. In this way, we define f on $\bigcup_a U_a$ as a function mapping into $\bigcup_{l\in L} V_l$. However, there may very well be finitely many $V_{a,i}$'s which f does not map onto. Each one of these is nevertheless contained in an interval U_a, which it shares with at least one $V_{a,j}$ (for $i,j \in L_a$) which is in the range of f. For each such a, it is straightforward to construct a strictly increasing map taking the union of those $V_{a,j}$ which are in the range of f onto an open dense subset of U_a. Adjust f by composing with such maps and we obtain a strictly increasing map of $\bigcup_a U_a$ into $[0,1]$ which contains $\bigcup_{l\in L} V_l$ in its range.

Since $\bigcup_{l\in L} V_l$ is dense and f is increasing, f extends to a continuous map of $[0,1]$ to itself. For if $x \in [0,1]$ then since f is increasing, the left- and right-hand limits of f at x exist. But they must be equal since otherwise a portion of the dense set $\bigcup_{l\in L} V_l$ would lie between them. By slight abuse, we also call this extended map f. Notice that f maps onto the endpoints of every $V_{a,i}$ except perhaps for the finitely many ones in U_a's on which f had to be adjusted in the last paragraph. Thus it only remains to check that f is a bijection of the unit interval to itself.

But our requirement that σ map extremal elements of A to extremal elements of L now yields $f(0) = 0$ and $f(1) = 1$ and so f maps onto $[0,1]$ by the Intermediate Value Theorem. It is clear from the construction that f is strictly increasing on $\bigcup_a U_a$. But $\bigcup_a U_a$ is dense in $[0,1]$ so f must be strictly increasing on the whole interval.

Suppose now K has interior. The case $K = [0,1]$ is trivial so we suppose that K, and hence S, is not the whole unit interval. Picking a point x_0 outside S and considering $[0, x_0]$ and $[x_0, 1]$ separately, we may suppose that $1 \notin S$. There are now three cases to distinguish: (i) 0 belongs to $\operatorname{Int} S$, (ii) 0 is an accumulation point of $\operatorname{Int} S$, and (iii) 0 does not belong to the closure of $\operatorname{Int} S$.

In the first case K contains an interval $[0,a]$ and excludes an interval $(b,1]$. Take f to be any increasing bijection of $[0,1]$ to itself which maps a to b. In the second case it is routine to find sequences

$$1 = b_1 > a_1 > b_2 > a_2 > \cdots$$

decreasing to zero such that $[a_i, b_i]$ lies in K for even i and in K^c for odd. One then constructs f so as to map a_{2i} to b_{2i+1} and b_{2i} to a_{2i-1}. In the last case, let a be the infimum of $\operatorname{Int} K$. The interior of $[0,a] \cap K$ is empty so by what has gone before there is a map of $[0,a]$ to itself with the appropriate properties for $[0,a] \cap K$ and $[0,a] \cap S$. Considering the parts of K and S in $[a,1]$ returns us to case (i) or (ii) above, and the maps obtained can be patched together. ∎

REFERENCES

1. F. Hausdorff, *Grundzüge einer Theorie der Geordneten Mengen*, Math. Ann. **65**(1908), 435–505.
2. C. Landraitis, *A combinatorial property of the homomorphism relation between countable order types*, J. Symbolic Logic **44**(1979), 403–411.
3. R. Laver, *On Fraisse's order type conjecture*, Ann. of Math. **93**(1971), 89–111.
4. C. St. John and A. Nash-Williams, *On well-quasi-ordering finite trees*, Math. Proc. Cambridge Philos. Soc. **61**(1965), 833–835.
5. ———, *On well-quasi-ordering infinite trees*, Math. Proc. Cambridge Philos. Soc. **61**(1965), 697–720.
6. J. L. Orr, *Stable ideals of nest algebras*, preprint.
7. J. R. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. (3) **15**(1965), 61–83,
8. J. G. Rosenstein, *Linear orderings*, Academic Press, New York, 1982.

Department of Mathematics and Statistics
University of Nebraska–Lincoln
Lincoln, Nebraska 68588-0323
U.S.A.
e-mail: jorr@math.unl.edu