COMPLETIONS OF QUADRANGLES IN PROJECTIVE PLANES II

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1. Introduction. This viewpoint of studying projective planes was given in my previous paper **(12)**. It is discussed in other papers: Hall **(4, 6, 7)**, Maisano **(16)**, Lombardo-Radice **(14, 15)**, Wagner **(19)**. In particular, we consider how to make identifications in the free plane, or how identifications are forced when one begins with a non-degenerate quadrangle and makes free extensions of this quadrangle of a known plane. We shall continue to develop this topic using the notations and definitions of the previous paper **(12)**. We consider the number of subplanes of certain planes, finding exact values in the cases of the known order-nine planes, and deriving a lower bound in a general case. We prove a theorem concerning the structure of all singly generated planes. We give an example to show that this structure is not definitive. Finally, we prove that a specific Walker-Knuth plane is singly generated.

2. Subplanes of geometries of order nine. A routine was written for the IBM 7090 which finds all the subplanes of a given plane of order nine. By Bruck's theorem **(7, 9)**, it is known that the subplanes must be of orders 2 and 3. The routine examines each non-degenerate quadrangle in turn by forming extensions and thereby determining whether the extensions complete to a subplane of order 2 or 3 or neither. Each result determined by the routine is tallied accordingly. The answers for the Desarguesian plane are easily calculated; hence, the Desarguesian plane was used for a preliminary test run. Then the Carmichael plane and the Veblen-Wedderburn plane were run. The dual to the Veblen-Wedderburn plane was not run because the counts for the Veblen-Wedderburn plane and its dual must be the same since the planes of orders 2 and 3 are self-dual.

An incidence matrix in the canonical form **(17)** of the given geometry and its transpose were the data of the computation. In the completion procedure one needs to know the point of intersection of two given lines. This point can be obtained by first performing the logical "and" upon the two matrix rows which represent the lines involved and then using the CAQ command to locate the resulting single bit, which represents the point, as quickly as possible. Since an incidence matrix consists of 91 by 91 ones and zeros, it is necessary to do this in triple precision. The transpose is used to perform the dual problem. These incidence matrices were produced by a SWAC programme which accepts the geometries in the form of eight Latin squares.

Received October 8, 1963.

https://doi.org/10.4153/CJM-1965-015-5 Published online by Cambridge University Press

There are seven distinct quadrangles in each Fano configuration—since one can choose any one of seven points, followed by any one of six, followed by one of four not on the line joining the first two chosen points, and this method produces every quadrangle in 24 forms. Likewise there are $13 \times 12 \times 9 \times 4/24 = 234$ distinct quadrangles of a plane of order 3, and $91 \times 90 \times 81 \times 64/24 = 1{,}769{,}040$ distinct quadrangles of a plane of order 9. If F is the number of Fano configurations, T is the number of subplanes of order 3, and S is the number of quadrangles from which the plane is singly generated, then $7F + 234T + S = 1{,}769{,}040$.

It is well known that the Desarguesian plane yields no Fano configurations, 7560 subplanes of order 3, and no quadrangle which singly generates the plane. Thus $7 \times 0 + 234 \times 7560 + 0 = 1{,}769{,}040$. The Carmichael plane yields 33,696 Fano configurations, 1080 subplanes of order 3, and 1,280,448 quadrangles which singly generate the plane. Now

$$7 \times 33696 + 234 \times 1080 + 1{,}280{,}448 = 1{,}769{,}040.$$

The Veblen-Wedderburn plane yields 51,840 Fano configurations, 1080 subplanes of order 3 (the same number as for the Carmichael plane), and 1,153,440 quadrilaterals which singly generate the plane. Again

$$7 \times 51840 + 234 \times 1080 + 1{,}153{,}440 = 1{,}769{,}040.$$

Whenever another non-Desarguesian plane of order 9 is discovered, one could calculate the number of its subplanes. Should this tally differ from the two given here, then it is clearly a new plane. On the other hand, an agreement of tallies between two planes can only suggest that one should look for an isomorphism between them. In general, then, it may be possible to use subplane counts as distinguishing invariants between non-isomorphic planes. (This routine has been run for this purpose five different times by either the W.D.P.C. or the Computing Facility IBM 7090 at U.C.L.A.)

Moreover, if one can find a quadrangle which generates the entire plane, then the plane can be characterized in terms of this completion. A second plane, possibly isomorphic to the first, could be examined to see if it has a quadrangle with the same extensions. This procedure is clearly valid theoretically. In practice, however, it is extremely time-consuming.

The collineation group for the Carmichael plane has the order 33,696 **(21)**. (It is interesting to note that this is the number of Fano configurations; however, they are not all in the same conjugate class determined by this group. The Fano configuration A_2, A_3, A_9, B_0, G_6, D_2, D_{11} is taken into itself by the non-identity transformation

$$x'_1 = x_1^{\alpha_6},$$

$$x'_2 = 2x_2^{\alpha_6},$$

$$x'_3 = 2x_2^{\alpha_6} + x_3^{\alpha_6},$$

https://doi.org/10.4153/CJM-1965-015-5 Published online by Cambridge University Press

where Zappa **(21)** asserts that this is a collineation and where the names of the points are found in **(2)**, α_6 is given in **(10)**, and the correspondence between the latter two papers is $0 \leftrightarrow 0$, $1 \leftrightarrow 1$, $2 \leftrightarrow 2$, $3 \leftrightarrow j$, $6 \leftrightarrow 2j$, $4 \leftrightarrow 1 + j$, $5 \leftrightarrow 1 + 2j$, $7 \leftrightarrow 2 + j$, $8 \leftrightarrow 2 + 2j$.) For each characteristic extension of a quadrangle, the collineation group takes a generating quadrangle to precisely every other generating quadrangle of the same extension. If it turns out that no generating quadrangle goes to itself under any collineation except the identity, then there are 38 different characterizations of the plane found by dividing the number of quadrangles by the order of the collineation group. If we count characterizations of a single quadrangle in 24 ways instead of one way, then the minimal number of characterizations is 38×24 instead of 38.

In the Veblen-Wedderburn plane, the order of the collineation group is 622,080, which does not divide the number of quadrangles which singly generate the plane. Thus in this case the collineation group does take some quadrangles singly generating the plane into themselves.

Note: The generators for the collineation group are given in **(4)**. It was learned from Hall by private communication that the order of the group reported in **(4)** was wrong. To see this, we note that $r = 2$, $s = 2$ is an element of order 2 generating a normal subgroup which leaves the points at infinity fixed. We have given the correct value of the group.

3. Subplanes of order p in Veblen-Wedderburn planes of order p^α. We notice that $1080 = (81 \times 80 \times 72)/(9 \times 8 \times 6)$. Thus in this particular case the number of subplanes of the Veblen-Wedderburn plane (as well as the Carmichael plane) could be formulated as

$$n^2(n^2 - 1)(n^2 - n)/p^2(p^2 - 1)(p^2 - p).$$

This leads one to the following

THEOREM. *In a Veblen-Wedderburn plane of order $n = p^\alpha$, there are at least $n^2(n^2 - 1)(n^2 - n)/p^2(p^2 - 1)(p^2 - p)$ subplanes of order p.*

Remark. The author discovered a proof of this theorem based on the underlying "vector space" **(1)** which involved several computations. The author reproduces below instead a shorter proof suggested by the referee.

Proof. Any quadrangle Q with two vertices on the axis can be used as a basis for co-ordinatizing the V–W plane by a V–W system **(9)**, p. 362. The additive subgroup of the V–W system which is generated by the multiplicative identity forms a field GF(p) under addition and multiplication. Hence the completion of Q is a (Desarguesian) subplane of order p. Therefore any quadrangle with two vertices on the axis of the V–W plane belongs to a subplane of order p. There are $n^2(n^2 - 1)(n^2 - n)$ such quadrangles, $p^2(p^2 - 1)(p^2 - p)$ of which occur in each subplane.

Now a natural question arises as to discovering when equality holds. The subplane count for Desarguesian planes shows that there are planes for which equality does not hold. If the nucleus of the V–W system exceeds the field F generated by the multiplicative identity, then there are some additional subplanes not counted. To see this, one realizes that in the Desarguesian subplane of order p^β, $\beta > 1$, obtained from the nucleus, there are more planes than those counted by the above process. We are led to the following open question: Does the fact that the inequality is not an equality guarantee that the nucleus is larger than the field F? One wonders also: What can be said of the subplanes of the Hughes planes? (The Carmichael plane is a special Hughes plane **(11)**.)

4. Identifications in the free plane. Starting with four points, A, B, C, D, as the first partial plane and then freely extending four times to the fifth partial plane we obtain nine sets of four collinear points: $ABEH$, $ACFI$, $ADGJ$, $BCGK$, $BDFL$, $CDEM$, $EFJK$, $EGIL$, $FGHM$. Then in the next extension we produce 24 lines, 12 of which have one of the original four points (AK, AL, AM, BI, BJ, BM, CH, CJ, CL, DH, DI, DK) and 12 more which have points H, I, J, K, L, M. More detail is given in the papers listed in the Introduction.

By a theorem in **(12)** it is unnecessary to consider identifications between elements adjoined in different partial planes. The axioms guarantee that no identifications take place in the first three partial planes; cf. **(8, 9)**. In the next partial plane there is only one possible identification and this produces a plane of order 2. Inspection will reveal that no identification in the fifth partial plane will yield new starts for planes. It is the possible identifications in the sixth partial planes among the 24 lines which warrant attention.

We define a *primitive category* as an ordered quadruple of numbers which indicate the possible identifications among the first 12 lines. The first number is 0 if AK, AL, AM are distinct, 1 if $AK = AL$ or $AK = AM$ or $AL = AM$, 2 if $AK = AL = AM$. Similarly, the second number is 0, 1, or 2 depending on the identifications occurring between BI, BJ, BM. Likewise the third and fourth numbers describe the identifications between the lines through C and through D respectively. The remaining 12 lines may have identifications forced on them, e.g. $AK = AL$ implies $AK = AL = KL$. In any event these remaining lines have few possibilities for further identifications. Either they may have identifications of the type $KL = KM = LM$ whether the forcing identification $AK = AL = AM$ occurs or not, or they may have identifications leading to Fano configurations, e.g. $HI = HJ = IJ$, when certain identifications do not occur among the first 12 lines.

The 24 permutations on A, B, C, D will produce isomorphic copies of a given structure. Two primitive categories are in the same equivalence class defined by the isomorphisms if and only if for each structure satisfying the first primitive category there is some structure in the second primitive category which is isomorphic to the first structure. We define a *category* as a primitive category

(a, b, c, d) with the property that $a \geqslant b \geqslant c \geqslant d$. Every primitive category is equivalent to some category and distinct categories are not equivalent. We abbreviate the notation for the category by dropping the parentheses and sometimes by dropping the zeros also. We order these 15 categories from the highest 2, 2, 2, 2 to the lowest 0, 0, 0, 0 (or NONE) as follows: 2, 2, 2, 2; 2, 2, 2, 1; 2, 2, 2; 2, 2, 1, 1; 2, 2, 1; 2, 2; 2, 1, 1, 1; 2, 1, 1; 2, 1; 2; 1, 1, 1, 1; 1, 1, 1; 1, 1; 1; NONE.

The 2, 2, 2, 2 category appears when completing quadrangles in planes of order 3. Conversely, if one makes the identifications indicated by the 2, 2, 2, 2 category, one obtains the plane of order 3. For related results and details on this, see **(14, 15)**. Some categories describe the structure uniquely, as in this 2, 2, 2, 2 category case. Some categories have several non-isomorphic structures within the same category. To illustrate this, we consider the following examples: (1) an extension with exactly these identifications: $AL = AM = LM$, $BJ = BM = JM$, $CJ = CL = JL$; (2) an extension with exactly these identifications: $AK = AM = KM$, $BI = BJ = IJ$, $CH = CL = HL$. Both examples are in the category 1, 1, 1; yet, by inspection, they are non-isomorphic structures in that the sets of points ALM, BJM, CJL have L, M, J in common whereas the sets AKM, BIJ, CHL have no points in common.

5. Theorems on identifications.

LEMMA 1. *In any singly generated plane of order greater than three which has a quadrangle which singly generates the plane in a way which falls in a* 2, 2, 2, x *category* (x *is* 0 *or* 1), *there exists another quadrangle which singly generates the plane in a way which does not fall in a* 2, 2, 2, x *category.*

Proof. Let A, B, C, D generate the plane so that in addition to the nine sets of collinear points given above we have the sets $AKLM$, $BIJM$, $CHJL$. We do not ascribe any particular identifications between the six lines DH, DI, DK, HI, HK, IK except, of course, to deny that $DHIK$ is a set of collinear points. Since $C = AF \cap BG$ and $D = AG \cap BF$, any element of the plane can be written as an expression in A, B, F, G; or, in other words, A, B, F, G singly generate the plane also. This is displayed below.

$ABHE$	$BFDL$	$HCJL$	ALM	GEL
$AFCP$	$BGCQ$	$HDPQ$	BJM	
$AGDJ$	$FGHM$	$CDEM$	FEJ	

Now to say that this quadrangle A, B, F, G singly generates the plane in a way which is in a 2, 2, 2, x category, then three of the following four sets must be collinear: $ALMQ$, $BJMP$, $FEJQ$, $GELP$. The collinearity of $ALMQ$ implies $Q = K$ since both Q and K lie on lines BG and AL. Similarly the collinearity of $FEJQ$ implies $Q = K$. Therefore, regardless of the choice of the three sets, $Q = K$. Similarly it can be shown that $P = I$. But this shows that $DHIK$ is a collinear set, contrary to the hypothesis.

LEMMA 2. *In any singly generated plane which has a quadrangle which singly generates the plane in a way which falls in a* 2, 2, x, y *category, where* $0 \leqslant y \leqslant x < 2$, *there exists another quadrangle which singly generates the plane in a way which does not fall in a* 2, 2, u, v *category where* $0 \leqslant v \leqslant u \leqslant 2$.

Proof. Let A, B, C, D be a quadrangle which singly generates the plane such that $AKLM$ and $BIJM$ are collinear sets whereas $CHJL$ and $DHIK$ are not collinear sets. Let us assume that the A, B, F, G singly generate this plane in a way which is in the 2, 2, u, v category and $v \neq 2$. Now we display the resulting collinear sets.

$ABHE$	$BFDR$	$HCQR$	AM	GE
$AFCP$	$BGCS$	$HDPS$	BM	
$AGDQ$	$FGHM$	$CDEM$	FE	

From our assumption, at least two of the following four sets must be collinear:

$$AMRS,\ BMPQ,\ FEQS,\ GEPR.$$

If $AMRS$ is a collinear set, then $R = L$, $S = K$;
if $BMPQ$ is a collinear set, then $P = I$, $Q = J$;
if $FEQS$ is a collinear set, then $Q = J$, $S = K$;
if $GEPR$ is a collinear set, then $P = I$, $R = L$.

In choosing any two of these conditions, either $CHJL$ or $DHIK$ is forced to be a collinear set. This fact can be found by inspecting the six ways it can occur. This fact contradicts the hypothesis.

From Lemmas 1 and 2, we obtain

THEOREM 1. *In any singly generated plane of order greater than three there exists a quadrangle which singly generates the plane in a way which falls in a category less than* 2, 2, 0, 0.

From our viewpoint, we are not concerned whether a given completion is for the plane or for its dual plane. In other words, we do not care whether we discuss extensions of quadrangles or quadrilaterals.

LEMMA 3. *In any singly generated plane which has a quadrangle which singly generates the plane in a way which falls in a* 2, x, y, z *category, where* $0 \leqslant z \leqslant y \leqslant x < 2$, *there exists a quadrangle or quadrilateral which singly generates the plane in a way which does not fall in a* 2, u, v, w *category where* $0 \leqslant w \leqslant v \leqslant u \leqslant 2$.

Proof. Let A, B, C, D be a quadrangle which singly generates the plane in a way which falls into one of the categories 2, 1, 1, 1; 2, 1, 1; 2, 1; 2. We can associate three quadrilaterals with this quadrangle. These quadrilaterals are: $a = AB$, $b = BC$, $c = CD$, $d = AD$; $a' = AB$, $b' = BD$, $c' = CD$, $d' = AC$; $a'' = AC$, $b'' = BC$, $c'' = BD$, $d'' = AD$. We note that A can be expressed as ad, $a'd'$, $a''d''$; B as ab, $a'b'$, $b''c''$; C as bc, $c'd'$, $a''b''$; D as cd, $b'c'$, $c''d''$. Thus these quadrilaterals have A, B, C, D in their extensions, and they singly generate the plane; for the method used to make this claim, see the proof of (**12**, Theorem 5.3). Furthermore, we can extend these quadrilaterals using the same notation in a dual fashion and obtain: $e = BD$, $f = EG$, $g = AC$, $h = BI$, $i = EF$, $j = AL$, $k = CL$, $l = FG$, $m = DI$; $e' = BC$, $f' = EF$, $g' = AD$, $h' = BJ$, $i' = EG$, $j' = AK$, $k' = DK$, $l' = FG$, $m' = CJ$; $e'' = CD$, $f'' = FG$, $g'' = AB$, $h'' = CH$, $i'' = EF$, $j'' = AM$, $k'' = BM$, $l'' = EG$, $m'' = DH$.

We are looking for a quadrangle or quadrilateral which singly generates the plane in a way which falls in a category less than 2, 0, 0, 0. Therefore, we consider the case when each of the three quadrilaterals introduced above has an extension in the 2, 0, 0, 0 category or higher category. Please note that we chose A, B, C, D so that $AKLM$ is a collinear set. Having made this choice, we cannot assert that a, b, c, d completing in the 2, 0, 0, 0 category would necessarily imply the existence of the concurrent set $aklm$. All we do assert is that $aklm$ or $bijm$ or $chjl$ or $dhik$ is a concurrent set. Of course this same assertion is true for a, b, c, d completing in any higher category than 2, 0, 0, 0 also. Likewise we now have $a'k'l'm'$ or $b'i'j'm'$ or $c'h'j'l'$ or $d'h'i'k'$ as a concurrent set and $a''k''l''m''$ or $b''i''j''m''$ or $c''h''j''l''$ or $d''h''i''k''$ as a concurrent set.

Suppose $aklm$ is a concurrent set. This says that the lines AB, CL, FG, and DI would be concurrent. But H is, by its definition, the point of intersection of AB and FG. Therefore, H would lie on CL and DI. Thus CHL and DHI would be collinear sets. Similarly, if instead $bijm$ is a concurrent set, the lines BC, EF, AL, DI would meet in the point K. Thus AKL and DIK would be collinear sets. Likewise, if $chjl$ is a concurrent set, then BIM and ALM are collinear sets; if $dhik$ is a concurrent set, then BIJ and CJL are collinear sets. The concurrency of $a'k'l'm'$, $b'i'j'm'$, $c'h'j'l'$, $d'h'i'k'$ implies respectively the collinearity of DHK, CHJ; AKL, CJL; BJM, AKM; BIJ, DIK. The remaining quadrilateral implies that one of the following four pairs of sets is a pair of collinear sets: BIM, DIH; AMK, DHK; CHL, AML; CHJ, BJM.

Suppose $aklm$ and $a'k'l'm'$ are identifications which exist for these quadrilaterals, then we have forced the collinear sets DIH, CLH and DHK, CHJ. To say that DIH and DHK are both collinear sets implies that $DHIK$ is a collinear set, but this is not allowed under the hypothesis that the quadrangle A, B, C, D extends to the plane in a category lower than 2, 2, 2, 2, which is the only category with $DHIK$ as a collinear set.

We wish to abbreviate the argument of the above paragraph so that we can apply it several times. First let us label the conditions as follows: 1 means $aklm$ concurrent, 2 means $bijm$ concurrent, 3 means $chjl$ concurrent, 4 means $dhik$ concurrent; similarly we make 1′ correspond to $a'k'l'm'$, 2′ to $b'i'j'm'$, 3′ to $c'h'j'l'$, 4′ to $d'h'i'k'$, and 1″ to $a''k''l''m''$, etc. Then the above argument will read "1, 1′ implies $DHIK$." Under our assumption, three numbers hold: one unprimed, one primed, and one double primed. Also we note that the conclusions $BIJM$, $CHJL$, $DHIK$ are contrary to the hypothesis.

Cases 1, 1′, x where x is any double-primed number are eliminated by 1, 1′ implies *DHIK*. Cases 1, 2′, x are eliminated by 1, 2′ implies *CHJL*. Cases 1, 3′, x, where $x \neq 3''$, are eliminated by 3′, 1″ implies *BIJM*; 1, 2″ implies *DHIK*; 1, 4″ implies *CHJL*. Cases 1, 4′, x are eliminated by 1, 4′ implies *DHIK*. In summary, if ***aklm*** is a concurrent set, then $c'h'j'l'$ and $c''h''j''l''$ are also. In fact, if ***aklm*** is a concurrent set, we have the collinear sets *AKLM*, *BJM*, *CHL*, *DHI*.

Similarly cases 4, 1′, x and 4, 3′, x are eliminated. Since 4, 1″; 4, 3″; 4, 4″ imply respectively *BIJM*, *CHJL*, *BIJM*, we consider 4, 2′, 2″ and 4, 4′, 2″ as the remaining cases for 4. Now 4′, 2″ implies *DHIK*. Therefore, if ***dhik*** is a concurrent set we have the collinear sets *AKLM*, *BIJ*, *CJL*, *DHK*.

Cases 2, 1′, x; 2, x, 1″; and 2, x, 2″ are eliminated. Since 2′, 3″ as well as 2′, 4″ implies *CHJL*, cases 2, 2′, x are eliminated. Now cases 2, 3′, 3″ and 2, 3′, 4″ imply respectively *AKLM*, *BJM*, *CHL*, *DIK* and *AKLM*, *BJM*, *CHJ*, *DIK*. Thus 2, 3′, x are accounted for. Now 4′, x for $x \neq 3''$ are eliminated, so that only 2, 4′, 3″ needs consideration. From 2, 4′, 3″ we obtain *AKLM*, *BIJ*, *CHL*, *DIK*.

Now we need only to consider the cases involving the concurrency of *chjl*. Cases 3, 3′, x and 3, 4′, x are eliminated. Then 3, 1′, 2″ is the only possible 3, 1′, x case. Also 3, 2′, 1″ and 3, 2′, 2″ are the only possible 3, 2′, x cases. Thus we obtain: *AKLM*, *BIM*, *CHJ*, *DHK*; *AKLM*, *BIM*, *CJL*, *DIH*; *AKLM*, *BIM*, *CJL*, *DHK*.

In summary, the only permissible combinations of collinear sets are displayed below in compact form:

AKLM	*AKLM*	*AKLM*	*AKLM*	*AKLM*	*AKLM*	*AKLM*	*AKLM*
BJM	*BIJ*	*BJM*	*BIJ*	*BIM*	*BIM*	*BIM*	*BJM*
CHL	*CJL*	*CHJ*	*CHL*	*CHJ*	*CJL*	*CJL*	*CHL*
DIH	*DHK*	*DIK*	*DIK*	*DHK*	*DIH*	*DHK*	*DIK*

Thus, in any event, if these quadrilaterals extend by the 2, 0, 0, 0 or higher category, then A, B, C, D extends by the 2, 1, 1, 1 category. By applying the permutations on A, B, C, D, one discovers that the first six displayed above are of the same type, and the last two are of the same type. Let us study the latter type first. Let us assume that *AKLM*, *BIM*, *CJL*, *DHK* are collinear sets. Then upon examining the quadrilateral a, b, c, d, we discover that ***chjl***, ***dik***, ***bij*** are concurrent sets. Now ***dhik*** and ***bijm*** are not concurrent sets. The only possible identification with the lines a, k, l, m is ***akm***. Thus this quadrilateral completes in category 2, 1, 1, 1 or 2, 1, 1. Suppose we have category 2, 1, 1, 1; then by considering the map $a \to C$, $b \to B$, $c \to A$, $d \to D$, this type of completion is dual to *AKLM*, *BIM*, *CHJ*, *DHI*. But applying the results already obtained in a dual fashion, the three quadrangles associated with the quadrilateral a, b, c, d (in a dual fashion to the three quadrilaterals associated with the quadrangle A, B, C, D) singly generate the plane but do not all complete in the 2, 0, 0, 0 or higher category since *AKLM*, *BIM*, *CHJ*, *DHI* is not displayed above. Suppose we have category 2, 1, 1 for the quadrilateral a, b, c, d, then by the same argument there would be some quadrangle associated with the quadrilateral which would not complete in the 2, 0, 0, 0 or any higher category.

Now we assume that $AKLM$, BIJ, CJL, DHK are collinear sets. We now introduce a new quadrilateral which we shall call a, b, c, d as follows: $a = CE$, $b = EF$, $c = FG$, $d = CG$. In this new notation $e = EG$, $f = KM$, $g = CF$ (note that $A = fg$), $h = AE$, $i = IM$, $j = CL$, $k = FL$, $l = IK$, $m = AG$. Immediately we have $akm = D$, $bjm = J$, $ch = H$, $dhk = B$. This quadrilateral singly generates the plane but none of the sets $aklm$, $bijm$, $chjl$, $dhik$ can be concurrent. Thus this quadrilateral lies in a category lower than 2, 0, 0, 0.

From Lemma 3 we can now obtain:

THEOREM 2. *In any singly generated plane of order greater than three there is a quadrangle or quadrilateral which extends to the whole plane in a way which does not fall in the* 2, 0, 0, 0 *category or any higher category.*

COROLLARY. *In any self-dual singly generated plane of order greater than three there is a quadrangle which extends in a way which does not fall in the* 2, 0, 0, 0 *category or any higher category.*

6. Limitations to this approach. Although the planes of orders 2 and 3 are easily removed from our discussion, and Desarguesian planes have special identifications, widely different types of planes may have some of their identifications alike. In particular, we shall now show an example where a dual Veblen-Wedderburn plane and a Hughes plane not only extend in the same category but this extension is of the same type within the category. We use the starred lines common to planes 1.44.1.1 and 1.44.1.2 given in **(10)**. In particular, let $A = (0, 0)$, $B = (1, 0)$, $C = (3, 1)$, $D = (4, 2)$. Then AB: $y = 0$, AC: $y = 6x$, AD: $y = 4x$, BC: $y = 7x + 5$, BD: $y = 3x + 6$, CD: $y = x + 7$. Please note that by $7x + 5$ we mean to imply that the line has slope 7 and y-intercept 5. We do not imply any underlying algebraic structure for multiplication or addition. Then $E = (5, 0)$, $F = (2, 3)$, $G = (5, 6)$. Then EF: $y = 2x + 5$, EG: $x = 5$, FG: $y = x + 4$. Then $H = (8, 0)$, $I = (5, 8)$, $J = (1, 4)$, $K = (0, 5)$, $L = (5, 7)$, $M = (1)$. The identifications to be made are CJ $= CL = JL$, $DI = DK = IK$, $KL = KM = KL$, $IJ = IM = JM$. No other identifications can be made in this partial plane.

The author wrote a SWAC routine to carry out the above computations. These computations are easily checked by hand but tedious in searching the original incidences. Therefore, the author adds: CJ: $y = 8x + 8$, DI: $y = 5x + 5$, KL: $y = x + 5$, IJ: $y = x + 3$.

7. The singly generated plane of Knuth and Walker. In **(12)** several specific planes were shown to be singly generated. Since the writing of that

https://doi.org/10.4153/CJM-1965-015-5 Published online by Cambridge University Press

paper, the author has learned by private communication from Lombardo-Radice that Cofman has shown that every finite Hall plane is singly generated. Therefore, hopefully many classes of finite non-Desarguesian planes may eventually be shown to be singly generated. The purpose of this example then is to show at least one more type of plane which has the possibility of being singly generated.

The example which we shall use is a geometry of order 32. By Bruck's theorem **(7, 9)**, any proper subplane is of order 2, 3, or 5. As in the previous examples, see **(12)**, we can soon discover if the quadrangle completes to a plane of order 2 or 3. In order to handle possible planes of order 5, we recall that the only such is Desarguesian **(2)**. Therefore, we point out an identification forced in Desarguesian geometries, already known in the study of Moebius nets **(18)**. Suppose A, B, C, D is a quadrangle and the points E, F, G obtained as elsewhere in this paper are not collinear. Then non-degenerate triangles ABC and GFE are perspective from D. Applying Desargues' theorem, HIK is a collinear set.

The plane due to Walker **(20)** and Knuth **(13)** and called $P(1)$ in the latter paper is the example. We display below part of the multiplication table for the algebraic system which co-ordinatizes the plane. The additive part of the system forms the elementary Abelian group of order 32. We represent the elements in binary notation; therefore, addition takes place without carries. Finally both distributive laws hold.

X	10000	01000	00100	00010	00001
10000	10000	01000	00100	00010	00001
01000	01000	00100	00010	00001	10100
00100	00100	00010	01001	10100	00101
00010	00010	00001	11010	11110	10111
00001	00001	10010	11011	10000	01110

In this case, the line $y = xm + b$ will mean the set of points (m) at infinity and (x, y) such that the product xm obtained from the table above (and the distributive laws) when added to b by the additive elementary Abelian group yields y. We take $A = (00000, 00000)$, $B = (00000, 10000)$, $C = (10000, 00000)$, $D = (00001, 00001)$. Then AB: $x = 00000$, AC: $y = 00000$, AD: $y = x$, BC: $y = x + 10000$, BD: $y = x(10010) + 10000$, CD: $y = x(10101) + 10101$. Then $E = (00000, 10101)$, $F = (00100, 00000)$, $G = (10000)$. And EF: $y = x(10011) + 10101$, EG: $y = x(10000) + 10101$, FG: $y = x(10000) + 00100$. Thus A, B, C, D does not complete to a plane of order 2. Finally $H = (00000, 00100)$, $I = (10101, 00000)$, $K = (10111, 00111)$. To show that A, B, C, D does not complete to a plane of order 3 or 5, it suffices to show that H, I, K are not collinear. The line joining H and I is: $y = x(01101) + 00100$. Also $(10111)(01101) + 00100 \neq 00111$. Therefore, this plane is singly generated.

Acknowledgments. The author wishes to express his gratitude to Professor Swift, who directed his thesis, which was published as his previous paper and Sections 2 and 3 of this paper. The author wishes to express his appreciation to the computing facility at U.C.L.A. for the use of SWAC and the IBM 7090, and to the Western Data Processing Center for the use of the IBM 7090. The author also wishes to thank the referee for his helpful comments.

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