SOLVABLE-BY-FINITE SUBGROUPS OF GL(2, F)

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1. Introduction. In a recent paper [**5**] Tits proves that a linear group over a field of characteristic zero is either solvable-by-finite or else contains a non-cyclic free subgroup. In this note we determine all the infinite irreducible solvable-by-finite subgroups of GL(2, F), where F is an algebraically closed field of characteristic zero. (Every reducible subgroup of GL(2, F) is metabelian.) In addition, we prove that an irreducible subgroup of GL(2, F) has an irreducible solvable-by-finite subgroup if and only if it contains an element of zero trace.

We use the flattened notation $(\alpha, \beta; \gamma, \delta)$ for the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We denote the 2×2 identity matrix by I, the group $\{\pm I\}$ by E, and the trace of a matrix x by $\operatorname{tr} x$.

2. We begin by listing all the finite non-abelian subgroups of GL(2, F). Dornhoff [**2**, p. 144] lists all the finite non-abelian subgroups of GL(2, $\mathbb{C}$), where $\mathbb{C}$ is the field of complex numbers. However, by [**1**, p. 81], any finite subgroup of GL(2, F) is isomorphic to a subgroup of GL(2, $\mathbb{C}$).

THEOREM 1. *Let G be a finite non-abelian subgroup of* GL(2, F). *Then one of the following holds.*

(a) *G has an abelian normal subgroup of index* 2.

(b) *$G/Z \cong A_4$, S_4 or A_5, where Z $(\neq 1)$ is the centre of G and consists of scalar matrices.*

From now on any group in the former category will be said to be of *type* (a).

COROLLARY 1. *Let H be a finite non-abelian subgroup of* PGL(2, F). *Then either H is of type (a) or else*

$$H \cong A_4, S_4 \quad \textit{or} \quad A_5.$$

Proof. Since PGL(2, F) and PSL(2, F) are isomorphic,

$$H \cong K/E,$$

where K is a finite non-abelian subgroup of SL(2, F). The result now follows from Theorem 1.

Although Corollary 1 is almost certainly well known, it does not appear to be readily accessible in the literature. Let P denote PSL(2, F).

LEMMA 1. $C_P(A_4') = A_4'$.

Proof. By means of a suitable similarity transformation we may assume that one of the involutions in A_4' is $x_0=\pm(\alpha_0, 0; 0, -\alpha_0)$, where $\alpha_0^2=-1$. It is readily verified that

$$C_P(x_0)=\{\pm(\alpha, 0; 0, \alpha^{-1}), \pm(0, \beta; -\beta^{-1}, 0): \alpha, \beta \in F\setminus\{0\}\}.$$

We may take generators of A_4' to be x_0 and $y_0=\pm(0, \gamma; -\gamma^{-1}, 0)$, for some non-zero γ. It follows that

$$C_P(x_0)\cap C_P(y_0)=A_4'.$$

LEMMA 2. *Let G be an irreducible subgroup of* GL(2, F) *containing an abelian normal subgroup N which does not consist entirely of scalar matrices. Then G is of type (a).*

Proof. By means of a suitable similarity transformation we may assume that N consists of diagonal matrices [**1**, p. 26]. By the above hypothesis, N contains an element $x=(\alpha, 0; 0, \beta)$, where $\alpha\neq\beta$.

Let $N_0=C_G(N)$. Then N_0 is a normal subgroup of G consisting of all the diagonal matrices in G. Hence, for every $y\in G$, we have $yxy^{-1}\in N_0$. It follows that either $y\in N_0$ or $y=(0, \gamma; \delta, 0)$, for some $\gamma, \delta\neq 0$. We conclude that $(G:N_0)=2$.

We note that the trace of any element in $G\setminus N_0$ is zero.

THEOREM 2. *Let G be an infinite irreducible solvable-by-finite subgroup of* GL(2, F). *Then either G is of type (a) or else*

$$G/Z\cong A_4, S_4 \quad \text{or} \quad A_5,$$

where Z (consisting of scalar matrices) is the centre of G.

Proof. By Malcev's theorem [**1**, p. 111], G has an abelian normal subgroup A of finite index, which we may assume contains Z. If $A\neq Z$ then G is of type (a) by Lemma 2.

If G/Z is abelian, then, by [**6**, p. 47], it follows that $G/Z\cong A_4'$, in which case G is of type (a).

By Corollary 1 and Lemma 2, we may suppose from now on that G is centre-by-finite, with $G'\not\leq E$, and that G/Z is of type (a), in which case $G''\leq E$. (We note that $G'\leq \mathrm{SL}(2, F)$.)

(i) If $G''=1$, then G is of type (a) by Lemma 2 (with $N=G'$).

(ii) If $G''=E$, then G' is nilpotent of class 2. By [**6**, p. 47], we have

$$G'Z/Z\cong G'/E\cong A_4'.$$

Let $L=G/Z$. Then $L/C_L(L')$ is a subgroup of Aut(L'). By Lemma 1, we deduce that L/L' is an abelian subgroup (of even order) of S_3. Hence $|L|=8$ and $|L'|=4$, which is impossible. Thus $G''\neq E$.

The proof of the theorem is now complete.

COROLLARY 2. *Let K be an infinite irreducible solvable-by-finite subgroup of* SL(2, F). *Then K has an abelian normal subgroup M (containing $-I$) of index 2 such that, for all $x\in K\setminus M$ and for all $y\in M$,*

$$\mathrm{tr}\, x=0 \quad \textit{and} \quad xyx^{-1}=y^{-1}.$$

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In particular, if $K=\langle a, b\rangle$, then precisely two of a^2, b^2, $(ab)^2$ are equal to $-I$ and K/E is the infinite dihedral group.

Proof. By considering the characteristic equation of an element $z\in \mathrm{SL}(2, F)$, we note that

$$\operatorname{tr} z=0 \Longleftrightarrow z^2=-I.$$

Using this fact, the first part of the corollary follows from Theorem 2 and the proof of Lemma 2.

If $K=\langle a, b\rangle$, then precisely two of a, b, ab are in $K\setminus M$ and hence, by Lemma 2, have zero traces. K/E is then the infinite dihedral group since any non-trivial factor of the latter group is finite.

Let $a, b\in \mathrm{SL}(2, F)$ with $\operatorname{tr} a=\alpha$, $\operatorname{tr} b=\beta$, $\operatorname{tr} ab=\gamma$, and let $F_{\alpha,\beta,\gamma}$ be the group generated by a, b. It has been shown [**3**] that $F_{\alpha,\beta,\gamma}$ is reducible if and only if

$$\alpha^2+\beta^2+\gamma^2-\alpha\beta\gamma-4=0.$$

The following result is an immediate consequence of Corollary 2 and Tits' theorem [**5**].

COROLLARY 3. *Let $F_{\alpha,\beta,\gamma}$ be infinite and irreducible.*

(a) *$F_{\alpha,\beta,\gamma}$ is solvable if and only if precisely two of α, β, γ are zero.*

(b) *$F_{\alpha,\beta,\gamma}$ contains a non-cyclic free subgroup if and only if at most one of α, β, γ is zero.*

THEOREM 3. *Let L be an irreducible subgroup* GL(2, F). *Then L contains an irreducible solvable-by-finite subgroup if and only if it contains an element x_0 such that* $\operatorname{tr} x_0=0$.

Proof. If L contains an irreducible solvable-by-finite subgroup then, by Theorems 1, 2 and Lemma 2, it contains a non-scalar matrix x_0 whose square is a scalar matrix. From the characteristic equation of x_0, it follows that $\operatorname{tr} x_0=0$.

Let L contain an element x_0 of zero trace. As before, we may assume that $x_0=(\alpha, 0; 0, -\alpha)$, for some $\alpha\neq 0$. We seek another element $y_0\in L$ of zero trace for which the group $\langle x_0, y_0\rangle$ is irreducible. In this case $\langle x_0, y_0\rangle$ is solvable since it is then, modulo its centre, a dihedral group.

Suppose that none of the conjugates of x_0 in L will suffice. Then, for all $g\in L$, x_0 and gx_0g^{-1} have a common eigenvector, which implies that g has at least one zero entry. Suppose further that no element of L has a zero $(1, 1)$ entry. Then since L is irreducible, there exist $x', y'\in L$ of the form

$$x'=(\beta, \gamma; 0, \delta) \quad \text{and} \quad y'=(\lambda, 0; \mu, \nu),$$

with $\beta, \gamma, \delta, \lambda, \mu, \nu\neq 0$. But the $(1, 2)$ and $(2, 1)$ entries of $x'y'$ are non-zero. It follows that L contains an element $y_0=(0, \varepsilon; \varepsilon', \rho)$, say. By considering the entries of y_0^2 and $(x_0y_0)^2$, we deduce that $\rho=0$. The irreducibility of $\langle x_0, y_0\rangle$ follows from a theorem of Maschke [**1**, p. 26].

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