108.40 A surprising coincidence between Pythagorean triples and an Euler-Cauchy differential equation

In this Note we shall discuss an unexpected aspect of an Euler-Cauchy second order linear differential equation

$$x^2 y'' - bxy' + cy = 0 \qquad (b \geqslant 0, c > 0) \tag{1}$$

having a regular singular point $x = 0$. For more information about the Euler-Cauchy differential equation and differential equations with a regular singular point (see [1, pp. 128 and 192]). Specifically, we will show when given certain Pythagorean triples (see [2, p. 41]), their values may be used in solving (1) with integer exponents provided the conditions on b and c are satisfied. We shall be examining triples (u, v, w) where $w - v = 1$. Our first step is to assume a solution of the form $y(x) = x^r$. Substituting this in (1) yields

$$r(r - 1)x^2 x^{r-2} - brxx^{r-1} + cx^r = 0. \tag{2}$$

Dividing (2) by x^r yields the quadratic equation

$$r^2 - (b + 1)r + c = 0. \tag{3}$$

Solving (3), yields two solutions r and s.

$$r = \frac{(b+1) + \sqrt{(b+1)^2 - 4c}}{2} \qquad \text{and} \qquad s = \frac{(b+1) - \sqrt{(b+1)^2 - 4c}}{2}.$$

Now there could be two real roots, double roots, or two complex roots. When the roots are distinct, the general solution is given by

$$f(x) = Ax^r + Bx^s.$$

When there is a double root, the general solution is

$$g(x) = Ax^r + Bx^r \ln(x).$$

In general, the solutions to (1) are not analytic for $x = 0$. In the case of Pythagorean triples, however, the solutions are positive integral powers of x. We now show the conditions when the two roots are positive integers derived from a Pythagorean triple (u, v, w) where $w - v = 1$. In addition, we show there is a 1-1 correspondence between these triples and the coefficients b and c. First we illustrate this by the following two examples below.

Example 1:

$$x^2 y'' - 4xy' + 4y = 0.$$

Here $b = 4$ and $c = 4$. The solutions to (3) are

$$\frac{1}{2}(4 + 1) \pm \frac{1}{2}\sqrt{(4 + 1)^2 - 4 \times 4} = \frac{5}{2} \pm \frac{3}{2}.$$

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So the general solution to (1) is

$$g(x) = Ax + Bx^4.$$

The Pythagorean triple is $(3, 4, 5)$ where $w = b + 1$, $v = 2\sqrt{c}$ and $u = \sqrt{(b + 1)^2 - 4c} = \sqrt{w^2 - v^2}$.

Example 2:

$$x^2y'' - 12xy' + 36y = 0.$$

In this case, $(5, 12, 13)$ is the Pythagorean triple and the general solution to (1) is therefore

$$g(x) = Ax^9 + Bx^4.$$

Remark 1: We see under these conditions $r = \frac{1}{2}(w + u)$ and $s = \frac{1}{2}(w - u)$.

Remark 2: These triples are created using two consecutive positive integers m and $n = m + 1$. Specifically, $w = m^2 + n^2 = 2m^2 + 2m + 1$, $v = 2mn = 2m(m + 1)$ and $u = n^2 - m^2 = 2m + 1$. This shows we have an infinite set of Pythagorean triples that are generated by any two consecutive positive integers which are directly linked to the exponents of the solution to (1) under suitable conditions on the coefficients.

Remark 3: Consider the case when $m = 6$ and $n = 7$. The corresponding triple is $(13, 84, 85)$ which generates the differential equation

$$x^2y'' - 84xy' + 1764 = 0. \tag{4}$$

Thus, $r = \frac{1}{2}(85 + 13) = 49$ and $s = \frac{1}{2}(85 - 13) = 36$. So the general solution of (4) is

$$y(x) = Ax^{36} + Bx^{49}.$$

Remark 4: There are also other solutions to (1) that do not follow this pattern. For example the equation

$$x^2y'' - 148xy' + 4900y = 0$$

gives rise to exponents $r = 100$ and $s = 49$. The Pythagorean triple in this instance is $(51, 140, 149)$.

What we have shown is this coincidence between a certain set of Pythagorean triples and a specific Euler differential equation occurs infinitely often.

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108.41 Diophantine approximations for a class of recursive sequences

Introduction: The canonical example of a divergent sequence is $\{(-1)^n\}_{n \geqslant 1}$. It is arguably the simplest example of a sequence $\{x_n\}_{n \geqslant 1}$ for which we can explicitly compute that $\overline{\lim_{n \to \infty}}\, x_n = 1 \neq -1 = \underline{\lim_{n \to \infty}}\, x_n$, where we recall that the limit superior and limit inferior are defined, respectively, by $\overline{\lim_{n \to \infty}}\, x_n = \lim_{n \to \infty} \left(\sup_{m \geqslant n} x_m\right)$ and $\underline{\lim_{n \to \infty}}\, x_n = \lim_{n \to \infty} \left(\inf_{m \geqslant n} x_m\right)$. Two closely related divergent sequences are given by $c_n = \cos(n)$ and $s_n = \sin(n)$, $n \geqslant 1$. Similarly, we have $\overline{\lim_{n \to \infty}}\, c_n = 1 \neq -1 = \underline{\lim_{n \to \infty}}\, c_n$, but these calculations are not nearly as simple as the ones for the canonical example $\{(-1)^n\}_{n \geqslant 1}$ since they essentially rely on a deeper fact regarding the equi-distribution modulo 2π of the positive integers.

A natural way to re-write the divergence of a bounded sequence such as $\{c_n\}_{n \geqslant 1}$ is by considering a slightly modified version of it that behaves monotonically. For example, let us define recursively the sequence $\{u_n\}_{n \geqslant 1}$ by

$$u_{n+1} = \max\{u_n, c_n\}, \qquad n \geqslant 1, \tag{1}$$

with $u_1 \in \mathbb{R}$ some fixed value. Proving the convergence of the recursive sequence (1) is a straightforward exercise found in the calculus textbook [1, Exercise 106, p. 505]. Clearly, if $u_1 \geqslant 1$, the sequence is constant and equal to u_1, hence convergent to u_1. Assuming $u_1 < 1$, we see that u_n is non-decreasing and bounded above by 1, therefore convergent by the Monotone Convergence Theorem. The really interesting question however, which is not asked in [1], is finding out *precisely which value* does the sequence $\{u_n\}_{n \geqslant 1}$ converge to. On a closer inspection, we discover that computing the exact value of $\lim_{n \to \infty} u_n$ propels us into the wonderful world of Diophantine approximations, the area of mathematics concerned with the approximation of real numbers by rational ones.