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ESTIMATES ON RENORMALIZATION GROUP TRANSFORMATIONS

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ABSTRACT. We consider a specific realization of the renormalization group (RG) transformation acting on functional measures for scalar quantum fields which are expressible as a polymer expansion times an ultra-violet cutoff Gaussian measure. The new and improved definitions and estimates we present are sufficiently general and powerful to allow iteration of the transformation, hence the analysis of complete renormalization group flows, and hence the construction of a variety of scalar quantum field theories.

1. **Introduction.** The present technical monograph contains the detailed analysis of a single RG transformation of a type general enough to use on scalar quantum field models of a broad class, including infra-red ϕ^4_4, and the non-Gaussian $\phi^4_{4-\epsilon}$ model. It is one of a series of papers by the authors ([BY90], [DH91], [DH93], [BDH94b], [BK94], [BDH94a], [BDH95], [BDH98]) in which we use rigorous renormalization group techniques to study the short and long distance behavior of various quantum field theories.

We consider a family of d-dimensional tori Λ, and scalar fields ϕ which are real valued functions on Λ. In its simplest form the problem is to study functional integrals over the fields of the form

$$\int e^{-V(\Lambda,\phi)}\, d\mu(\phi) \tag{1}$$

where μ is a Gaussian measure on the fields over Λ. The covariance v of μ may be a smoothed inverse Laplacian or more generally given by a sum over scales

$$v(x,y) = \sum_{i=0}^{\infty} L^{-2i\dim\phi} C_i(L^{-i}x, L^{-i}y). \tag{2}$$

Each piece $C_i(x,y)$ is to be a smooth positive-definite function with good decay as $|x-y| \longrightarrow \infty$ (uniformly in the size of Λ) which is almost independent of the index i. The scaling factor L is a large positive integer, and the "scaling dimension" $\dim\phi$ is some real number determined by the model. The potential $V(\Lambda,\phi)$ is some local function of ϕ, for example of the form

$$V(\Lambda,\phi) = \int_\Lambda \Big[\lambda : \phi^4 : + \zeta : (\partial\phi)^2 : + \mu : \phi^2 :\Big]\, d^d x \tag{3}$$

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where the coupling constants λ, ζ, μ are small, and $\lambda > 0$.

The decomposition (2) has the consequence that convolution by the Gaussian measure $\mu * F(\phi) = \int F(\phi+\zeta)\, d\mu(\zeta)$ can be written as a sequence of convolutions $\ldots \mu_2 * \mu_1 * \mu_0 * F$ where μ_i has covariance $L^{-2i \dim \phi} C_i(L^{-i}x, L^{-i}y)$. Therefore integration with respect to μ can be expressed as a sequence of convolutions.

The renormalization group analysis carries out this sequence of convolutions, expressing such an integral in terms of more general integrals

$$\int \mathcal{Z}(\phi)\, d\mu(\phi) \tag{4}$$

with new densities $\mathcal{Z}(\phi)$ that are more complicated but less singular. A characteristic feature of our program is that we keep careful track of the localization of the densities by expressing them in terms of polymer expansions of the form

$$\mathcal{Z}(\phi) = \sum_{\{X_i\}} \prod_i A(X_i, \phi). \tag{5}$$

Here the sum is over collections $\{X_i\}$ of polymers X defined to be unions of unit blocks. The polymer activities $A(X, \phi)$ are required to have their ϕ dependence localized in X and to show decay with the "size of X". The polymer activities generally have more structure, and are expressed in the form

$$A(X, \phi) = \square(X) e^{-V(X,\phi)} + K(X, \phi) \tag{6}$$

where $\square$ is the characteristic function of unit blocks, and V is a local potential similar to the original potential. If $K = 0$ we recover $\mathcal{Z} = e^{-V}$, so K describes the deviation from a strictly local potential.

A single renormalization group transformation replaces A or (K, V) by new activities A' or (K', V'). This happens in three steps. The first step is called "fluctuation": a Gaussian convolution is applied to the density $\mathcal{Z}(\phi)$, and the result is expressed as a new polymer expansion. The essential properties of Gaussian integration we need for this are summarized in the Appendix. The second step is extraction and consists of localizing relevant pieces of K and transferring them to V. One can think of this as the step in which coupling constants are renormalized, and the resulting "renormalization cancellations" are exhibited. The third step is scaling which returns the Gaussian measure to its original form (on a smaller torus). In this way, the RG transformation has been realized in a form ready for iteration. The complete analysis of a RG problem now proceeds by iterating these three steps and tracking the flow of the activities.

The purpose of this paper is to estimate the effect of each of these steps on the polymer activities. In the initial section (Section 2) we describe polymer expansions and the norms we use, and in Section 3 a small norm condition is proved for the specific case of the local ϕ_d^4 potential. Then in Section 4 we give definitions and estimates for the three parts of a single RG transformation: the fluctuation step, the extraction step, and the scaling step. Finally, we include an appendix which states important properties of Gaussian integrals.

Our theorems are variations on earlier proofs of similar theorems, see especially [BDH94a], [BDH95]. However we have technical improvements which are of such a wide scope that a complete new treatment seems necessary. The main changes are:

1. Formerly the detailed estimates on $K(X,\phi)$, particularly in the scaling step, required that the dependence on ϕ be explicitly separated in a dependence on ϕ and a few low order derivatives. Thus polymer activities might be written in the form $K(X,\phi) = \hat{K}(X,\phi,\partial\phi,\partial^2\phi)$. Doing this consistently was a nuisance. The present treatment does away with this extra structure and works directly with $K(X,\phi)$.
2. We have introduced a new notion of "dimension" which applies to polymer activities. With this definition, the split of activities into relevant (dimension $\leq d$) and irrelevant (dimension $> d$) parts becomes more systematic.
3. Formerly the large field behavior of the polymer activities $K(X,\phi)$ was required to be no worse than $\exp(\kappa\|\partial\phi\|_X^2)$ where the norm is a suitable Sobolev norm. This was supposed to be more or less preserved through each step. For infrared problems this causes a lot of trouble because it leads to the introduction of boundary terms, closed polymers, hybrid polymers, *etc*. For problems in which the background potential e^{-V} supplies a stabilizing factor $\exp(-\kappa\|\phi\|^2)$ (such as (3)) we find that it is sufficient make the weaker requirement that large field behavior be no worse than $\exp(\kappa\|\partial^2\phi\|_X^2)$. This decreases under scaling and so is easily preserved. This idea also appears in Lemma 19 of [AR96]. With no boundary terms we are free to take all polymers to be open which is the simplest possibility.
4. Formerly in the extraction step one was allowed to remove pieces from $K(X,\phi)$ only if X was a small set. The new treatment allows extractions for any X. This makes it possible to track more cleanly the leading contributions to $K(X,\phi)$ in low order perturbation theory, something that is essential for good control.
5. We make no assumption of translation invariance, or that fields have their canonical scaling dimension.

The new theorems are especially designed for a problem on non-Gaussian infrared fixed points in $4-\epsilon$ dimensions [BDH98]. However they are quite general and should be appropriate tools for any problem with a scalar field and potential similar to (3). This should be true in any dimension and for both infrared and ultraviolet problems. With modifications we are hopeful that they are useful for more than just scalar field theories.

In this paper, we adopt the convention for constants that $O(1)$ signifies a number which is independent of the parameters. By C we denote numbers which may depend on L, but not on other parameters.

2. Polymers and Norms.

2.1. *Polymer expansions.* The *base space* Λ is the torus $\mathbf{R}^d/L^N\mathbf{Z}^d$ for N an integer. A *polymer* X is a possibly empty union of blocks where a *block*, Δ, is an open *unit* hypercube in Λ centered on a point of the lattice $\mathbf{Z}^d/L^N\mathbf{Z}^d$. Every set considered in what follows will be a polymer unless otherwise specified. For example, Λ is now identified with the polymer $\cup\{\Delta : \Delta\subset\Lambda\}$. An L-block is an open hypercube of side L centered on a point of the lattice $L\mathbf{Z}^d/L^N\mathbf{Z}^d$. An *L-polymer* is a union of L-blocks.

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Polymer activities are complex valued functions $K(X)$ defined on polymers, including the empty set, although one should assume that $K(\emptyset) = 0$, unless cautioned otherwise. Our polymer activities are also functions $K(X, \phi)$ of the fields ϕ.

On the space of functions $A(X), B(X), \ldots$ defined on polymers there is a commutative product [BY90], [GMLMS71], [Rue69]

$$(A \circ B)(X) = \sum_{Y \subset X} A(Y) B(X \setminus Y)$$

and an $\mathcal{E}$xponential

$$\mathcal{E}xp\,(A) = \mathcal{I} + A + \frac{1}{2!} A \circ A + \cdots$$

where $\mathcal{I}(\emptyset) = 1$ and otherwise $\mathcal{I}(X) = 0$. The domain of the $\mathcal{E}$xponential is all functions that vanish on the empty set and

$$\mathcal{E}xp\,(A)(X) = \sum_{\{X_j\}} \prod_j A(X_j) \tag{7}$$

where the sum is over partitions of X into a set of polymers $\{X_j\}$. The $\mathcal{E}$xponential is a terminating series. It deserves attention because $\mathcal{E}xp\,(A + B) = \mathcal{E}xp\,(A) \circ \mathcal{E}xp\,(B)$.

An example of a function on polymers is the ordinary exponential of a local interaction $(e^{-V})(X, \phi)$ where $V(X, \phi)$ is the local potential (3). Note that $(e^{-V})(X, \phi)$ is independent of the values $\phi(x)$ taken on the complement, *i.e.*: $(e^{-V})(X, \phi_1) = (e^{-V})(X, \phi_2)$ if $\phi_1(x) = \phi_2(x)$ for all $x \in X$. All polymer activities we consider will have this localization property. Note also that the function $(e^{-V})(X)$ is *multiplicative* which means that $(e^{-V})(X \cup Y) = (e^{-V})(X)(e^{-V})(Y)$ whenever $X \cap Y = \emptyset$.

Another example is the function

$$\square(X) = \begin{cases} 1 & \text{if } X \text{ is a unit block} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Since every polymer has a unique decomposition into blocks, it follows from (7) that $\mathcal{E}xp\,(\square) = 1$, the function that is identically one on all polymers, and more generally $\mathcal{E}xp\,(\square e^{-V}) = e^{-V}$.

Thus, the initial density of a local field theory has the form $\mathcal{Z} = \big(\mathcal{E}xp\,(\square e^{-V})\big)(\Lambda)$. The renormalization group does not preserve this form, but $\big(\mathcal{E}xp\,(\square e^{-V} + K)\big)(\Lambda)$, where K is a field dependent polymer activity, is preserved in form. Note that

$$\big(\mathcal{E}xp\,(\square e^{-V} + K)\big)(Y) = \sum_{\{X_j\}} \exp\big(-V(Y \setminus X)\big) \prod_j K(X_j) \tag{9}$$

where now the sum is over sets $\{X_j\}$ of disjoint polymers in Y and $X = \cup_j X_j$.

In general the polymer activities $K(X, \phi)$ we need to consider have certain decay properties depending on the "size" of X, certain growth and decay behavior depending on the value of ϕ and its derivatives, and finally analyticity in the variable ϕ. All three properties are controlled by imposing a finite norm condition on K, for one of a general family of norms we now introduce.

2.2. *Decay in X: the large set regulator* Γ. Let $K(X)$ be a polymer activity (with possible ϕ dependence suppressed). The decay of K in the "size" of X is controlled by a norm of the following type

$$\|K\|_{\Gamma_n} = \sup_{\Delta} \sum_{X \supset \Delta} |K(X)| \Gamma_n(X) \tag{10}$$

Here the *large set regulators* $\Gamma_n(X)$ are defined in dimension d by

$$\begin{aligned} \Gamma_n(X) &= 2^{n|X|}\Gamma(X) \\ \Gamma(X) &= L^{(d+2)|X|}\Theta(X) \\ \Theta(X) &= \inf_{\tau} \prod_{b \in \tau} \theta(|b|). \end{aligned} \tag{11}$$

The *volume* $|X|$ of X is the number of blocks in X. The infimum is over trees τ composed of bonds b connecting the centers of the blocks in X. The length $|b|$ of a bond $b = xy$ is defined by the ℓ^∞-metric $\sup_{1 \le j \le d} |x_j - y_j|$. θ is a rapidly increasing function described in Lemma 1 below.

DEFINITION 1. 1. A polymer X is called a *small set* if its closure $\bar{X}$ is connected and it has volume $|X| \le 2^d$. Otherwise it is a *large set*.

2. The L-*closure* $\bar{X}^L$ of a polymer X is the smallest union of L-blocks containing X.

LEMMA 1. *Let the function θ be chosen so that $\theta(s) = 1$ for $s = 0, 1$ and*

$$\theta\big(\{s/L\}\big) \ge L^{-d-1}\theta(s), \quad s = 2, 3, \ldots \tag{12}$$

where $\{x\}$ denotes the smallest integer greater than or equal to x. Then, for each $q, p = 0, 1, 2, \ldots$ there is a constant $c_{q,p}$ such that for L sufficiently large and for any polymer X,

$$\Gamma_q(L^{-1}\bar{X}^L) \le c_{q,p}\Gamma_{q-p}(X). \tag{13}$$

For any large set X, there is a stronger bound

$$\Gamma_q(L^{-1}\bar{X}^L) \le c_{q,p}L^{-d-1}\Gamma_{q-p}(X). \tag{14}$$

The proof of the related Lemma 3.2 in [BY90] also gives this bound, (but the definition of small set given in [BY90] is incorrect and was corrected in [DH92]).

2.3. *Smoothness in the fields.* Functionals of ϕ are defined on the Banach space $C^r(\Lambda)$ of r-times continuously differentiable fields with the norm

$$\|f\|_{C^r(\Lambda)} = \max_{|\alpha| \le r} \sup_x |\partial^\alpha f(x)|. \tag{15}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_a \alpha_a$, and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. Derivatives with respect to ϕ are symmetric multilinear functionals $f_1, \ldots, f_n \longrightarrow K_n(X, \phi; f_1, \ldots, f_n)$ on this Banach space defined by

$$\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_n} K\big(X, \phi + \sum s_i f_i\big)\big|_{s=0} = K_n(X, \phi; f_1, \cdots, f_n).$$

Note that such multilinear functionals define distributions on Λ^n by the kernel theorem. The choice of r is a restriction on how singular these distributions are allowed to be and is determined by the model being considered.

We have further conditions on the polymer activities $K(X,\phi)$:

1. Each $K(X,\phi)$ should be Frèchet-analytic in ϕ in a complex strip around the real space $C^r(\bar{\Lambda})$. It is equivalent to the condition that they are continuous functions on $C^r(\bar{\Lambda})$ and that the finite dimensional functions $s \longrightarrow K(X,\phi+\sum s_i f_i)$ are all analytic in a strip.
2. We assume that the ϕ dependence of $K(X,\phi)$ is localized in $\bar{X}$ in the sense that it is actually a function on $C^r(\bar{X})$ which is evaluated on $\phi \in C^r(\bar{\Lambda})$ by first taking the restriction of ϕ to $\bar{X}$. Then $K_n(X,\phi;f_1,\ldots,f_n)$ is also defined for $f_j \in C^r(\bar{X})$ and for $f_j \in C^r(\bar{\Lambda})$ by restriction. The derivative vanishes if any f_j vanishes on $\bar{X}$. ($C^r(\bar{X})$ is all functions in $C^r(X)$ such that partial derivatives have continuous boundary values. The norms $\|\ \|_{C^r(X)}$ and $\|\ \|_{C^r(\bar{X})}$ coincide).

The size of the derivatives $K_n(X,\phi)$ is naturally measured by the norm

$$\|K_n(X,\phi)\| = \sup\Big\{|K_n(X,\phi;f_1,\ldots,f_n)| : f_j \in C^r(\bar{X}), \|f_j\|_{C^r(X)} \le 1\Big\}. \tag{16}$$

for $n>0$ and $\|K_0(X,\phi)\| = |K_0(X,\phi)|$.

However, in the fluctuation step we find we need a localized version. Therefore we consider derivatives restricted to neighborhoods

$$\tilde{\Delta} = \big\{x : \mathrm{dist}(x,\Delta) < 1/4\big\} \tag{17}$$

of blocks Δ. Let $\Delta^{\times n} = (\Delta_1,\ldots,\Delta_n)$ be an n-tuple of blocks. The localized norm is

$$\|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} = \sup\Big\{|K_n(f_1,\ldots,f_n)| : f_j \in C^r(\bar{X}), \|f_j\|_{C^r(X)} \le 1, \mathrm{supp}\, f_j \subset \tilde{\Delta}_j \cap \bar{X}\Big\}. \tag{18}$$

A connection between the natural norm (16) and the localized version is given if we select a smooth partition of unity χ_Δ indexed by unit blocks Δ such that $\mathrm{supp}\,\chi_\Delta \subset \tilde{\Delta}$. We assume that each χ_Δ is a translate of a fixed function χ. We define $\|\chi\|$ as the best constant such that

$$\|\chi_\Delta f\| \le \|\chi\|\,\|f\|. \tag{19}$$

LEMMA 2. *For any ϕ and any polymer activity K*

$$\|K_n(X,\phi)\| \le \|\chi\|^n \sum_{\Delta^{\times n}} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}}. \tag{20}$$

PROOF.

$$\begin{aligned} \|K_n(X,\phi)\| &= \sup_f |K_n(X,\phi;f^{\times n})| \\ &\le \sup_f \sum_{\Delta^{\times n}} \Big|K_n\big(X,\phi;(\chi_\Delta f)^{\times n}\big)\Big| \\ &\le \sum_{\Delta^{\times n}} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} \|\chi\|^n_{C^r}. \end{aligned}$$

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2.4. *Growth in the fields: the large field regulator G.* The growth of $K(X,\phi)$ as a function of (derivatives of) ϕ is controlled by a *large field regulator* which is a functional $G(X,\phi)$ with properties:

G1 $$G(X,0) \geq 1$$

G2 $$G(X \cup Y, \phi) \geq G(X,\phi)G(Y,\phi) \quad \text{if } X \cap Y = \emptyset.$$

Our standard choice will be $G = G(\kappa) = G(\kappa, X, \phi)$ where

$$G(\kappa, X, \phi) = \exp\Big(\kappa \|\phi\|^2_{X,2,\sigma}\Big). \tag{21}$$

Here

$$\|\phi\|^2_{X,a,b} = \sum_{a \leq |\alpha| \leq b} \|\partial^\alpha \phi\|^2_X. \tag{22}$$

and $\|\phi\|_X$ is the $L^2(X)$ norm. We take σ large enough so that this norm can be used in Sobolev inequalities for any low order derivative $\partial^\alpha\phi$, a point we discuss shortly.

For any such G define a norm on derivatives $K_n(X,\phi)$ by

$$\|K_n(X)\|_G = \sum_{\Delta^{\times n}} \sup_{\phi \in C^r} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} G^{-1}(X,\phi) \tag{23}$$

where $\Delta^{\times n} = (\Delta_1, \ldots, \Delta_n)$.

For these norms to be useful, we will need further properties for the regulators G. The fluctuation step involves convolution with a Gaussian measure μ_C with a covariance operator C which has a kernel $C(x,y)$ with good decay and regularity properties. We discuss general properties of Gaussian convolution in the Appendix.

To control the fluctuation step we will need a family of regulators $G_t(X,\phi)$ that are integrable with respect to μ_C in the sense that

G3 $$\mu_{(t-s)C} * G_s(X,\phi) \leq G_t(X,\phi).$$

These will generally have the form

$$G_t(X,\phi) = \Big(2^{|X|} G^\#(X,\phi)\Big)^t G(X,\phi)^{1-t} \tag{24}$$

for some regulator $G^\#$. If we choose $G(X,\phi) = G(\kappa, X, \phi)$ and

$$G^\#(\kappa, X, \phi) = G(2\kappa, X, \phi)$$

then the following lemma shows that for κ sufficiently small the integrability is satisfied.

LEMMA 3. *There exists $\kappa_0 > 0$ depending only on norms of C such that for all $\kappa \in [0,\kappa_0]$ property* **G3** *holds for all $s < t \in [0,1]$.*

PROOF. Let $U(s,\phi) = \log G_s(\phi)$. It is enough to prove that

$$\frac{\partial U}{\partial s} - \Delta_C U - \frac{1}{2} C\left(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi}\right) \geq 0 \tag{25}$$

where the functional Laplacian Δ_C is defined in the Appendix, and

$$C\left(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi}\right) = \int dx\, dy C(x,y) \frac{\delta U}{\delta\phi(x)} \frac{\delta U}{\delta\phi(y)}.$$

This is because of the implications

$$\begin{aligned}
\frac{\partial U}{\partial s} - \Delta_C U - \frac{1}{2} C\left(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi}\right) \geq 0 &\Rightarrow \frac{\partial G_s}{\partial s} - \Delta_C G_s \geq 0 \\
&\Rightarrow \mu_{(t-s)C} * \left(\frac{\partial G_s}{\partial s} - \Delta_C G_s\right) \geq 0 \\
&\Rightarrow \frac{\partial}{\partial s} \mu_{(t-s)C} * G_s(X) \geq 0 \text{ for } s \in (0,t) \\
&\Rightarrow \mu_{(t-s)C} * G_s(X) \leq G_t(X).
\end{aligned}$$

where we have used the functional heat equation discussed in the Appendix.

From the definitions

$$U = t\log(2)|X| + \kappa \sum_{2\leq|\alpha|\leq\sigma} \int_X |\partial^\alpha \phi|^2 \cdot \big((1-t) + 2t\big) \tag{26}$$

we verify (25). For example if we choose κ_0 small so that

$$\kappa_0 \sup_{x,y} |\partial_x^\alpha \partial_y^\beta C(x,y)| \tag{27}$$

is small for $2 \leq |\alpha|, |\beta| \leq \sigma$ then the ϕ independent term in $\partial U/\partial t$ dominates $\Delta_C U$. To dominate $C(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi})$ let $\|C\|$ be the operator norm on the (matrix-valued) kernels $|\partial_x^\alpha \partial_y^\beta C(x,y)|$, $2 \leq |\alpha|, |\beta| \leq \sigma$. Then we have

$$\begin{aligned}
\left| C\left(\frac{\partial U}{\partial \phi}, \frac{\partial U}{\partial \phi}\right)\right| &= 2(1+t)^2\kappa^2 \sum_{\alpha,\beta} \int_{X\times X} (\partial^{\alpha+\beta} C)(x,y)(\partial^\alpha\phi)(x)(\partial^\beta\phi)(y)\, dx\, dy \\
&\leq 8\kappa^2 \|C\|\, \|\phi\|_{X,2,\sigma}^2
\end{aligned}$$

and this is smaller than the ϕ dependent terms in $\partial U/\partial t$ when $\kappa_0\|C\|$ is sufficiently small, because κ^2 is small compared with κ. ∎

Next we need some special Sobolev inequalities in which intermediate derivatives are omitted.

LEMMA 4. *Let $\sigma > d/2 + 2$. Let X be any polymer and let Y be an L^{-1}-scale polymer (possibly empty) contained in X such that $\Delta \cap Y$ is a small polymer for all unit blocks $\Delta \subset X$. Then for $|\alpha| \leq 1$ and $x \in X$*

$$|\partial^\alpha \phi(x)| \leq O(1)\big(\|\phi\|_{X\setminus Y} + \|\phi\|_{X,2,\sigma}\big). \tag{28}$$

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PROOF. It suffices to prove the lemma for a unit block $X = \Delta$. Let $\rho \geq 0$ be a smooth function compactly supported in $X \setminus Y$ with $\int \rho = 1$ chosen so that $\|\partial^\alpha \rho\|_{\Delta\setminus Y} \leq O(1)$ for $|\alpha| \leq 1$. This is possible by the condition on $\Delta \cap Y$. We write

$$\partial^\alpha \phi(x) = \int_{\Delta\setminus Y} dy \rho(y) \partial^\alpha \phi(y) + \int_{\Delta\setminus Y} dy \rho(y) \big(\partial^\alpha \phi(x) - \partial^\alpha \phi(y)\big). \tag{29}$$

First suppose $|\alpha| = 1$. By integration by parts and the Schwarz inequality the first term is bounded by

$$\|\partial^\alpha \rho\|_{\Delta\setminus Y} \|\phi\|_{\Delta\setminus Y} \leq O(1) \|\phi\|_{\Delta\setminus Y}. \tag{30}$$

Joining x, y by a path in Δ we find the second term is bounded by

$$O(1) \sup_{\substack{\beta:|\beta|=2 \\ z\in\Delta}} |\partial^\beta \phi(z)| \leq O(1) \|\phi\|_{\Delta,2,\sigma} \tag{31}$$

by a Sobolev inequality. Thus the bound holds for $|\alpha| = 1$. For $|\alpha| = 0$ we again use (29). Now the first term is bounded by

$$\|\rho\|_{\Delta\setminus Y} \|\phi\|_{\Delta\setminus Y} \leq O(1) \|\phi\|_{\Delta\setminus Y}. \tag{32}$$

The second term is bounded by

$$O(1) \sup_{\substack{\beta:|\beta|=1 \\ z\in\Delta}} |\partial^\beta \phi(z)| \leq O(1) \big(\|\phi\|_{\Delta\setminus Y} + \|\phi\|_{\Delta,2,\sigma} \big) \tag{33}$$

by the first result and the desired bound follows. ∎

2.5. *Norms.* We introduce the final ingredient, *the derivative parameter* $h > 0$, and construct norms on $K = K(X, \phi)$. Our preferred choice is

$$\begin{aligned} \|K(X)\|_{G,h} &= \sum_n \frac{h^n}{n!} \|K_n(X)\|_G \\ \|K\|_{G,h,\Gamma} &= \big\| \|K(\cdot)\|_{G,h} \big\|_\Gamma . \end{aligned} \tag{34}$$

However we sometimes want to change the order and take

$$\begin{aligned} \|K_n\|_{G,\Gamma} &= \big\| \|K_n(\cdot)\|_G \big\|_\Gamma \\ \|K\|_{G,\Gamma,h} &= \sum_n \frac{h^n}{n!} \|K_n\|_{G,\Gamma} . \end{aligned} \tag{35}$$

There is also a limiting case of the norms $\|K\|_{G,\Gamma,h}$ in which G^{-1} is concentrated at $\phi = 0$. These are called *kernel norms* and are defined by

$$\begin{aligned} |K_n(X,0)| &= \sum_{\Delta^{\times n}} \|K_n(X,0)\|_{\tilde{\Delta}^{\times n}} \\ |K|_{h,\Gamma} &= \Big\| \sum_n \frac{h^n}{n!} |K_n(\cdot,0)| \Big\|_\Gamma \\ |K|_{\Gamma,h} &= \sum_n \frac{h^n}{n!} \big\| |K_n(\cdot,0)| \big\|_\Gamma . \end{aligned} \tag{36}$$

Let $G_0(\alpha, X, \phi) = \exp(\alpha \int_X |\phi|^2)$. Then $G_0(\alpha) G(\alpha)$ is a regulator and we have

LEMMA 5. *Suppose* $\|K\|_{G_0(1)G(1),h,\Gamma} < \infty$. *Then*

$$|K|_{h,\Gamma} = \lim_{\alpha\longrightarrow\infty} \|K\|_{G_0(\alpha)G(\alpha),h,\Gamma} \tag{37}$$

and similarly for $|K|_{\Gamma,h}$.

PROOF. We show that

$$\lim_{\alpha\longrightarrow\infty} \sup_{\phi} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} G_0(\alpha,X,\phi)^{-1} G(\alpha,X,\phi)^{-1} = \|K_n(X,0)\|_{\tilde{\Delta}^{\times n}} \tag{38}$$

assuming that

$$C = \sup_{\phi} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} G_0(1,X,\phi)^{-1} G(1,X,\phi)^{-1} < \infty. \tag{39}$$

The supremum is greater than or equal to the value at zero which is $\|K_n(X,0)\|_{\tilde{\Delta}^{\times n}}$. We claim that for α sufficiently large the supremum is taken on the set $\|\phi\|_{C_r} \leq \alpha^{-1/4}$. To see this note that

$$\begin{aligned} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} G_0(\alpha,X,\phi)^{-1} G(\alpha,X,\phi)^{-1} &\leq CG_0(\alpha-1,X,\phi)^{-1} G(\alpha-1,X,\phi)^{-1} \\ &\leq C\exp\bigl(-O(1)(\alpha-1)\|\phi\|^2_{X,0,\sigma}\bigr) \\ &\leq C\exp\bigl(-O(1)(\alpha-1)\|\phi\|^2_{C_r}\bigr). \end{aligned} \tag{40}$$

Here $O(1)$ depends on X, and we have used Lemma 4 and a Sobolev inequality. For $\|\phi\|_{C_r} \geq \alpha^{-1/4}$ this goes to zero as $\alpha \longrightarrow \infty$ and hence is smaller than $\|K_n(X,0)\|_{\tilde{\Delta}^{\times n}}$ for α sufficiently large. Hence the claim.

Having established the claim, it now suffices to prove the lemma with the supremum taken over $\|\phi\|_{C^r} \leq \alpha^{-1/4}$. However we have

$$\begin{aligned} \|K_n(X,0)\|_{\tilde{\Delta}^{\times n}} &\leq \sup_{\|\phi\|_{C^r}\leq\alpha^{-1/4}} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}} G_0(\alpha,X,\phi)^{-1} G(\alpha,X,\phi)^{-1} \\ &\leq \sup_{\|\phi\|_{C^r}\leq\alpha^{-1/4}} \|K_n(X,\phi)\|_{\tilde{\Delta}^{\times n}}. \end{aligned} \tag{41}$$

The right end of this string of inequalities converges to the left end by the continuity of the function at $\phi = 0$. Hence the result. ∎

In the case where $K(X,\phi)$ is invariant under translations $X \longrightarrow X + a$, $\phi(x) \longrightarrow \phi(x-a)$ by lattice vectors a we have,

$$\|K\|_{G,\Gamma,h} = \|K\|_{G,h,\Gamma}; \quad |K|_{h,\Gamma} = |K|_{\Gamma,h}. \tag{42}$$

Otherwise there is

LEMMA 6. *For any* $h' < h$

$$\|K\|_{G,h',\Gamma} \leq \|K\|_{G,\Gamma,h'} \leq \left(1 - \frac{h'}{h}\right)^{-1} \|K\|_{G,h,\Gamma}$$

together with an analogous result for the kernel norms.

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PROOF. The first inequality is easy. For the second let $j_n(\Delta) = \sum_{X\supset\Delta}\Gamma(X)\|K_n(X)\|_G$

$$\begin{aligned}\|K\|_{G,\Gamma,h'} &= \sum_n \frac{h'^n}{n!}\sup_\Delta j_n(\Delta) \le \Big(\sum_n \frac{h'^n}{h^n}\Big)\sup_{\Delta,n}\frac{h^n}{n!}j_n(\Delta)\\ &\le \left(1-\frac{h'}{h}\right)^{-1}\sup_\Delta\sum_n\frac{h^n}{n!}j_n(\Delta) = \left(1-\frac{h'}{h}\right)^{-1}\|K\|_{G,h,\Gamma}.\end{aligned}$$

The bound for the kernel norm is a corollary by (37). ■

Property **G2** implies the following lemma.

LEMMA 7. *For all disjoint polymers X, Y*

$$\begin{aligned}&\|K_n(X)K'_m(Y)\|_G \le \|K_n(X)\|_G\|K'_m(Y)\|_G\\ &\|K(X)K'(Y)\|_{G,h} \le \|K(X)\|_{G,h}\|K'(Y)\|_{G,h}\end{aligned} \tag{43}$$

where G is evaluated on $X\cup Y$ on the left side. If X, Y are permitted to intersect then

$$\begin{aligned}&\|K_n(X)K'_m(Y)\|_{G_1G_2} \le \|K_n(X)\|_{G_1}\|K'_m(Y)\|_{G_2}\\ &\|K(X)K'(Y)\|_{G_1G_2,h} \le \|K(X)\|_{G_1,h}\|K'(Y)\|_{G_2,h}\\ &|K(X)K'(Y)|_h \le |K(X)|_h|K'(Y)|_h.\end{aligned} \tag{44}$$

3. **Bounds on densities of the form $e^{-V(\phi)}$.** An appropriate illustration of all of the preceding formulation is to analyze densities of the form

$$Z(\Lambda,\phi) = e^{-V}(\Lambda,\phi) = \big(\mathcal{E}xp\,(\Box e^{-V})\big)(\Lambda,\phi) \tag{45}$$

for some specific potential V. This is the usual starting point of the renormalization group in quantum field theory, and is moreover, in most elementary examples, the form of the leading approximation to the flow of the RG.

We shall give a bound on $\|e^{-V(X)}\|_{G,h}$ for the general ϕ^4 potential in d-dimensions:

$$V(X) = \int_\Lambda\big[\lambda : \phi^4 : +\zeta_1 : (\partial\phi)^2 : +\zeta_2 : \phi(-\Delta)\phi : +\mu : \phi^2 :\big]\,dx \tag{46}$$

and the regulator $G = G(\kappa) = G(\kappa, X, \phi)$ defined in Section 2.4. Since we have an ultraviolet cutoff this is not a deep result. The proof we present here is a slight variation on the proof in [BDH95].

In fact we prove a stronger bound with $G(\kappa)G_0^{-1}(\kappa_0)$ where

$$G_0(\kappa_0, X, \phi) = \exp\big(\kappa_0\|\phi\|_X^2\big) \tag{47}$$

and then use

$$\|e^{-V(X)}\|_{G(\kappa),h} \le \|e^{-V(X)}\|_{G(\kappa)G_0^{-1}(\kappa_0),h}. \tag{48}$$

The G_0^{-1} means that the norm remembers the stabilizing effect of ϕ^4 at large ϕ.

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The bound is proved under the following hypotheses. $\mathrm{Re}(\lambda)h^4$ is positive and bounded by a sufficiently small constant, and $\mathrm{Im}(\lambda)/\mathrm{Re}(\lambda)$ is bounded by a constant. Furthermore we assume

$$|\mu|h^2 \le \mathrm{Re}(\lambda)h^4, \quad |\zeta|h^2 \le \mathrm{Re}(\lambda)h^4, \quad \kappa_0 h^2 \le \mathrm{Re}(\lambda)h^4 \tag{49}$$

where $|\zeta| = \max(\zeta_1, \zeta_2)$ and finally that $h^{-2}v(0)$, $h^{-2}\partial^2 v(0)$, $h^{-2}\kappa^{-1}$, $h^{-2}\kappa_0^{-1}$ are all bounded by constants. In all the above, constants only depend on the dimension d.

THEOREM 1. *Under the above hypotheses for any polymer X:*

$$\|e^{-V(X)}\|_{G(\kappa)G_0^{-1}(\kappa_0),h} \le 2^{|X|}; \quad |e^{-V(X)}|_h \le 2^{|X|}. \tag{50}$$

If X is a subset of a unit block Δ, *then*

$$\|e^{-V(X)}\|_{G(\kappa,\Delta)G_0^{-1}(\kappa_0,\Delta),h} \le 2; \quad |e^{-V(X)}|_h \le 2. \tag{51}$$

REMARK. The theorem also holds if the coupling constants are moved inside the integrals and are permitted to have x dependence but are bounded as described above.

PROOF. We first prove the result when X is a single block Δ with $V(\phi) = V(\Delta, \phi)$. Let $f^{\times n} = (f_1, \ldots, f_n)$ be $C^r(\bar{\Delta})$ functions with norm one. We compute the derivatives of e^{-V} by

$$\frac{h^n}{n!}(e^{-V})_n(\phi; f^{\times n}) = \frac{h^n}{n!}\sum_\pi (-1)^{|\pi|}\prod_j V_{n_j}(\phi; f^{\times \pi_j})e^{-V(\phi)}.$$

Here $\pi = \{\pi_j\}$ is any partition of $1, \ldots, n$ and $n_j = |\pi_j|$, and $f^{\times \pi_j}$ denotes the set of functions f_i with $i \in \pi_j$. We use $|V_{n_j}(\phi; f^{\times \pi_j})| \le \|V_{n_j}(\phi)\|$. Furthermore classify the partitions by the number of elements r and order the elements in the partition which overcounts by a factor of $r!$. Finally use the fact that there are $n!/n_1! \ldots n_r!$ ordered partitions with given n_j. This yields

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\| \le \sum_r \frac{1}{r!}\sum_{\mathbf{n}}\prod_{j=1,\ldots,r}\left[\frac{h^{n_j}}{n_j!}\|V_{n_j}(\phi)\|\right]e^{-\mathrm{Re}(V(\phi))}.$$

Dropping the constraint $\sum_j n_j = n$ gives

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\| \le \exp\Big(-\mathrm{Re}\big(V(\phi)\big) + \sum_{n\ge 1}\frac{h^n}{n!}\|V_n(\phi)\|\Big).$$

Now we consider monomials in V_n separately. If $q(\phi) = \phi^m$, then

$$h^n\Big\|\Big(\int_\Delta q(\phi)\,dx\Big)_n\Big\| \le h^n\int_\Delta q^{(n)}(|\phi|)\,dx = h^m\int_\Delta q^{(n)}(|\phi|/h)\,dx. \tag{52}$$

Since $:\phi^4:_v = \phi^4 - 6v(0)\phi^2 + 3v(0)^2$, we apply the calculation to each term and obtain

$$-\mathrm{Re}(\lambda)\int_\Delta :\phi^4:_v + \sum_{n\ge 1}\frac{h^n}{n!}\Big\|\Big(\lambda\int_\Delta \phi^4:_v\Big)_n\Big\| \le -\mathrm{Re}(\lambda)h^4\int_\Delta p(h^{-1}|\phi|) \tag{53}$$

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where $p(t) = t^4+$ terms of lower degree in t. The coefficients of lower order terms are of order one because $h_0^{-2}v(0) \leq 1$. Clearly this is bounded above by $O\big(\mathrm{Re}(\lambda)h^4\big)$. Furthermore, since $|\mu|h^2 \leq \mathrm{Re}(\lambda)h^4$, the term $\int :\phi^2:_v$ can be included without affecting this upper bound. In the same way we can include a term like $\kappa_0\|\phi\|_\Delta^2$ from G_0. For the

$$W = \zeta_1 \int :(\partial\phi)^2: + \zeta_2 \int :\phi(-\Delta)\phi:$$

term in V, bounds such as

$$h\Big\|\Big(\int_\Delta (\partial\phi)^2\,dx\Big)_1\Big\| \leq 2h^2 \int_\Delta \sum_i |\partial_i\phi| h^{-1} \tag{54}$$

together with $|\zeta|h^2 < \mathrm{Re}(\lambda)h^4$ assure us that

$$-W(\phi) + \sum_{n\geq 1} \frac{h^n}{n!}\|W_n(\phi)\| \leq O\big(\mathrm{Re}(\lambda h^4)\big)(1 + \|\phi\|_{\Delta,0,\sigma}^2/h^2). \tag{55}$$

We estimate $\|\partial\phi\|_\Delta^2$ using Lemma 4, and estimate $h^{-2}\|\phi\|_\Delta^2 \leq O(1)\kappa_0\|\phi\|_\Delta^2$ as before to obtain

$$-\mathrm{Re}\big(V(\phi)\big) + \sum_{n\geq 1} \frac{h^n}{n!}\|V_n(\phi)\| \leq O\big(\mathrm{Re}(\lambda)h^4\big)(1 + \|\phi\|_{\Delta,2,\sigma}^2/h^2). \tag{56}$$

In the second factor on the right hand side, replace h^{-2} by κ and bound it by G. From all the above it follows that

$$\frac{h^n}{n!} \sup_\phi \|(e^{-V(\phi)})_n\| G^{-1}(\phi)G_0(\phi) \leq \exp\Big(O\big(\mathrm{Re}(\lambda)h^4\big)\Big). \tag{57}$$

This argument was valid for arbitrary large h. Therefore we can replace h by $4ah$ with $a \geq 1$ and dominate the norm on $\tilde{\Delta}^{\times n}$ by the full norm to conclude

$$\frac{(ah)^n}{n!} \sup_\phi \|(e^{-V(\phi)})_n\|_{\tilde{\Delta}^{\times n}} G^{-1}(\phi)G_0(\phi) \leq 4^{-n}\exp\Big(O\big(\mathrm{Re}(\lambda)h^4\big)\Big). \tag{58}$$

Since the $\tilde{\Delta}^{\times n}$ norm vanishes unless every block neighborhood in $\tilde{\Delta}^{\times n}$ intersects Δ, and since there are $a = 3^d$ tilde blocks that intersect Δ, we also have

$$\frac{h^n}{n!}\|(e^{-V(\phi)})_n\|_{GG_0^{-1}} \leq \frac{(ah)^n}{n!} \sup_\phi \|(e^{-V(\phi)})_n(\phi)\| G^{-1}(\phi)G_0(\phi). \tag{59}$$

We combine these and take the parameters sufficiently small, so that the sum over n is bounded by 2 as required.

In the general case, where X is no longer a single block, we write

$$e^{-V(X)} = \prod_{\Delta\in X} e^{-V(\Delta)}.$$

By the multiplicative property (43),

$$\|e^{-V(X)}\|_{GG_0^{-1},h} \leq \prod_{\Delta\in X} \|e^{-V(\Delta)}\|_{GG_0^{-1},h} \leq 2^{|X|}.$$

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The bound for the kernel norm is a corollary by (37), taking $\kappa_0 = -\alpha$ and $\kappa = \alpha$ and then the limit $\alpha \longrightarrow \infty$. ∎

We now estimate the norms of certain classes of functionals which will arise later. $P(X, \phi)$ is said to be a polynomial of degree r if derivatives of higher order than r vanish.

LEMMA 8. *Suppose that $G = G(\kappa, X, \phi)$ is as defined in (2.4), $\kappa = O(\lambda^{1/2})$ and $h = O(\lambda^{-1/4})$. For any polynomial P of degree r there is a constant $O(1)$ (depending on r) such that*

$$\|Pe^{-V}\|_{G,h,\Gamma} \leq O(1)|P|_{h,\Gamma_1}. \tag{60}$$

PROOF. Let $G_0 = G_0(\kappa, X, \phi)$ as above. Then we claim that

$$\|P(X)\|_{GG_0,h} \leq O(1)|P(X)|_h. \tag{61}$$

To prove this suppose we are given $\Delta^{\times n}$. Let $f^{\times n}$ be functions with support in $\tilde{\Delta}^{\times n}$ and with $\|f_j\|_{C^r(X)} \leq 1$. Expand $P_n(X, \phi; f^{\times n})$ in powers of ϕ

$$P_n(X, \phi; f^{\times n}) = \sum_{k=n}^{r} 1/(k-n)!\, P_k(X, 0; f^{\times n}, \phi^{\times k-n}).$$

Localize ϕ using the partition of unity χ_Δ introduced in Section 2.3. Then we have

$$\|P_n(X, \phi)\|_{\tilde{\Delta}^{\times n}} \leq \sum_{k=n}^{r} 1/(k-n)! \sum_{\Delta^{\times k-n}} \|P_k(X, 0)\|_{\tilde{\Delta}^{\times k}} \|(\chi_\Delta \phi)^{\times k-n})\|_{C^r(X)}.$$

The ϕ term is bounded using Lemma 4 to obtain

$$\|\phi\|_{C^r(X)} \leq \sqrt{O(1)r/\kappa}\,(GG_0)^{1/r}(X, \phi)$$

and this leads to

$$\|P_n(X)\|_{GG_0} \leq \sum_{k=n}^{r} 1/(k-n)!\, |P_k(X, 0)| \left(\sqrt{O(1)r/\kappa}\right)^{k-n}.$$

We multiply by $h^n/n!$, sum over n and use the binomial theorem with the result

$$\begin{aligned} \|P(X)\|_{GG_0,h} &\leq |P(X)|_{h+\sqrt{O(1)r/\kappa}} \\ &\leq \left(1 + \sqrt{O(1)r/\kappa h^2}\right)^r |P(X)|_h \end{aligned}$$

which proves (61).

By Lemma 7, Theorem 1 and (61) we have

$$\|Pe^{-V}(X)\|_{G^2,h} \leq \|P(X)\|_{GG_0,h}\|e^{-V(X)}\|_{GG_0^{-1},h} \leq O(1)|P(X)|_h 2^{|X|}. \tag{62}$$

The result follows by taking the $\|\ \|_\Gamma$ norm of both sides, because $G^2 = G(2\kappa)$ and $2\kappa = O(\lambda^{1/2})$. ∎

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4. **The Renormalization Group Map.** In this section we compute and estimate changes in the polymer activities under RG transformations. Applying the RG transformation to a density $Z(\Lambda, \phi) = (\mathcal{E}xp\, A)(\Lambda, \phi)$ expressed in polymer activities A leads to a new density

$$Z'(\Lambda', \phi) = \text{const}\,(\mu_C * \mathcal{E}xp\, A)(\Lambda, \phi_{L^{-1}})$$

on a new torus Λ' where $\Lambda = L\Lambda'$. We want to express this new density in the form $Z'(\Lambda', \phi) = \text{const}\,(\mathcal{E}xp\, A')(\Lambda', \phi)$ with new polymer activities A'. We treat this problem in three steps: A fluctuation step which is the convolution with μ_C, an extraction step which is a rearrangement of the polymer expansion and, thirdly, the scaling step.

In the models of interest, it will be necessary to know more about A than just its norm: it will be necessary to "guess" an approximate form of A, and to estimate the norm of the error. The leading order guess is the form $A = \Box e^{-V} + K$ where $\Box$ is given by (8), V is of the form (3) for some parameter values, and the error K is suitably small. For each of the three steps making up the RG map, we will state and prove a theorem which maps an A of this type to a new A' of the same form. More refined guesses can be expressed as a further breakdown of the form $K = Qe^{-V} + R$ where Qe^{-V} are some leading contributions to K and R is very small. Expressed another way, we have $A = B + R$ where $B = (\Box + Q)e^{-V}$ describes the leading form of A. We then want to write the new activities in the form $A' = B' + R'$ with B' known and R' very small.

Estimates on polymer activities will be given in terms of the norms $\|\cdot\|_{G,\Gamma,h}$, $\|\cdot\|_{G,h,\Gamma}$ introduced in Section 2. Unless otherwise noted the (G, Γ, h) will be of the general form discussed in that section.

4.1. *Fluctuation.* The fluctuation step is the map induced on polymer activities by Gaussian convolution with respect to a measure with covariance $C = C(x, y)$. In applications, the covariance C is usually a smooth Euclidean or toral invariant function rapidly decaying in the separation $|x - y|$. The technical hypotheses on C needed to control the fluctuation step turns out to be smoothness and finiteness of the following norm:

$$\|C\|_\theta = 3^d \sup_{\Delta_1} \sum_{\Delta_2} C(\Delta_1, \Delta_2)\theta\big(d(\Delta_1, \Delta_2)\big) \tag{63}$$

$$C(\Delta_1, \Delta_2) = \|\chi_{\Delta_1} C \chi_{\Delta_2}\|_{C^{2r}} \tag{64}$$

and $\theta(s)$ is the function given by (12). $\chi_\Delta(x)$ is the "bump" function chosen in Section 2. For Theorem 3 and Theorem 4 we also require the condition on C:

$$\sup_{0 \le |\alpha| \le r + d/2 + 1} |(\partial^\alpha \partial^\alpha C)(x, x)| \le O(1). \tag{65}$$

The main fluctuation theorem refers to norms which involve $G(t, X, \phi)$, a one-parameter family of regulators $G(t, X, \phi)$ that satisfy **G3** in addition to the basic properties **G1** and **G2**:

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THEOREM 2. *For any polymer activity A and any $t \in [0,1]$, there is a unique polymer activity $A(t)$ so that*

$$\mu_t * (Exp\, A) = Exp\big(A(t)\big) \tag{66}$$

where $\mu_t = \mu_{tC}$. The map $\bar{F}_t(A) \equiv A(t)$ is analytic. If $h' < h$ and

$$\|A\|_{G(0),\Gamma,h} \le D \equiv \frac{(h-h')^2}{16\|C\|_\theta} \tag{67}$$

then

$$\|\bar{F}_t(A)\|_{G(t),\Gamma,h'} \le \|A\|_{G(0),\Gamma,h}. \tag{68}$$

REMARK. A consequence of this theorem is that if $A(s)$ is any polymer activity with $\|A(s)\|_{G(s),\Gamma,h} \le D$, then for $0 \le s \le t$,

$$\|A(t)\|_{G(t),\Gamma,h'} \le \|A(s)\|_{G(s),\Gamma,h}$$

because the theorem says

$$\|\bar{F}_{t-s}(\tilde{A})\|_{\tilde{G}(t-s),\Gamma,h'} \le \|\tilde{A}\|_{\tilde{G}(0),\Gamma,h}$$

when $\tilde{A} = A(s)$ and $\tilde{G}(t) = G(s+t)$.

PROOF. We define $A(t)$ by $A(t) = Log\big(\mu_t * Exp\,(A)\big)$ where the Log is the inverse to the Exp and is given by a terminating series[1]. Then (66) is satisfied. To estimate $A(t)$ we derive an integral equation for it.

An essential characteristic of Gaussian convolution, discussed in detail in the Appendix, is that for any sufficiently smooth functional $F(\phi)$, the function $t \longrightarrow F_t \equiv \mu_{tC} * F$ solves the functional heat equation

$$\frac{\partial}{\partial t} F_t = \Delta_C F_t$$

with initial condition $F_0 = F$. The functional Laplacian is formally

$$\Delta_C F(\phi) = \frac{1}{2}\int F_2(\phi; x, y) C(x,y)\, dx\, dy. \tag{69}$$

As the precise definition we take

$$\Delta_C F(\phi) = \frac{1}{2}\int F_2(\phi; \zeta, \zeta)\, d\mu_C(\zeta). \tag{70}$$

Now differentiating (66) and using the definition of the Laplacian we have

$$\begin{aligned}\frac{\partial A(t)}{\partial t} \circ Exp\big(A(t,\phi)\big) \;&=\; \Delta_C A(t,\phi) \circ Exp\big(A(t,\phi)\big) \\ &\quad + \frac{1}{2}\int d\mu_1(\zeta) A_1(t,\phi;\zeta) \circ A_1(t,\phi;\zeta) \circ Exp\big(A(t,\phi)\big)\end{aligned}$$

[1] $Log(A) = (A - I) - \frac{1}{2}(A-I)\circ(A-I) + \cdots$
Domain: $A(\emptyset) = 1$.

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We cancel out the $Exp\big(A(t,\phi)\big)$ and obtain

$$\frac{\partial}{\partial t}A(t,\phi) = \Delta_C A(t,\phi) + \frac{1}{2}B_C(A,A)(t,\phi) \tag{71}$$

where the bilinear operator on activities B_C is defined by

$$B_C(A,B)(\phi) = \int d\mu_1(\zeta) A_1(\phi;\zeta) \circ B_1(\phi;\zeta). \tag{72}$$

This equation can be converted to an integral equation by convolving and integrating:

$$\int_0^t ds \mu_{t-s} * \frac{\partial}{\partial s}A(s) = \int_0^t ds \mu_{t-s} * \Delta_C A(s) + \frac{1}{2}\int_0^t ds \mu_{t-s} * B_C\big(A(s),A(s)\big).$$

The integrands in the first two terms combine to give a total derivative in s, so this is the same as

$$A(t) = \mu_t * A(0) + \frac{1}{2}\int_0^t ds \mu_{t-s} * B_C\big(A(s),A(s)\big). \tag{73}$$

This is the desired integral equation satisfied by $A(t)$.

The next step is to take norms in this equation, which in a manner reminiscent of the proof of the Cauchy-Kowaleska existence theorem of partial differential equations, replaces the ϕ dependence in the integral equation by a single parameter h. Using

$$|\mu_{\delta t} * F(\phi)| = \left| \int d\mu_{\delta t}(\zeta) G(s,\phi+\zeta) G(s,\phi+\zeta)^{-1} F(\phi+\zeta)\right|$$

and $|\mu_{\delta t} * G(s,\phi)| \leq G(s+\delta t,\phi)$ we obtain general formulas

$$\|\mu_{\delta t} * F_n\|_{G(s+\delta t)} \leq \|F_n\|_{G(s)} \tag{74}$$

$$\|\mu_{\delta t} * F\|_{G(s+\delta t),\Gamma,h} \leq \|F\|_{G(s),\Gamma,h}. \tag{75}$$

These enable us to take the norm $\| \ \|_t = \| \ \|_{G(t),\Gamma,h}$ of both sides of the integral equation (73), and obtain

$$\|A(t)\|_t \leq \|A(0)\|_0 + \frac{1}{2}\int_0^t ds \big\|B_C\big(A(s),A(s)\big)\big\|_s. \tag{76}$$

We combine this with the following lemma

LEMMA 9. *For any regulator G*

$$\frac{1}{2}\|B_C(A,B)\|_{G,\Gamma,h} \leq \|C\|_\theta \left[\frac{\partial}{\partial h}\|A\|_{G,\Gamma,h}\right]\left[\frac{\partial}{\partial h}\|B\|_{G,\Gamma,h}\right]$$

together with the same bound with $\partial^p/\partial h^p$ *applied to both sides.*

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This gives an integral inequality

$$\|A(t)\|_t \le \|A(0)\|_0 + \|C\|_\theta \int_0^t ds \left(\frac{\partial}{\partial h}\|A(s)\|_s\right)^2$$

together with the same bound with $\partial^p/\partial h^p$ applied to both sides. In this integral inequality the ϕ dependence of the original integral equation has been replaced by a single variable h.

Iteration of these inequalities results in an upper bound (a "majorant" series) for $\|A(t)\|_t$ which is the unique formal power series in t, h solution to the corresponding integral equality. This majorant series solves the initial value problem

$$\frac{\partial k(t,h)}{\partial t} = \|C\|_\theta \left(\frac{\partial k(t,h)}{\partial h}\right)^2; \quad k(0,h) = \|A\|_{G(0),\Gamma,h}.$$

By the action principle applied to this Hamilton-Jacobi equation there is a solution which is analytic in t, h near $t, h = 0$. By uniqueness the majorant series must be the power series in t, h that represents this solution and therefore the majorant series is convergent for t, h sufficiently small depending on the initial data $\|A\|_{G(0),\Gamma,h}$. The bounds in Theorem 2 are obtained by exploiting explicit solutions for this Hamilton-Jacobi equation obtained by the action principle. The details are in Lemma 8.4 in [BY90]. ∎

PROOF (LEMMA 9). For the derivative in the direction $f^{\times n} = (f_1, \ldots, f_n)$ we have

$$(77)\qquad \big(B_C(A,B)\big)_n(\phi; f^{\times n}) = \sum_{\sigma\subset\{1,\ldots,n\}} B_C\big(A_{|\sigma|}(\phi; f^{\times\sigma}), B_{|\sigma^c|}(\phi; f^{\times\sigma^c})\big).$$

This can be written as

$$(78)\qquad \sum_\sigma \sum_{\Delta_x,\Delta_y} \int d\mu_C(\zeta) A_{1+|\sigma|}(\phi; f^{\times\sigma}, \chi_{\Delta_x}\zeta) \circ B_{1+|\sigma^c|}(\phi; f^{\times\sigma^c}, \chi_{\Delta_y}\zeta).$$

After taking the supremum over functions $f^{\times n}$ supported in $\tilde\Delta^{\times n}$ with unit C^r norms one finds

$$(79)\quad \big\|\big(B_C(A,B)\big)_n(\phi)\big\|_{\tilde\Delta^{\times n}} \le \sum_\sigma \sum_{\Delta_x,\Delta_y} C(\Delta_x,\Delta_y)\|A_{1+|\sigma|}(\phi)\|_{\tilde\Delta_x,\tilde\Delta^{\times\sigma}} \circ \|B_{1+|\sigma^c|}(\phi)\|_{\tilde\Delta_y,\tilde\Delta^{\times\sigma^c}}.$$

(We digress to explain the justification of this step. Let L, M be linear functionals on C^r defined by $L(\zeta) = A_{1+|\sigma|}(\phi; f^{\times\sigma}, \zeta)$ and $M(\zeta) = B_{1+|\sigma^c|}(\phi; f^{\times\sigma^c}, \zeta)$. We need to show that

$$(80)\qquad \int d\mu_C(\zeta) L(\chi_{\Delta_x}\zeta) M(\chi_{\Delta_y}\zeta) \le \|L\|_{\tilde\Delta_x}\|M\|_{\tilde\Delta_y}\|\chi_{\Delta_x} C \chi_{\Delta_y}\|_{C^{2r}}.$$

To see this we use the fact that any continuous linear functional L on $C^r(\bar\Lambda)$ can be written in the form

$$L(f) = \sum_{|\alpha|\le r} \int \partial^\alpha f \, d\mu_\alpha$$

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where each μ_α is a bounded Borel measure on Λ. The idea of the proof can be found in [SR72, p. 176]. Actually they consider the case $\Lambda = \mathbf{R}^n$, but the case Λ= torus which we quote here is even easier. Given this representation it is straightforward to show for any function $f(x,y)$ in $C^{2r}(\bar\Lambda \times \bar\Lambda)$ that $L\big(f(\cdot,y)\big)$ which we denote $L_x\big(f(x,y)\big)$ is a C^r function of y and that $\partial_y^\beta L_x\big(f(x,y)\big) = L_x\big(\partial_y^\beta f(x,y)\big)$. Thus expressions like $M_y\big(L_x f(x,y)\big)$ are defined. Furthermore one can show that

$$\int d\mu_C(\zeta) L(\chi_{\Delta_x}\zeta) M(\chi_{\Delta_y}\zeta) = M_y\Big(L_x\big(\chi_{\Delta_x} C(x,y)\chi_{\Delta_y}\big)\Big). \tag{81}$$

The estimate now follows by dominating this expression by the $\|M\|$ norm and then the $\|L\|$ norm.)

Returning to the main proof we evaluate on Z, take the supremum over ϕ weighted by G^{-1}, and sum over the n-tuple of cubes $\tilde\Delta^{\times n}$.

$$\begin{aligned} &\Big\|\big(B_C(A,B)\big)_n(Z)\Big\|_G \\ &\quad \le \sum_\sigma \sum_{\tilde\Delta^{\times n}} \sum_{X,\Delta_x,Y,\Delta_y} C(\Delta_x,\Delta_y)\, \|A_{1+|\sigma|}(X)\|_{\tilde\Delta_x,\tilde\Delta^{\times\sigma},G}\, \|B_{1+|\sigma^c|}(Y)\|_{\tilde\Delta_y,\tilde\Delta^{\times\sigma^c},G}. \end{aligned} \tag{82}$$

Here $\|A_n\|_{\tilde\Delta^{\times n},G} = \sup_\phi \|A_n\|_{\tilde\Delta^{\times n}} G(\phi)^{-1}$ still needs the sum over $\tilde\Delta^{\times n}$ to become the G-norm. To obtain this bound we needed the property **G2** and the fact that the sum is over disjoint sets X, Y with $X \cup Y = Z$. Note that X, Δ_x must be nearest neighbors and so must Y, Δ_y.

Shortly we want to sum over Z containing a fixed Δ. This forces one of X or Y to contain Δ. By including a factor of two we can restrict to the case where it is X which contains Δ. (We drop the other constraint on X). Keep the sum over Δ_y but use

$$\sum_{\tilde\Delta^{\sigma^c}} \|B_{1+|\sigma^c|}(Y)\|_{\tilde\Delta_y,\tilde\Delta^{\times\sigma^c},G} \le \|B_{1+|\sigma^c|}(Y)\|_G$$

and obtain

$$\|B_C(A,B)_n(Z)\|_G \le 2\sum_\sigma \sum_{\tilde\Delta^\sigma} \sum_{X,\Delta_x,Y,\Delta_y} C(\Delta_x,\Delta_y)\|A_{1+|\sigma|}(X)\|_{\tilde\Delta_x,\tilde\Delta^{\times\sigma},G}\|B_{1+|\sigma^c|}(Y)\|_G.$$

Now multiply by $\Gamma(Z)$ and sum over $Z \supset \Delta$. On the right hand side the sum over Z combined with the sum over X, Y constrained to have union Z is the same as summing over X, Y without a constraint on their union. On the right hand side we use the property $\Gamma(Z) \le \Gamma(X)\theta\big(d(\Delta_x,\Delta_y)\big)\Gamma(Y)$. Since Y must contain a block that is a nearest neighbor to Δ_y, and since Δ_y has 3^d neighbors including itself, the sum over Y gives $3^d\|B_{1+|\sigma^c|}\|_{G,\Gamma}$. Thus

$$\begin{aligned} \sum_{Z\supset\Delta} \|B_C(A,B)_n(Z)\|_G\Gamma(Z) &\le 2\sum_\sigma \sum_{X\supset\Delta} \sum_{\tilde\Delta^\sigma} \sum_{\Delta_x,\Delta_y} 3^d C(\Delta_x,\Delta_y)\theta\big(d(\Delta_x,\Delta_y)\big)\Gamma(X) \\ &\qquad \times \|A_{1+|\sigma|}(X)\|_{\tilde\Delta_x,\tilde\Delta^{\times\sigma},G}\|B_{1+|\sigma^c|}\|_{G,\Gamma} \\ &\le 2\sum_\sigma \|C\|_\theta \|A_{1+|\sigma|}\|_{G,\Gamma}\|B_{1+|\sigma^c|}\|_{G,\Gamma}. \end{aligned}$$

In the second step we have used that the sum over Δ_y gives $\|C\|_\theta$ and then identified $\|A_{1+|\sigma|}\|_{G,\Gamma}$. Taking the supremum over Δ gives the Γ-norm.

Now note that the sum over σ is the same as summing over $l = |\sigma|$ with a factor of "n choose l" and so

$$\|B_C(A,B)_n\|_{G,\Gamma} \leq 2 \sum_{l,m:l+m=n} \frac{n!}{l!\,m!} \|C\|_\theta \|A_{1+l}\|_{G,\Gamma} \|B_{1+m}\|_{G,\Gamma}. \tag{83}$$

We multiply by $h^n/n!$ and sum over n

$$\|B_C(A,B)\|_{G,\Gamma,h} \leq 2\|C\|_\theta \left[\frac{\partial}{\partial h}\|A\|_{G,\Gamma,h}\right]\left[\frac{\partial}{\partial h}\|B\|_{G,\Gamma,h}\right].$$

A similar conclusion giving $\partial^p/\partial h^p$ of this inequality follows if we multiply (83) by $h^{n-p}/(n-p)!$ and sum over $n \geq p$. ∎

For the more refined versions of Theorem 2 which now follow, we will need some ideas from its proof. Since $\bar{F}_t(A) = A(t)$ is the evolution under the fluctuation flow equation (71), we have

$$E\big(A(t)\big) \equiv \frac{\partial}{\partial t}A(t) - \Delta_C A(t) - \frac{1}{2}B_C\big(A(t), A(t)\big) = 0. \tag{84}$$

If we are given some approximate evolution $t \longrightarrow B(t)$, we can measure how well it matches an exact evolution by the *error* $E\big(B(t)\big)$. The following theorem tracks the growth of the *remainder* $R(t)$ under the fluctuation step $A \longrightarrow A(t)$, when $A(t) = B(t) + R(t)$ with $B(t)$ known and $R(0)$ small.

THEOREM 3. *Let $B(t)$ be a continuously differentiable function of $t \in [0,1]$ and define $R(t) = \bar{F}_t\big(B(0) + R(0)\big) - B(t)$ so that*

$$\mu_t * \mathcal{E}xp\,(B + R) = \mathcal{E}xp\big(B(t) + R(t)\big). \tag{85}$$

Suppose $h > h'$ and $\|R(0)\|_{G(0),\Gamma,h}$, $\sup_{0\leq t\leq 1}\|B(t)\|_{G(t),\Gamma,h} \leq \frac{1}{4}D$ where $D = (h-h')^2/(16\|C\|_\theta)$ as in (67). Then

1.

$$\|R(t)\|_{G(t),\Gamma,h'} \leq 2\Big(\|R(0)\|_{G(0),\Gamma,h} + t\sup_{s\leq t}\big\|E\big(B(s)\big)\big\|_{G(s),\Gamma,h}\Big). \tag{86}$$

2. If we suppose further that $\|R(0)\|_{G(0),\Gamma,h}$, $\sup_{0\leq t\leq 1}\|B(t)\|_{G(t),\Gamma,h} \leq h'/(2\|C\|_\theta)$ and $h' \geq 2$, then for any $M \geq 0$,

$$\begin{aligned} |R(t)|_{\Gamma_{-1},1/2} \leq\ & O(1)\big(|R(0)|_{\Gamma_{-1},1} + (h')^{-M}\|R(0)\|_{G(0),\Gamma,h}\big) \\ & + O(1)\sup_{s\leq t}\Big(\big|E\big(B(s)\big)\big|_{\Gamma_{-1},1} + (h')^{-M}\big\|E\big(B(s)\big)\big\|_{G(s),\Gamma,h}\Big) \end{aligned} \tag{87}$$

where $O(1)$ depends on M.

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PROOF. Let us introduce notation for derivatives of $\bar{F}$ evaluated at A, namely:

$$(\bar{F})_n(A; B_1, \dots, B_n)) = \frac{d}{d\beta_1} \cdots \frac{d}{d\beta_n} \bar{F}(A + \beta_1 B_1 + \dots + \beta_n B_n) \Big|_{\beta_1 = \dots \beta_n = 0} . \tag{88}$$

We claim that

$$R(t) = \int_0^1 (\bar{F}_t)_1 \big(B(0) + sR(0); R(0) \big) \, ds - \int_0^t (\bar{F}_{t-s})_1 \big(B(s); E(s) \big) \, ds. \tag{89}$$

This follows from

$$\begin{aligned} R(t) &= A(t) - B(t) \\ &= \Big[\bar{F}_t \big(B(0) + R(0) \big) - \bar{F}_t \big(B(0) \big) \Big] + \Big[\bar{F}_t \big(B(0) \big) - B(t) \Big] \\ &= \int_0^1 (\bar{F}_t)_1 \big(B(0) + sR(0); R(0) \big) \, ds - \int_0^t \frac{d}{ds} \bar{F}_{t-s} \big(B(s) \big) \, ds \end{aligned}$$

and

$$\begin{aligned} \frac{d}{ds} \bar{F}_{t-s} \big(B(s) \big) &= -\frac{d}{dr} \bar{F}_{t-s+r} \big(B(s-r) \big) \Big|_{r=0} \\ &= -\frac{d}{dr} \Big[\bar{F}_{t-s} \Big(\bar{F}_r \big(B(s) \big) \Big) + \bar{F}_{t-s} \big(B(s-r) \big) \Big] \Big|_{r=0} \\ &= (\bar{F}_{t-s})_1 \big(B(s); E(s) \big) \end{aligned}$$

because $E(s) = (-d/dr) \bar{F}_r \big(B(s) \big) + \partial B(s) / \partial s$.

Theorem 3 follows from (89) in conjunction with the following theorem on the linearized fluctuation operator (part (2) taken with $\eta = 1, \eta' = 1/2$). ∎

THEOREM 4.

1. *Suppose $h > h'$ and $\|A\|_{G(0),\Gamma,h} \leq \frac{1}{2} D$ where $D = (h - h')^2 / (16 \|C\|_\theta)$ as in Theorem 2. Then for $0 \leq s \leq t \leq 1$*

$$\|(\bar{F}_{t-s})_1(A; B)\|_{G(t),\Gamma,h'} \leq 2 \|B\|_{G(s),\Gamma,h}. \tag{90}$$

2. *Suppose in addition we have η, η' so that $0 < \eta - \eta' \leq 1$ and $h' - \eta' \geq 1$. Also suppose that $G(t, X, \phi = 0) \leq 2^{|X|}$ for all $t \in [0, 1]$ and that $(h' - \eta')^{-1} \|A\|_{G(0),\Gamma,h} \leq \|C\|_\theta^{-1}$. Then for any $M \geq 0$ and for $0 \leq s \leq t \leq 1$*

$$|(\bar{F}_{t-s})_1(A; B)|_{\Gamma_{-1},\eta'} \leq O(1) \Big[(\eta - \eta')^{-2M} |B|_{\Gamma_{-1},\eta} + (h' - \eta')^{-M} \|B\|_{G(s),\Gamma,h} \Big] \tag{91}$$

where $O(1)$ depends on M.

REMARK. The idea is that $\|B\|_{G(s),\Gamma,h}$ enters the kernel estimates with a large negative power of h to reduce its contribution.

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PROOF. We give the proofs for $s = 0$. The remark below Theorem 2 shows why this is sufficient.

(1) Let $B(t) = (\bar{F}_t)_1(A; B)$. The bound is a consequence of the Cauchy integral formula:

$$B(t) = (2\pi i)^{-1} \oint \frac{d\beta}{\beta^2} \bar{F}_t(A + \beta B).$$

We integrate over the contour $|\beta| = \frac{1}{2} D \|B\|^{-1}_{G(0),\Gamma,h}$ and use $\|\bar{F}_t(A + \beta B)\|_{G(t),\Gamma,h'} \leq \|A + \beta B\|_{G(0),\Gamma,h} \leq D/2 + D/2$ which follows by Theorem 2.

(2) The difficulty here is that there is no straightforward version of Theorem 2 for the kernel norm. Instead we work directly from the flow equation (71) for $A(t)$ which in integral form says:

$$A(t) = A + \int_0^t ds \Delta_C A(s) + \frac{1}{2} \int_0^t ds \mathcal{B}_C\big(A(s), A(s)\big). \tag{92}$$

Differentiate with respect to ϵ when $A \longrightarrow A + \epsilon B$ and obtain the linearized equation:

$$B(t) = B + \int_0^t ds \Delta_C B(s) + \int_0^t ds \mathcal{B}_C\big(A(s), B(s)\big). \tag{93}$$

We take seminorms $|\;|^{(p)}_{\Gamma,\eta} = (d/d\eta)^p |\;|_{\Gamma,\eta}$ of this equation. To bound the second term note that by (65)

$$\begin{aligned} |(\Delta_C B)(X,0)| &\leq \sum_{\Delta_x,\Delta_y} \|B_2(X,0)\|_{\tilde{\Delta}_x \times \tilde{\Delta}_y} \int \|\chi_{\Delta_x}\zeta\|_r \|\chi_{\Delta_y}\zeta\|_r \, d\mu_C(\zeta) \\ &\leq \mathcal{O}(1) |B_2(X,0)| \end{aligned}$$

because the $\|\;\|_r$ norm is bounded by a Sobolev norm which is integrable. Similar estimates hold for higher derivatives and this leads to

$$|\Delta_C B|^{(p)}_{\Gamma,\eta} \leq \mathcal{O}(1) |B|^{(p+2)}_{\Gamma,\eta}. \tag{94}$$

The third term is bounded by Lemma 9, provided we use the singular G concentrated at 0. Altogether we have the bound

$$\begin{aligned} |B(t)|^{(p)}_{\Gamma_{-1},\eta'} &\leq |B|^{(p)}_{\Gamma_{-1},\eta'} + \mathcal{O}(1) \int_0^t ds |B(s)|^{(p+2)}_{\Gamma_{-1},\eta'} \\ &\quad + 2\|C\|_\theta \int_0^t ds \sum_{n=0}^p \frac{p!}{(p-n)!\,n!} |A(s)|^{(n+1)}_{\Gamma_{-1},\eta'} |B(s)|^{(p+1-n)}_{\Gamma_{-1},\eta'}. \end{aligned} \tag{95}$$

By use of a Cauchy bound one finds

$$|A(s)|^{(n+1)}_{\Gamma_{-1},\eta'} \leq (n+1)!\,(h' - \eta')^{-n-1} |A(s)|_{\Gamma_{-1},h'} \tag{96}$$

and we combine this with the bound

$$|A(s)|_{\Gamma_{-1},h'} \leq \|A(s)\|_{G(s),\Gamma,h'} \leq \|A\|_{G(0),\Gamma,h} \tag{97}$$

and the condition $\|C\|_\theta \|A\|_{G(0),\Gamma,h} \leq (h' - \eta')$. This leads to the inequality (with new constants)

$$
\begin{aligned}
|B(t)|^{(p)}_{\Gamma_{-1},\eta'} \leq\ & |B|^{(p)}_{\Gamma_{-1},\eta'} + O(1) \int_0^t ds\, |B(s)|^{(p+2)}_{\Gamma_{-1},\eta'} \\
& + O(1) \sum_{n=0}^{p} \frac{p!\,(n+1)}{(p-n)!} \int_0^t ds (h'-\eta')^{-n} |B(s)|^{(p+1-n)}_{\Gamma_{-1},\eta'}.
\end{aligned}
\tag{98}
$$

Now we claim that for all $p = 0, 1, \ldots$ and all $M = 0, 1, \ldots$ there exists $C^{(p)}_M$ independent of L so that

$$
|B(t)|^{(p)}_{\Gamma_{-1},\eta'} \leq C^{(p)}_M \big[(\eta - \eta')^{-p-2M} |B|_{\Gamma_{-1},\eta} + (h'-\eta')^{-p-M} \|B\|_{G(0),\Gamma,h} \big].
$$

The theorem is the case $p = 0$.

The proof of the claim is by induction on M. The case $M = 0$ is true since as in (96), (97)

$$
|B(t)|^{(p)}_{\Gamma_{-1},\eta'} \leq p!\,(h'-\eta')^{-p} \|B\|_{G(0),\Gamma,h}.
$$

Now suppose it is true for M. Inserting the bound in (98) we find:

$$
\begin{aligned}
|B(t)|^{(p)}_{\Gamma_{-1},\eta'} \leq\ & |B|^{(p)}_{\Gamma_{-1},\eta'} + O(1) C^{(p+2)}_M \\
& \times \big[(\eta-\eta')^{-p-2-2M} |B|_{\Gamma_{-1},\eta} + (h'-\eta')^{-p-2-M} \|B\|_{G(0),\Gamma,h} \big] \\
& + O(1) \sum_{n=0}^{p} \frac{p!\,(n+1)}{(p-n)!} \int_0^1 ds (h'-\eta')^{-n} C^{(p+1-n)}_M \\
& \times \big[(\eta-\eta')^{n-p-1-2M} |B|_{\Gamma_{-1},\eta} + (h'-\eta')^{n-p-1-M} \|B\|_{G(0),\Gamma,h} \big].
\end{aligned}
\tag{99}
$$

From this we identify the bound for $M + 1$. The first term is bounded by

$$
|B|^{(p)}_{\Gamma_{-1},\eta'} \leq p!\,(\eta-\eta')^{-p} |B|_{\Gamma_{-1},\eta}
$$

which suffices since $\eta - \eta' < 1$. The second term has the correct form once we use $h' - \eta' \geq 1$. The third term also has the correct form since

$$
(h'-\eta')^{-n} \big((\eta-\eta')^{n+1} + (h'-\eta')^n \big) \leq 2
$$

which follows from $\eta - \eta' < 1$ and $h' > \eta$. ■

4.2. *Extraction.* Now suppose that the polymer activity has the form $A = \square e^{-V} + K$. The extraction step consists in removing terms F from the K's and compensating by shifts in the potential V. This will be used to put the relevant parts of K in V.

We assume F satisfies the following *localization* property: $F(X, \phi)$ is defined on polymers and has the decomposition

$$
F(X,\phi) = \sum_{\Delta \subset X} F(X, \Delta, \phi)
\tag{100}
$$

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where Δ is summed over open blocks, and $F(X,\Delta,\phi)$ has the ϕ dependence localized in Δ, *i.e.*, $F(X,\Delta,\phi)$ is a functional on $C^r(\overline{\Delta})$.

For example we might have $F(X,\phi) = \alpha(X)\int_X P\big(\phi(x)\big)\,dx$ in which case

$$F(X,\Delta,\phi) = \alpha(X)\int_\Delta P\big(\phi(x)\big)\,dx. \tag{101}$$

However we also want to consider the more general case

$$F(X,\Delta,\phi) = \int_\Delta \alpha(X,\Delta,x)P\big(\phi(x)\big)\,dx. \tag{102}$$

For the estimates we also assume the following *stability* of V relative to the perturbation F: there are positive numbers $f(X)$ independent of ϕ and a regulator G such that for all Δ

$$\Big\|\exp\Big\{-V(\Delta) - \sum_{X\supset\Delta} z(X)F(X,\Delta)\Big\}\Big\|_{G,h} \le 2 \tag{103}$$

for all complex $z(X)$ with $|z(X)|\,f(X) \le 2$.

THEOREM 5. *If K is a polymer activity and F satisfies the localization assumption (100) then there is a new polymer activity $\mathcal{E}(K,F)$ such that*

$$\mathcal{E}xp\,(\Box e^{-V} + K)(\Lambda) = \mathcal{E}xp\big(\Box e^{-V(F)} + \mathcal{E}(K,F)\big)(\Lambda), \tag{104}$$

where $V(F)$ is defined on each cell Δ by

$$\big(V(F)\big)(\Delta) = V(\Delta) - \sum_{Y\supset\Delta} F(Y,\Delta). \tag{105}$$

The linearization $\mathcal{E}_1$ of $\mathcal{E}$ in K and F is

$$\mathcal{E}_1(K,F) = K - Fe^{-V}. \tag{106}$$

Suppose in addition the stability assumption (103) holds and $\|f\|_{\Gamma_3}$, and $\|K\|_{G,h,\Gamma_1}$ are sufficiently small. Then $\mathcal{E}$ is jointly analytic in K, F and there is $\mathcal{O}(1)$ such that

$$\begin{aligned} \|\mathcal{E}(K,F)\|_{G,h,\Gamma} &\le \mathcal{O}(1)(\|K\|_{G,h,\Gamma_1} + \|f\|_{\Gamma_3});\\ |\mathcal{E}(K,F)|_{h,\Gamma} &\le \mathcal{O}(1)(|K|_{h,\Gamma_1} + \|f\|_{\Gamma_3}). \end{aligned} \tag{107}$$

REMARK. The proof is postponed. Note the distinguished role of Λ in this theorem. To illustrate how we are going to use this theorem suppose that $V(X) = \lambda\int_X :\phi^4:$ and $F(X,\Delta) = \alpha(X)\int_\Delta :\phi^4:$ where $\alpha(X)$ vanishes on polymers X with three or more blocks. Then the stability bound holds by Theorem 1 provided

$$\sum_{X\supset\Delta} |z(X)|\,|\alpha(X)| \le \lambda/2. \tag{108}$$

So we could take $f(X) = C|\theta(X)|\,|\alpha(X)|\lambda^{-1}$ with $C = 4\sum_{X\supset\Delta}\theta^{-1}(X)$, which is finite for X summed over all polymers with $|X| \le 2$. The smallness condition is now that $\|f\|_{\Gamma_3} = C\lambda^{-1}\|\theta\alpha\|_{\Gamma_3}$ be sufficiently small.

The next theorem is a variation on these results in which a constant term $F_0(X)$ (independent of ϕ) is also removed from K and factored out front.

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THEOREM 6. *If K is a polymer activity and $F_0(X), F_1(X,\phi)$ satisfy the localization hypothesis (100)*[2]*, then there exists a new polymer activity $\mathcal{E}(K, F_0, F_1)$ so that:*

$$\mathcal{E}xp\,(\Box e^{-V} + K)(\Lambda) = e^{\sum_X F_0(X)} \mathcal{E}xp\big(\Box e^{-V(F_1)} + \mathcal{E}(K, F_0, F_1)\big)(\Lambda), \tag{109}$$

where the linearization $\mathcal{E}_1$ of $\mathcal{E}$ in K, F_0, F_1 is

$$\mathcal{E}_1(K, F_0, F_1) = K - (F_0 + F_1)e^{-V}. \tag{110}$$

If in addition F_1 satisfies stability hypothesis (103), $\|f\|_{\Gamma_4}$ and $\|K\|_{G,h,\Gamma_2}$ are sufficiently small, and $\sum_{Y\supset\Delta} |F_0(X,\Delta)| \leq \log 2$ then $\mathcal{E}$ is jointly analytic in K, F_0, F_1 and there is $O(1)$ such that

$$\begin{aligned} \|\mathcal{E}(K, F_0, F_1)\|_{G,h,\Gamma} &\leq O(1)(\|K\|_{G,h,\Gamma_2} + \|f\|_{\Gamma_4}); \\ |\mathcal{E}(K, F_0, F_1)|_{h,\Gamma} &\leq O(1)(|K|_{h,\Gamma_2} + \|f\|_{\Gamma_4}). \end{aligned} \tag{111}$$

PROOF. Define

$$\Phi(X) = \prod_{\Delta\subset X} e^{\sum_{Y\supset\Delta} F_0(Y,\Delta)} \tag{112}$$

and $\Phi(\emptyset) = 1$. Since $\Phi(X \cup Y) = \Phi(X)\Phi(Y)$ whenever X, Y are disjoint we have with $F = F_0 + F_1$

$$\begin{aligned} \mathcal{E}xp\,(\Box e^{-V} + K)(\Lambda) &= \mathcal{E}xp\big(\Box e^{-V(F)} + \mathcal{E}(K, F)\big)(\Lambda) \\ &= \Phi(\Lambda)\mathcal{E}xp\big(\Box\Phi^{-1}e^{-V(F)} + \Phi^{-1}\mathcal{E}(K, F)\big)(\Lambda). \end{aligned} \tag{113}$$

But $\Phi^{-1}e^{-V(F)} = e^{-V(F_1)}$ so we may define $\mathcal{E}(K, F_0, F_1) = \Phi^{-1}\mathcal{E}(K, F)$ to obtain (109). By the hypothesis on F_0 we have $\Phi^{-1}(X) \leq 2^{|X|}$ and so

$$\|\mathcal{E}(K, F_0, F_1)\|_{G,h,\Gamma} = \|\mathcal{E}(K, F)\|_{G,h,\Phi^{-1}\Gamma} \leq \|\mathcal{E}(K, F)\|_{G,h,\Gamma_1}. \tag{114}$$

Therefore (111) follows from (107). ∎

COROLLARY 7. *For $F = (F_0, F_1)$ the quantity $\mathcal{E}_{\geq 2}(K, F) = \mathcal{E}(K, F) - \mathcal{E}_1(K, F)$ satisfies*

$$\begin{aligned} \|\mathcal{E}_{\geq 2}(K, F)\|_{G,h,\Gamma} &\leq O(1)\|K\|_{G,h,\Gamma_2}\|f\|_{\Gamma_4} \\ |\mathcal{E}_{\geq 2}(K, F)|_{\Gamma} &\leq O(1)|K|_{\Gamma_2}\|f\|_{\Gamma_4}. \end{aligned}$$

PROOF. See [BDH95], Corollary 2.

The proof of Theorem 5 is given after the following lemmas have established a formula for $\mathcal{E}(K, F)$.

DEFINITION 2. $\{X_i : i = 1, \ldots, n\}$ *is overlap connected iff the graph G is connected, where G is the graph whose vertices are $1, \ldots, n$ and whose bonds are the pairs ij such that $X_i \cap X_j \neq \emptyset$.*

[2] For $F_0(X)$ this means that (100) holds with $F_0(X,\Delta,\phi)$ independent of ϕ.

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Overlap connected is not the same as $\cup X_i$ being connected because the polymers X_i need not be connected. Given a polymer activity J define

$$J^+(X) = \sum_{\{X_i\}\to X} \prod_i J(X_i) \tag{115}$$

where the sum is over overlap connected sets of distinct polymers whose union is X.

LEMMA 10.

$$\sum_{\{X_i\}} \prod_i J(X_i) = \mathcal{E}xp\,(\Box + J^+)(X), \tag{116}$$

where the sum is over sets of distinct polymers contained in X.

PROOF. Group the $\{X_i\}$ into disjoint overlap connected sets. ■

LEMMA 11. *Let F be any polymer activity and let*

$$\Omega(X) = \sum_{Y\subset X} F(Y). \tag{117}$$

Then

$$e^{\Omega} = \mathcal{E}xp\big(\Box + (e^F - 1)^+\big). \tag{118}$$

PROOF. Write $e^{\Omega}(X) = \prod_{Y\subset X}(e^{F(Y)} - 1 + 1)$, expand the product and use Lemma 10 with $J = e^F - 1$. ■

LEMMA 12. *Let K, F be any polymer activities and let*

$$\tilde{K}(X) = K(X) - (e^F - 1)^+(X)e^{-V(X)}. \tag{119}$$

Then

$$e^{-V} \circ \mathcal{E}xp\,(K) = e^{-V+\Omega} \circ \mathcal{E}xp\,(\tilde{K}) \tag{120}$$

with Ω as in Lemma 11.

PROOF. $e^{-V} \circ \mathcal{E}xp\,(K) = \mathcal{E}xp\,(\Box e^{-V} + K)$ because V has the multiplicativity property $\exp\big(-V(X\cup Y)\big) = \exp\big(-V(X)\big)\exp\big(-V(Y)\big)$ whenever X, Y are disjoint. $\mathcal{E}xp\,(\Box e^{-V} + K) = \mathcal{E}xp\big(\Box e^{-V} + (e^F - 1)^+ e^{-V}\big) \circ \mathcal{E}xp\,(\tilde{K})$ by the definition of $\tilde{K}$. By Lemma 11, $\mathcal{E}xp\big(\Box e^{-V} + (e^F - 1)^+ e^{-V}\big) = e^{-V}\mathcal{E}xp\big(\Box + (e^F - 1)^+\big) = e^{-V+\Omega}$. ■

Since Ω is not additive, we cannot immediately rewrite $e^{-V+\Omega} \circ \mathcal{E}xp\,(\tilde{K})$ in the form $\mathcal{E}xp\,(\Box e^{-V'} + \tilde{K})$ for some $V' = V(F)$. We are now going to absorb this non-additivity by reorganizing $e^{-V+\Omega} \circ \mathcal{E}xp\,(\tilde{K})$ into new polymers.

LEMMA 13. *Let $F(Z, Y) = \sum_\Delta F(Z, \Delta)$ and $V' = V(F)$. Then formula (104) holds with $\mathcal{E}(K, F)$ given by*

$$\mathcal{E}(K, F)(W) = \sum_{\{X_i\},\{Z_k\}\to W} \exp\big(-V'(W \setminus X)\big) \prod_i \tilde{K}(X_i) \prod_k \Big(\exp\big(-F(Z_k, Z_k \setminus X)\big) - 1\Big). \tag{121}$$

Here $X = \cup_i X_i$, and the sum is over collections of disjoint subsets $\{X_i\}$ and collections of distinct subsets $\{Z_k\}$ so that

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1. *the union over* $\{X_i\}$ *and* $\{Z_k\}$ *is* W*;*
2. *each* Z_k *intersects both* X *and* $X^c = \Lambda \setminus X$*;*
3. *the polymers* $\{X_i\}, \{Z_k\}$ *are overlap connected.*

PROOF. Let $X^c = \Lambda \setminus X$. We have

$$\begin{aligned}\Omega(X^c) &= \sum_{Z\subset X^c} F(Z)\\ &= \sum_{Z\subset X^c}\sum_{\Delta\subset Z} F(Z,\Delta)\\ &= \sum_{\Delta\subset X^c}\Big\{\sum_{Z\supset\Delta} - \sum_{Z\supset\Delta, Z\not\subset X^c}\Big\} F(Z,\Delta).\end{aligned}$$

Subtract $V(X^c) = \sum_{\Delta\subset X^c} V(\Delta)$ from both sides. Recalling the definition of $V' = V(F)$ in (105) we find

$$\begin{aligned}(V-\Omega)(X^c) &= V'(X^c) + \sum_{\Delta\subset X^c}\sum_{Z\supset\Delta, Z\not\subset X^c} F(Z,\Delta)\\ &= V'(X^c) + \sum_{Z\not\subset X, Z\not\subset X^c} F(Z, Z\setminus X).\end{aligned}$$

Therefore

$$\begin{aligned}e^{-V+\Omega}(X^c) &= e^{-V'}(X^c)\cdot \prod_{Z\not\subset X, Z\not\subset X^c} e^{-F(Z,Z\setminus X)}\\ &= e^{-V'}(X^c)\cdot\sum_{\{Z_k\}}\prod_k \big(e^{-F(Z_k,Z_k\setminus X)}-1\big) \qquad (122)\end{aligned}$$

with $Z \in \{Z_j\}$ required to intersect X and X^c. Substitute Eqs. (122) and the definition of $\mathcal{E}xp(\tilde K)$ into

$$e^{-V+\Omega}\circ \mathcal{E}xp\,(\tilde K)(\Lambda) = \sum_{X\subset\Lambda} e^{-V+\Omega}(X^c)\mathcal{E}xp\,(\tilde K)(X). \qquad (123)$$

Then group the polymers in the sum over $\{X_i\}, \{Z_k\}$ into disjoint overlap connected sets. One finds that $e^{-V+\Omega}\circ\mathcal{E}xp\,(\tilde K)(\Lambda) = \mathcal{E}xp\big(\Box e^{-V'} + \mathcal{E}(K)\big)(\Lambda)$ with $\mathcal{E}(K) = \mathcal{E}(K,F)$ as claimed in the lemma. ∎

PROOF (THEOREM 5). Now consider the bounds (107). We prove the first bound. The second bound is a limiting case of the first in which the large field regulator G^{-1} is concentrated at $\phi = 0$. Starting with (121)

$$\begin{aligned}\mathcal{E}(K,F)(W) = &\sum_{\{X_i\},\{Z_k\}\to W} \exp\big(-V'(W\setminus X)\big)\\ &\prod_i \tilde K(X_i)\prod_k \frac{1}{2\pi i}\int\frac{dz_k}{z_k(z_k-1)}\exp\big\{-z_kF(Z_k, Z_k\setminus X)\big\}. \qquad (124)\end{aligned}$$

Here the integral is over the circles $|z_k| = 2/f(Z_k)$. We take the norm using the multiplicative property and obtain

$$\begin{aligned}\|\mathcal{E}(K,F)(W)\|_{G,h} \le &\sum_{\{X_i\},\{Z_k\}\to W}\prod_i\|\tilde K(X_i)\|_{G,h}\prod_k f(Z_k)\\ &\sup_{\mathbf{z}}\Big\|\exp\big\{-V'(W\setminus X) - \sum_k z_kF(Z_k,Z_k\setminus X)\big\}\Big\|_{G,h}. \qquad (125)\end{aligned}$$

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Next we bound the norm by

$$\prod_{\Delta\subset W\setminus X}\Big\|\exp\big\{-V'(\Delta)-\sum_k z_k F(Z_k,\Delta)\big\}\Big\|_{G,h}\leq 2^{|W\setminus X|}\leq\prod_k 2^{|Z_k|}. \tag{126}$$

We used the stability hypothesis without concern for the difference between V and V' because f is sufficiently small and there is a factor of 2 in the stability hypothesis. These two points also are used in estimating the Cauchy integral as if $z-1$ were z. Next we write

$$\sum_{\{X_i\},\{Z_k\}}=\sum_N\frac{1}{N!\,M!}\sum_{(X_1,\ldots,X_N),(Z_1,\ldots,Z_M)}$$

where the sum is over ordered sets, but otherwise the restrictions apply.

We multiply by $\Gamma(W)$ and use $\Gamma(W)\leq\prod_i\Gamma(X_i)\prod_k\Gamma(Z_k)$ which follows from the overlap connectedness. Then sum over W with a pin, and use a spanning tree argument[3] and the small norm hypotheses to obtain

$$\begin{aligned}\|E(K,F)\|_{G,h,\Gamma}&\leq\sum_{\substack{N,M\\N+M\geq1}}\frac{(N+M)!}{N!\,M!}\big(O(1)\big)^{N+M}\|\tilde K\|^N_{G,h,\Gamma_1}\|f\|^M_{\Gamma_2}\\&\leq O(1)(\|\tilde K\|_{G,h,\Gamma_1}+\|f\|_{\Gamma_2}).\end{aligned}\tag{127}$$

Recall that $\tilde K=K+(e^{-F}-1)^+e^{-V}$. Since

$$\begin{aligned}(e^{-F}-1)^+e^{-V}(X)&=e^{-V(X)}\sum_{\{X_i\}}\prod_i(e^{-F}-1)(X_i)\\&=\sum_{\{X_i\}}\prod_i\frac{1}{2\pi i}\int\frac{dz_i}{z_i(z_i-1)}\exp\big\{-V(X)-z_iF(X_i)\big\}\end{aligned}\tag{128}$$

we may use the same argument again with Γ replaced by Γ_1 to prove that

$$\|\tilde K\|_{G,h,\Gamma_1}\leq\|K\|_{G,h,\Gamma_1}+O(1)\|f\|_{\Gamma_3}. \tag{129}$$

The theorem follows by combining (127,129). ∎

4.3. *Scaling.* The scaled field is

$$\phi_{L^{-1}}(x)=L^{-\dim\phi}\phi(x/L) \tag{130}$$

where $\dim\phi$ is the scaling dimension of the field ϕ. Canonically $\dim\phi=(d-2)/2$ but we do not restrict ourselves to this choice. Functionals scale by

$$K_{L^{-1}}(X,\phi)=K(LX,\phi_{L^{-1}}). \tag{131}$$

Rescaled polymer activities $S(K)=S(K,V)$ are defined by the equation

$$\mathcal{E}xp\,(\Box e^{-V}+K)(LX,\phi_{L^{-1}})=\mathcal{E}xp\big((\Box e^{-V})_{L^{-1}}+S(K)\big)(X,\phi). \tag{132}$$

[3] Described in the proof of Lemma 5.1 of [BrYa90].

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One finds the explicit formula

$$
\begin{aligned}
S(K)(Z,\phi) &= \sum_{\{X_j\}\to LZ} \exp\big(-V(LZ\setminus X,\phi_{L^{-1}})\big)\prod_j K(X_j,\phi_{L^{-1}})\\
&= \sum_{\{X_j\}\to LZ} \exp\big(-V_{L^{-1}}(Z\setminus L^{-1}X,\phi)\big)\prod_j K_{L^{-1}}(L^{-1}X_j,\phi).
\end{aligned} \tag{133}
$$

Here the sum is over disjoint 1-polymers $\{X_j\}$ with union X such that the L-block closures $\bar{X}_j^L$ are overlap connected[4] and have union LZ.

We continue to assume that for all open L^{-1}-scale polymers $X \subset$ some block Δ

$$\|(e^{-V})_{L^{-1}}(X)\|_{G,h} \le 2. \tag{134}$$

For example if G is given by equation (21) then Theorem 1 verifies the bound for a specific choice of V.

Now define

$$
\begin{aligned}
h_L &= L^{-\dim\phi}h\\
a &= 2^d\|\chi\|_{C^r}
\end{aligned} \tag{135}
$$

where $\chi(x)$ is the bump function which defines the partition of unity in Section 2.3.

THEOREM 8. *Let $p,q \ge 0$ be non-negative integers. Let V satisfy the stability assumption (134) and suppose $\|K\|_{G_L,ah_L,\Gamma_{q-p}}$ is sufficiently small. Then*

$$
\begin{aligned}
\|S(K)\|_{G,h,\Gamma_q} &\le O(1)L^d\|K\|_{G_L,ah_L,\Gamma_{q-p}}\\
|S(K)|_{h,\Gamma_q} &\le O(1)L^d|K|_{ah_L,\Gamma_{q-p}}
\end{aligned} \tag{136}
$$

$O(1)$ depends on q,p.

We also need a sharper estimate on the linearization S_1 of S

$$
\begin{aligned}
S_1(K)(Z,\phi) &= \sum_{X:\bar{X}^L=LZ} (e^{-V})(LZ\setminus X,\phi_{L^{-1}})K(X,\phi_{L^{-1}})\\
&= \sum_{X:\bar{X}^L=LZ} (e^{-V})_{L^{-1}}(Z\setminus L^{-1}X,\phi)K_{L^{-1}}(L^{-1}X,\phi).
\end{aligned} \tag{137}
$$

The new estimate needs the stronger bound for L^{-1} scale polymers X:

$$\|(e^{-V})_{L^{-1}}(X)\|_{g,h} \le 2 \tag{138}$$

where

$$
\begin{aligned}
g(X,\phi) &= G_0^{-1}(\kappa_0,X,\phi)G(\kappa/2,X,\phi)\\
&= \exp\big[-\kappa_0\|\phi\|_X^2 + \kappa/2\|\partial\phi\|_{X,2,\sigma}^2\big].
\end{aligned} \tag{139}
$$

Again, Theorem 1 proves this for a choice of V.

Next we define the scaling dimension of a polymer activity K. We set

[4] This notion was defined in Section 4.2.

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DEFINITION 3.

$$
\begin{aligned}
\dim(K_n) &= r_n + n \dim \phi; \\
\dim(K) &= \inf_n \dim(K_n)
\end{aligned}
\tag{140}
$$

where the infimum is taken over n such that $K_n(X,0) \neq 0$. Here r_n is defined to be the largest integer satisfying $r_n \leq r$ and $K_n(X, \phi = 0; p^{\times n}) = 0$ whenever $p^{\times n}$ is an n–tuple of polynomials of total degree less than r_n.

Roughly r_n gives the number of derivatives in K_n. Omitting the condition $r_n \leq r$ would give a more intrinsic concept, but adding the restriction is necessary because K is a functional on C^r.

As an example of how this definition works we compute the dimension of

$$K(X,\phi) = \int_X (\partial \phi)^2(x)\, dx.$$

We have

$$K_2(X, 0; f_1, f_2) = 2 \int_X (\partial f_1)(x)(\partial f_2)(x)\, dx.$$

This vanishes if either f_1 or f_2 is a constant and so $r_2 = 2$. Since $K_n(X,0) = 0$ for $n \neq 2$ we have $\dim(K) = \dim(K_2) = 2 + 2 \dim \phi$.

THEOREM 9. *Let $p, q \geq 0$ be non-negative integers. Let V satisfy (138).*

1. If $K(X)$ is supported on large sets, then

$$
\begin{aligned}
\|S_1(K)\|_{G,h,\Gamma_q} &\leq O(1) L^{-1} \|K\|_{G_L, ah_L, \Gamma_{q-p}} \\
|S_1(K)|_{h,\Gamma_q} &\leq O(1) L^{-1} |K|_{ah_L,\Gamma_{q-p}}.
\end{aligned}
\tag{141}
$$

2. If $K(X)$ is supported on small sets, and in addition $\kappa_0 h^2 \geq O(1)$ and $\kappa h^2 \geq O(1)$, then

$$
\begin{aligned}
\|S_1(K)\|_{G,h,\Gamma_q} &\leq O(1) L^{d-\dim(K)} \|K\|_{G_L, h/2, \Gamma_{q-p}} \\
|S_1(K)|_{h,\Gamma_q} &\leq O(1) L^{d-\dim(K)} |K|_{h/2,\Gamma_{q-p}}.
\end{aligned}
\tag{142}
$$

$O(1)$ depends on p, q.

The proof of these two theorems is given after the following lemmas.

LEMMA 14. *For any regulator G*

$$
\begin{aligned}
\|K_{L^{-1},n}(L^{-1}X)\|_G &\leq L^{-n\dim(\phi)} a^n \|K_n(X)\|_{G_L} \\
\|K_{L^{-1}}(L^{-1}X)\|_{G,h} &\leq \|K(X)\|_{G_L, ah_L}.
\end{aligned}
$$

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PROOF. Given $\Delta_f^{\times n}$ let $f^{\times n}$ be n functions supported in $\tilde{\Delta}_f^{\times n}$ with $\|f_j\|_{C^r} \leq 1$. Then the left hand side is given by

$$\begin{aligned}
&\sum_{\Delta_f^{\times n}} \sup_{\phi,f} |K_{L^{-1},n}(L^{-1}X,\phi;f^{\times n})| G^{-1}(L^{-1}X,\phi) \\
&\quad = \sum_{\Delta_f^{\times n}} \sup_{\phi,f} |K_n(X,\phi_{L^{-1}};f_{L^{-1}}^{\times n})| G_L^{-1}(X,\phi_{L^{-1}}) \\
&\quad \leq \sum_{\Delta_f^{\times n},\Delta^{\times n}} \sup_{\phi,f} |K_n(X,\phi_{L^{-1}};(\chi_\Delta f_{L^{-1}})^{\times n}| G_L^{-1}(X,\phi_{L^{-1}}) \\
&\quad \leq \sum_{\Delta_f^{\times n},\Delta^{\times n}} \sup_{\phi,f} \|K_n(X,\phi_{L^{-1}})\|_{\tilde{\Delta}^{\times n}} \|(\chi_\Delta f_{L^{-1}})^{\times n}\|_{C^r} G_L^{-1}(X,\phi_{L^{-1}}) \\
&\quad \leq L^{-n\dim\phi}(2^d\|\chi\|)^n \|K_n(X)\|_{G_L}.
\end{aligned}$$

Here we inserted the partition of unity $\chi_{\Delta^{\times n}}$ to localize the scaled $f_{L^{-1}}$ back in blocks of unit scale. Note that $\chi_\Delta f_{L^{-1}} = 0$ unless $L\tilde{\Delta}_f$ intersects Δ, and for fixed Δ there are at most 2^d blocks Δ_f satisfying this constraint. Thus doing the sum over $\Delta_f^{\times n}$ first in the last step gives rise to a factor 2^{dn}. In the last step we have also estimated in C^r:

$$\|\chi_\Delta f_{L^{-1}}\| \leq L^{-\dim\phi}\|\chi\|\,\|f\| \leq L^{-\dim\phi}\|\chi\|. \tag{143}$$

Now the first inequality is proved and the second is an immediate corollary. ∎

LEMMA 15. *Let X be a small set. Then for a constant $O(1)$ depending on r*

$$|K_n(X,0;f_{L^{-1}}^{\times n})| \leq O(1)^n L^{-\dim(K_n)} \|K_n(X,0)\| \prod_j \|f_j\|_{C^r(L^{-1}X)}$$

where $\|K_n(X,0)\|$ is the norm in (16).

PROOF. We pick $z \in X$ and expand the functions $f_{j,L^{-1}}$ in $K_n(X,0;f_{L^{-1}}^{\times n})$ in a Taylor series

$$\begin{aligned}
f_{j,L^{-1}}(x) &= \sum_{q=0}^{r_n-1} \sum_{\alpha:|\alpha|=q} (\alpha!)^{-1}(x-z)^\alpha (\partial^\alpha f_{j,L^{-1}})(z) + R_{j,r_n}(x) \\
&\equiv \sum_{q=0}^{r_n} g_{j,q}(x)
\end{aligned} \tag{144}$$

where r_n appears in Definition 3.

We claim that

$$\|g_{j,q}\|_{C^{r_n}(X)} \leq O(1) L^{-q-\dim\phi} \|f_j\|_{C^{r_n}(L^{-1}X)}. \tag{145}$$

For $q < r_n$, note that

$$\partial^\beta g_{j,q}(x) = \sum_{\alpha:|\alpha|=q} \frac{(x-z)^{\alpha-\beta}}{(\alpha-\beta)!} (\partial^\alpha f_{j,L^{-1}})(z) \tag{146}$$

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and since

$$(\partial^{\alpha} f_{j,L^{-1}})(z) = L^{-|\alpha|-\dim\phi}(\partial^{\alpha} f_j)(L^{-1}z)$$

one obtains (145). For $q = r_n$ note that for $|\beta| \leq r_n$, $\partial^{\beta} g_{j,r_n}(x)$ is equal to the Taylor remainder for the expansion of $\partial^{\beta} f_{j,L^{-1}}$ to order $r_n - |\beta|$ and is given by

$$\frac{1}{(r_n - |\beta| - 1)!}\int_0^1 ds(1-s)^{r_n-|\beta|-1}\frac{d^{r_n-|\beta|}}{ds^{r_n-|\beta|}}(\partial^{\beta} f_{j,L^{-1}})\big(z - s(x-z)\big). \tag{147}$$

Now $\bar{X}$ is connected and if we also assume that it is convex then the path $z - s(x_j - z)$ stays entirely in $\bar{X}$ and it follows that (145) is also true for $q = r_n$. If $\bar{X}$ is not convex we have to use another representation for the remainder which is discussed at the end of the proof.

The first inequality follows from the definition of r_n: only the terms with total degree $\geq r_n$ contribute to K_n. Using (145) we have

$$\begin{aligned}
|K_n(X,0;f_{L^{-1}}^{\times n})| &= \Big|\sum_{q_i \leq r_n} \chi\big(\textstyle\sum q_j \geq r_n\big) K_n(X,0;g_{1,q_1}\times\cdots\times g_{n,q_n})\Big| \\
&\leq \sum_{q_j \leq r_n} \chi\big(\textstyle\sum q_j \geq r_n\big)\|K_n(X,0)\| \prod_j O(1) L^{-q_j-\dim\phi}\|f_j\|_{C^{r_n}(L^{-1}X)} \\
&\leq O(1)^n L^{-r_n - n\dim\phi}\|K_n(X,0)\| \prod_j \|f_j\|_{C^r(L^{-1}X)}.
\end{aligned} \tag{148}$$

X not convex: For any sufficiently smooth function $f(x)$ let $T(x,z)$ be the Taylor polynomial of order $r-1$ around $x = z$ and let $R(x,z)$ be the remainder so $f(x) = T(x,z) + R(x,z)$. Usually the remainder is expressed in terms of derivatives of order r along a line from z to x. Here we argue that instead one can express the remainder in terms of derivatives of order r along any piecewise linear curve from z to x.

Suppose for simplicity that we have a curve from z to z' to x. We define

$$G(x,z,z') = R(x,z) - R(x,z') = -T(x,z) + T(x,z').$$

Since $R(x,z')$ has the properties we want it suffices to consider $G(x,z,z')$. Since $G(x,z',z') = 0$ we have

$$\begin{aligned}
G(x,z,z') &= \int_0^1 \frac{d}{ds} G\big(x, z' + s(z-z'), z'\big)\, ds \\
&= \sum_{|\beta|=1}\int_0^1 (z-z')^{\beta}(\partial_z^{\beta} G)\big(x, z' + s(z-z'), z'\big)\, ds.
\end{aligned}$$

But for $|\beta| = 1$

$$(\partial_z^{\beta} G)(x,z,z') = -(\partial_z^{\beta} T)(x,z) = -\sum_{|\alpha|=r,\alpha\geq\beta}\frac{(\partial^{\alpha} f)(z)(x-z)^{\alpha-\beta}}{(\alpha-\beta)!}.$$

Thus $G(x,z,z')$ only involves derivatives of order r along the curve from z to z'. ∎

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The next lemma refers to a regulator $\bar{G}$ defined on L^{-1}-scale polymers by

$$\begin{aligned}\bar{G}(L^{-1}X) &= G(\kappa, L^{-1}\bar{X}^L)\, g^{-1}(L^{-1}\bar{X}^L \setminus L^{-1}X)\\ &= G(\kappa, L^{-1}X)G(\kappa/2, L^{-1}\bar{X}^L \setminus L^{-1}X)G_0(\kappa_0, L^{-1}\bar{X}^L \setminus L^{-1}X)\end{aligned}$$

where $g = G(\kappa/2)G_0^{-1}(\kappa_0)$ is the regulator appearing in (138). Note that

$$G(\kappa, L^{-1}X) \leq \bar{G}(L^{-1}X). \tag{149}$$

LEMMA 16. *For any small set X and $\kappa_0 h^2 \geq O(1)$ and $\kappa h^2 \geq O(1)$*

$$\|K_{L^{-1}}(L^{-1}X)\|_{\bar{G},h} \leq O(1)L^{-\dim(K)}\, \|K(X)\|_{G_L,h/2}$$

$O(1)$ depends on $\dim(K)$*.*

PROOF. Take p large enough so that $p\dim\phi \geq \dim K$. For $n < p$ we expand the n-th derivative $K_{L^{-1},n}(L^{-1}X, t\phi; f^{\times n})$ in a Taylor series in t to order $p-n$.

$$\begin{aligned}K_{L^{-1},n}(L^{-1}X, \phi; f^{\times n}) &= \sum_{q=n}^{p-1} \frac{1}{(q-n)!} K_{L^{-1},q}(L^{-1}X, 0; f^{\times n} \times \phi^{\times q-n})\\ &\quad + \int_0^1 dt \frac{(1-t)^{p-n-1}}{(p-n-1)!} K_{L^{-1},p}(L^{-1}X, t\phi; f^{\times n} \times \phi^{\times p-n}).\end{aligned} \tag{150}$$

To proceed, we define for $n \leq q \leq p$

$$J_{n,q,t}(L^{-1}X, \phi; f^{\times n}) = K_{L^{-1},q}(L^{-1}X, t\phi; f^{\times n} \times \phi^{\times q-n})$$

and will show that

$$\|J_{n,q,t}(L^{-1}X)\|_{\bar{G}} \leq O(1)L^{-q\dim\phi}h^{q-n}(1-t^2)^{(n-q)/2}\|K_q(X)\|_{G_L} \tag{151}$$

while for $t = 0$

$$\|J_{n,q,0}(L^{-1}X)\|_{\bar{G}} \leq O(1)L^{-\dim(K_q)}h^{q-n}\|K_q(X)\|_{G_L}. \tag{152}$$

In these bounds $O(1)$ depends on p.

Note that with these bounds the t integral in the remainder term of (150) is integrable. From this, and the fact that $L^{-\dim(K_q)}, L^{-p\dim\phi} \leq L^{-\dim(K)}$, it follows that

$$\begin{aligned}\sum_{n=0}^{p} \frac{h^n}{n!}\|K_{L^{-1},n}(L^{-1}X)\|_{\bar{G}} &\leq \sum_{n=0}^{p} \frac{h^n}{n!}\left(\sum_{q=n}^{p} \frac{O(1)}{(q-n)!} L^{-\dim(K)}h^{q-n}\|K_q(X)\|_{G_L}\right)\\ &\leq O(1)L^{-\dim(K)}\sum_{q=0}^{p} 4^q\frac{(h/2)^q}{q!}\|K_q(X)\|_{G_L}\\ &\leq O(1)L^{-\dim(K)}\|K(X)\|_{G_L,h/2}.\end{aligned} \tag{153}$$

For $p > n$ we use the first bound in Lemma 14 and $\bar{G}_L^{-1} \leq G_L^{-1}$ to obtain

$$\sum_{n=p+1}^{\infty} \frac{h^n}{n!} \|K_{L^{-1},n}(L^{-1}X)\|_{\bar{G}} \leq O(1)L^{-\dim(K)}\|K(X)\|_{G_L,h/2}. \tag{154}$$

Combining the two proves the lemma.

To bound J we proceed as in the proof of Lemma 14:

$$\begin{aligned} \|J_{n,q,t}(L^{-1}X)\|_{\bar{G}} &= \sum_{\Delta_f^{\times n}} \sup_{\phi,f} |K_q(X, t\phi_{L^{-1}}; f_{L^{-1}}^{\times n} \times \phi_{L^{-1}}^{\times q-n})|\bar{G}^{-1}(L^{-1}X, \phi) \\ &\leq O(1)L^{-q\dim\phi} a^q \sum_{\Delta^{\times n}, \Delta_\phi^{\times q-n}} \sup_{\phi} \|\phi\|_{C^r(L^{-1}X)}^{q-n} \qquad (155) \\ &\quad \times \|K_q(X, t\phi_{L^{-1}})\|_{\tilde{\Delta}^{\times n} \times \tilde{\Delta}_\phi^{\times q-n}} \bar{G}^{-1}(L^{-1}\bar{X}, \phi). \qquad (156) \end{aligned}$$

Now write

$$\bar{G}^{-1}(L^{-1}X, \phi) = \bar{G}_L^{-1}(X, t\phi_{L^{-1}})\bar{G}^{-1}\big(L^{-1}X, (1-t^2)^{1/2}\phi\big). \tag{157}$$

The first factor is paired with $\|K_q(X, t\phi_{L^{-1}})\|$ and the second factor is paired with $\|\phi\|^{q-n}$. Using Lemma 4, the fact that a small set has $O(1)$ blocks and the hypotheses on κ, κ_0, one finds that

$$\begin{aligned} \|\phi\|_{C^r(L^{-1}X)} &\leq \|\phi\|_{C^r(L^{-1}\bar{X}^L)} \\ &\leq O(1)(\|\phi\|_{L^{-1}\bar{X}^L \setminus L^{-1}X} + \|\partial\phi\|_{L^{-1}\bar{X}^L,2,\sigma}) \\ &\leq O(1)h\big(\kappa_0^{1/2}\|\phi\|_{L^{-1}\bar{X}^L\setminus L^{-1}X} + (\kappa/2)^{1/2}\|\partial\phi\|_{L^{-1}\bar{X}^L,2,\sigma}\big) \qquad (158) \end{aligned}$$

and hence

$$\begin{aligned} \|\phi\|_{C^r(L^{-1}X)}^{q-n} &\leq O(1)h^{q-n}G_0(\kappa_0, L^{-1}\bar{X}^L \setminus L^{-1}X, \phi)G(\kappa/2, L^{-1}\bar{X}^L, \phi) \\ &\leq O(1)h^{q-n}\bar{G}(L^{-1}X, \phi). \qquad (159) \end{aligned}$$

This leads to the bound

$$\|J_{n,q,t}(L^{-1}X)\|_{\bar{G}} \leq O(1)L^{-q\dim\phi}a^q h^{q-n}(1-t^2)^{(n-q)/2}\|K_q(X)\|_{\bar{G}_L} \tag{160}$$

and since $\bar{G}_L^{-1} \leq G_L^{-1}$ this gives (151).

When $t = 0$ we use Lemma 15 and have instead of (156)

$$\begin{aligned} \|J_{n,q,0}(L^{-1}X)\|_{\bar{G}} &= \sum_{\Delta_f^{\times n}} \sup_{\phi,f} |K_q(X, 0; f_{L^{-1}}^{\times n} \times \phi_{L^{-1}}^{\times q-n})|\bar{G}^{-1}(L^{-1}X, \phi) \\ &\leq O(1)L^{-\dim(K_q)} \sum_{\Delta_f^{\times n}} \sup_{\phi} \big(\|\phi\|_{C^r(L^{-1}X)}^{q-n}\bar{G}^{-1}(L^{-1}X, \phi)\big)\|K_q(X, 0)\|. \qquad (161) \end{aligned}$$

The sum over $\Delta_f^{\times n}$ has at most $O(1)^n$ terms because X is a small set and $K_q(X, 0; f_{L^{-1}}^{\times n} \times \phi^{\times q-n}) = 0$ if, for any Δ_f, $L\tilde{\Delta}_f \cap X = \emptyset$. By Lemma 2 and again using (159) we have

$$\|J_{n,q,0}(L^{-1}X)\|_{\bar{G}} \leq O(1)L^{-\dim K}h^{q-n}\|K_q(X)\|_{\bar{G}_L} \tag{162}$$

which gives (152). ∎

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PROOF (THEOREM 8). The bound on the kernels is a corollary of the first bound by letting the large field regulator G become concentrated at $\phi = 0$ as in (37). We rewrite (133) as

$$S(K)(Z,\phi) = \sum_N 1/N! \sum_{(X_1,\dots,X_N)} (e^{-V})_{L^{-1}}(Z \setminus L^{-1}X,\phi) \prod_i K_{L^{-1}}(L^{-1}X_i,\phi), \tag{163}$$

where the X_i are disjoint but the L-closures $\overline{X}_i^L$ overlap and fill LZ. Using

$$G(Z,\phi)^{-1} = G(Z \setminus L^{-1}X,\phi)^{-1} \prod_i G(L^{-1}X_i,\phi)^{-1} \tag{164}$$

we obtain by the multiplicative property of the norm (7)

$$\|S(K)(Z)\|_{G,h} \leq \sum_N 1/N! \sum_{(X_1,\dots,X_N)} \times \|(e^{-V})_{L^{-1}}(Z \setminus L^{-1}X)\|_{G,h} \prod_i \|K_{L^{-1}}(L^{-1}X_i)\|_{G,h}.$$

By the multiplicative property of the norm and the small V hypothesis (138),

$$\|(e^{-V})_{L^{-1}}(Z \setminus L^{-1}X)\|_{G,h} \leq \prod_{\Delta \subset Z} \|(e^{-V})_{L^{-1}}(\Delta \setminus L^{-1}X)\|_{G(\Delta\setminus L^{-1}X),h} \leq 2^{|Z|}. \tag{165}$$

By Lemma 14,

$$\|K_{L^{-1}}(L^{-1}X_i)\|_{G,h} \leq \|K(X_i)\|_{G_L,ah_L}.$$

Now multiply by Γ_q and note that $\Gamma_q(Z)2^{|z|} = \Gamma_{q+1}(Z)$. By the connectedness we have $\Gamma_{q+1}(Z) \leq \prod_i (\Gamma_{q+1})(L^{-1}\overline{X}_i^L)$. Furthermore we have the bound (13) for some constant $O(1)$:

$$(\Gamma_{q+1})(L^{-1}\overline{X}^L) \leq O(1)(\Gamma_{q-p})(X).$$

Summing over Z with a pin and using a spanning tree argument[5] we obtain

$$\|S(K)\|_{G,\Gamma_q,h} \leq \sum_{N=1}^{\infty} O(1)^{N-1}(L^d\|K\|_{G_L,\Gamma_{q-p},ah_L})^N.$$

This gives the result. ∎

PROOF (THEOREM 9). (1. Large sets) Proceeding as in the proof of Theorem 8 we obtain

$$\|S_1(K)(Z)\|_{G,h} \leq 2^{|Z|} \sum_{X:\overline{X}^L = LZ} \|K_{L^{-1}}(L^{-1}X)\|_{G,h}.$$

We take the $\|\ \|_{\Gamma_q}$ norm of both sides using

$$\sum_{Z\supset\Delta} \sum_{X:\overline{X}^L=LZ} \cdots = \sum_{X:\overline{X}^L\supset L\Delta} \cdots \leq \sum_{\Delta_0\subset L\Delta} \sum_{X\supset\Delta_0} \cdots$$

which leads to

$$\|S_1(K)\|_{G,h,\Gamma_q} \leq L^d \sup_{\Delta_0} \sum_{X\supset\Delta_0} (\Gamma_{q+1})(L^{-1}\overline{X}^L)\|K(X)\|_{G_L,ah_L}$$

[5] Described in the proof of Lemma 5.1 of [BY90].

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because $\|K_{L^{-1}}(L^{-1}X)\|_{G,h} \leq \|\underline{K}(X)\|_{G_L,ah_L}$ by Lemma 14. But for X large, by Lemma 1 we have the bound $(\Gamma_{q+1})(L^{-1}\bar{X}^L) \leq O(1)L^{-d-1}(\Gamma_{q-p})(X)$ which gives the result.

(2. Small sets) We have

$$G(Z,\phi) = \bar{G}(L^{-1}X,\phi)g(Z \setminus L^{-1}X,\phi)$$

from which we obtain

$$\begin{aligned}\|S_1(K)(Z)\|_{G,h} &\leq \sum_{X:\bar{X}^L=LZ} \|K_{L^{-1}}(L^{-1}X)\|_{\bar{G},h}\|(e^{-V})_{L^{-1}}(Z\setminus L^{-1}X)\|_{g,h} \\ &\leq 2^{|Z|}\sum_{X:\bar{X}^L=LZ} \|K_{L^{-1}}(L^{-1}X)\|_{\bar{G},h}\end{aligned}$$

so that

$$\|S_1(K)\|_{G,h,\Gamma_q} \leq L^d \sup_{\Delta_0} \sum_{X\supset\Delta_0} (\Gamma_{q+1})(L^{-1}\bar{X}^L)\|K_{L^{-1}}(L^{-1}X)\|_{\bar{G},h}.$$

Now use the Lemma 16 and the bound $(\Gamma_{q+1})(L^{-1}\bar{X}^L) \leq O(1)(\Gamma_{q-p})(X)$ from Lemma 1 to complete the proof. ∎

A. **Gaussian integration.** We recall some facts about Gaussian measures (see for example [Sim79]). Let $\langle\cdot,\cdot\rangle_C$ be an inner product on the real Sobolev space H_{-s}. By general probability theory there is an abstract measure space $(\Omega, \mathcal{F}, \mu)$ and a linear map $f \longmapsto \Phi_f$ from $f \in H_{-s}$ to random variables (functions on Ω) such that

$$\int d\mu(\phi)e^{\alpha\Phi_f(\phi)} = e^{\alpha^2\langle f,f\rangle_C/2} \tag{166}$$

for all $\alpha \in \mathbf{C}$. This family of random variables is called the Gaussian process indexed by H_{-s} with mean zero and covariance C.

One can make the specific choice $\Omega = H_r$ provided $r < s$ is such that the injection $H_s \longrightarrow H_r$ is trace class. In this case $\Phi_f(\phi) = \langle\phi, f\rangle$ for $f \in H_{-r}$ and is defined by L^p limits for $f \in H_{-s}$.

The Gaussian processes of interest to us are derived from inner products of the form

$$\langle f,g\rangle_C = \int_{\Lambda\times\Lambda} dx\,dy\,C(x,y)f(x)g(x)$$

where $C(x,y)$ is a C^∞ function on $\Lambda\times\Lambda$. This defines an inner product on H_{-s} for any s, and so we can get a process on any H_r.

Addition principle: If ϕ,ζ are two Gaussian processes with covariance B, C respectively, then the sum $\phi+\zeta$ is a Gaussian process with covariance $B+C$:

$$\int d\mu_B(\phi)\,d\mu_C(\zeta)F(\phi+\zeta) = \int d\mu_{B+C}(\psi)F(\psi) \tag{167}$$

Convolution: When Fubini's theorem holds on the left side of (167), the ζ integral can be done first, and the result is a measurable function of ϕ called the μ_C–convolution of F denoted by $\mu_C * F$:

$$(\mu_C * F)(\phi) = \int d\mu_C(\zeta)F(\phi+\zeta) \tag{168}$$

and

$$\int d\mu_B(\phi)\,d\mu_C(\zeta)F(\phi+\zeta) = \int d\mu_B(\phi)(\mu_C * F)(\phi) = \int d\mu_C(\zeta)(\mu_B * F)(\zeta). \tag{169}$$

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Semi-group property: A consequence of the addition principle is that the Gaussian convolution (168) can be broken up into steps. For any $t \in [0, 1]$, define the convolution function $F \longmapsto F_t = \mu_{tC} * F$. This function satisfies the semi-group property:

$$F_t = (F_s)_{t-s}, \quad \text{for all } s \in [0, t]. \tag{170}$$

Let us now define the *functional Laplacian of F* with respect to the measure μ_C

$$\Delta_C F(\phi) = \frac{1}{2} \int d\mu_C(\zeta) F_2(\phi; \zeta, \zeta) \tag{171}$$

where F_2 denotes the second functional derivative (*cf.* Section 2.3). The next proposition states conditions under which Gaussian convolution leads to solutions of the functional heat equation derived from Δ_C.

PROPOSITION 10. *Let μ_C be a Gaussian measure on $H_s(\Lambda)$, and F be a smooth functional of the Gaussian field ϕ whose third derivative F_3 is uniformly bounded pointwise in ϕ:*

$$\sup_{\zeta_1, \zeta_2, \zeta_3 \in H_s} \frac{|F_3(\phi; \zeta_1, \zeta_2, \zeta_3)|}{\|\zeta_1\|_{H_s} \|\zeta_2\|_{H_s} \|\zeta_3\|_{H_s}} \leq K.$$

Then the one-parameter family of functionals $F_t = \mu_{tC} * F$ parametrized by $t \in [0, 1]$ solves the functional heat equation

$$\frac{\partial F_t}{\partial t} = \Delta_C F_t \tag{172}$$

with the initial condition $F_0 = F$.

PROOF. We note that

$$F_{\delta t}(\phi) = \int d\mu_{\delta t}(\zeta) F(\phi + \zeta) = \int d\mu_1(\zeta) F(\phi + \delta t^{1/2} \zeta) \tag{173}$$

where we use the notation $\mu_t = \mu_{tC}$. Into this we insert the Taylor expansion in powers of $\delta t^{1/2}$

$$F(\phi + \delta t^{1/2} \zeta) = F(\phi) + \delta t^{1/2} F_1(\phi; \zeta) + \frac{\delta t}{2} F_2(\phi; \zeta, \zeta) + \delta t^{3/2} R(\phi, \zeta)$$

and obtain

$$\frac{1}{\delta t} \big(\mu_{\delta t} * F(\phi) - F(\phi) \big) - \Delta_C F(\phi) = \delta t^{1/2} \int d\mu_1(\zeta) R(\phi, \zeta).$$

The F_1 term vanished because it is odd in ζ. By the Taylor remainder formula,

$$\begin{aligned} \int d\mu_1(\zeta)\, |R(\phi, \zeta)| &\leq \sup_{\alpha \in [0, \delta t]} \int d\mu_1(\zeta)\, |F_3(\phi + \alpha^{1/2} \zeta; \zeta, \zeta, \zeta)| \\ &\leq K \int d\mu_1(\zeta)\, \|\zeta\|_{H_s}^3 \\ &\leq K'. \end{aligned}$$

Thus the remainder is uniformly bounded pointwise in ϕ for all small δt so that

$$\lim_{\delta t \longrightarrow 0} \frac{1}{\delta t}\big(\mu_{\delta t} * F(\phi) - F(\phi)\big) - \Delta_C F(\phi) \longrightarrow 0. \tag{174}$$

Combining this with the semigroup property $\mu_{t+\delta t} * = \mu_{\delta t} * \mu_t *$ shows that the function $t \longrightarrow \mu_t * F$ satisfies the functional heat equation

$$\frac{\partial}{\partial t}\mu_t * F = \Delta_C \mu_t * F$$

pointwise in ϕ and $t \longrightarrow \mu_t * F$ is smooth in t and ϕ. ∎

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