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GLOBAL ASYMPTOTIC STABILITY IN AN ALMOST-PERIODIC LOTKA-VOLTERRA SYSTEM

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Abstract

Sufficient conditions are obtained for the existence of a globally asymptotically stable strictly positive (componentwise) almost-periodic solution of a Lotka-Volterra system with almost periodic coefficients.

1. Introduction

This article is concerned with the derivation of a set of "easily verifiable" sufficient conditions for the existence of a globally asymptotically stable strictly-positive (componentwise) almost-periodic solution of the Lotka-Volterra system

$$\frac{dx_i(t)}{dt} = x_i(t)\left\{b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right\};$$

$$t > t_0;\ t_0 \in (-\infty, \infty);\ i = 1, 2, \dots, n, \quad (1.1)$$

where b_i, a_{ij} $(i, j = 1, 2, \dots, n)$ are nonnegative almost periodic functions defined for $t \in (-\infty, \infty)$. In mathematical ecology, the system (1.1) denotes the dynamics of an n-species population system, in which each individual competes with all others of the system for a common pool of resources. The assumption of almost-periodicity of the parameters b_i, a_{ij} $(i, j = 1, 2, \dots, n)$ in (1.1) is a way of incorporating the time dependent variability of the environment, especially when the various components of the environment are periodic with not necessarily commensurate periods (e.g. seasonal effects of weather, food supplies, mating habits, harvesting etc.). Mathematically, (1.1) will denote a generalisation of an autonomous and a periodic system; our result will, as a special case, provide

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results corresponding to such special cases of (1.1). We refer to Fink [1] or Yoshizawa [7] for the relevant definitions and properties of almost-periodic functions.

We shall need the following preparation; let $\mathbf{R}$ and $\mathbf{R}_n$ respectively denote the set of all real numbers and the n-dimensional real Euclidean space; $\mathbf{R}_n^+$ will denote the nonnegative cone of $\mathbf{R}_n$. Since almost-periodic functions defined on $\mathbf{R}$ are bounded, we define the constants b_i^l, b_i^u, a_{ij}^l, a_{ij}^u $(i, j = 1, 2, \ldots, n)$ by the following:

$$\inf_{t\in\mathbf{R}} b_i(t) = b_i^l; \quad \sup_{t\in\mathbf{R}} b_i(t) = b_i^u, \qquad i, j = 1, 2, \ldots, n.$$
$$\inf_{t\in\mathbf{R}} a_{ij}(t) = a_{ij}^l; \quad \sup_{t\in\mathbf{R}} a_{ij}(t) = a_{ij}^u.$$

We shall study the almost-periodic system (1.1) with the following assumption on the coefficients of (1.1);

$$b_i^l > 0; \qquad a_{ii}^l > 0, \tag{1.2}$$
$$b_i^l > \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}^u\left(b_j^u/a_{ij}^l\right). \tag{1.3}$$

Since the solutions of (1.1) corresponding to nonnegative initial conditions remain nonnegative subsequently, it will follow that

$$\frac{dx_i}{dt} \leqslant x_i\left\{b_i^u - a_{ii}^l x_i\right\}; \qquad t > t_0,\ i = 1, 2, \ldots, n, \tag{1.4}$$

as a consequence of which we will have

$$0 < x_i(t_0) \leqslant b_i^u/a_{ii}^l = x_i^u \Rightarrow x_i(t) \leqslant b_i^u/a_{ii}^l, \qquad i = 1, 2, \ldots, n;\ t \geqslant t_0. \tag{1.5}$$

(1.1) and (1.5) together lead to

$$\frac{dx_i}{dt} \geqslant x_i\left\{b_i^l - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}^u\left(b_j^u/a_{jj}^l\right) - a_{ii}^u x_i\right\}, \qquad i = 1, 2, \ldots, n. \tag{1.6}$$

Now (1.2), (1.3) and (1.6) lead to

$$x_i(t_0) \geqslant \left(b_i^l - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}^u\left(b_j^u/a_{jj}^l\right)\right)(1/a_{ii}^u) = x_i^l$$
$$\Rightarrow x_i(t) \geqslant x_i^l \quad \text{for } t \geqslant t_0,\ i = 1, 2, \ldots, n. \tag{1.7}$$

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From the foregoing preparation we have the following:

LEMMA 1.1. *Assume that* b_i, a_{ij} $(i, j = 1, 2, \ldots, n)$ *are scalar nonnegative almost periodic functions defined on* $\mathbf{R}$ *such that* (1.2) *and* (1.3) *hold. Then the set* S *defined by*

$$S = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}_n | x_* \leqslant x_i \leqslant x^*;\ i = 1, 2, \ldots, n \right\} \tag{1.8}$$

where

$$x_* = \min_{1 \leqslant i \leqslant n} x_i^l; \qquad x^* = \min_{1 \leqslant i \leqslant n} x_i^u \tag{1.9}$$

is invariant with respect to (1.1).

2. Existence of an almost-periodic solution

We shall note the following facts on almost-periodic functions before formulating our existence theorem.

DEFINITION 2.1. A function $f(t, x)$, where f is an m-vector, t is a real scalar and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in X \subset \mathbf{R}_n$, if $f(t, x)$ is continuous in $t \in \mathbf{R}$ and $x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a τ such that the inequality

$$\| f(t + \tau, x) - f(t, x) \| = \sum_{i=1}^{m} | f_i(t + \tau, x) = f_i(t, x) | < \varepsilon \tag{2.1}$$

is satisfied for all $t \in (-\infty, \infty)$, $x \in X$. The number τ is called an ε-translation number of $f(t, x)$.

Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$\lim_{T \to \infty} T^{-1} \int_0^T f(t, x) \exp(-i \lambda_j t)\, dt \not\equiv 0 \quad \text{for } x \in X;\ i = \sqrt{-1}. \tag{2.2}$$

It is known that when the set X is separable, the set of numbers $\{\lambda_j\}$ in (2.2) is countable (Yoshizawa [7], page 6). The set $\{\Sigma_1^N n_j \lambda_j\}$ for all integers N and integers n_j is called the module of $f(t, x)$.

The following result concerned with module containment will be used in the proof of our existence theorem below.

MODULE CONTAINMENT THEOREM (Yoshizawa [7], page 18). *Let* $f(t, x)$ *and* $g(t, x)$ *be almost periodic in* t *uniformly for* $x \in D \subset \mathbf{R}_n$. *If for any compact set* $S \subset D$ *and for any sequence of real numbers* $\{\tau_k\}$ *having a limit as* $k \to \infty$ *(the limit being finite or* $\pm\infty$*) for which* $\{f(t + \tau_k, x)\}$ *is uniformly convergent on* $\mathbf{R} \times S$, $\{g(t + \tau_k, x)\}$ *is also uniformly convergent on* $\mathbf{R} \times S$ *then the module of* $g(t, x)$ *is contained in the module of* $f(t, x)$.

We can now formulate our existence theorem.

THEOREM 2.1. *Assume that the almost-periodic parameters* b_i, a_{ij} $(i, j = 1, 2, \ldots, n)$ *of* (1.1) *satisfy* (1.2)–(1.3). *Furthermore assume that there exists a positive number* μ *such that*

$$\inf_{t \in \mathbf{R}} a_{ii}(t) > \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\sup_{t \in \mathbf{R}} a_{ji}(t) \right) + \mu, \qquad i = 1, 2, \ldots, n. \tag{2.3}$$

Then the system (1.1) *has a strictly positive* (*componentwise*) *almost-periodic solution, say* $w(t) = w_1(t), \ldots, w_n(t)$, $t \in \mathbf{R}$, *whose module is contained in that of*

$$f(t, x) = \{ f_1(t, x_1, \ldots, x_n), f_2(t, x_1, \ldots, x_n), \ldots, f_n(t, x_1, \ldots, x_n) \}$$

where

$$f_i(t, x_1, \ldots, x_n) = x_i \left\{ b_i(t) - \sum_{j=1}^{n} a_{ij}(t) x_j \right\},$$
$$i = 1, 2, \ldots, n; \ (x_1, x_2, \ldots, x_n) \in S. \tag{2.4}$$

PROOF. The set S defined by (1.8)–(1.9) is compact in $\mathbf{R}_n$. Let $\{\tau_k\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$\sum_{i=1}^{n} | f_i(t + \tau_k, x_1, \ldots, x_n) - f_i(t, x_1, \ldots, x_n) | \to 0 \quad \text{as } k \to \infty \tag{2.5}$$

uniformly in $(x_1, \ldots, x_n) \in S$ for $t \in (-\infty, \infty)$. For any real number β, let $k_0(\beta)$ be the smallest value of k such that $\tau_{k_0} + \beta \geqslant t_0$. Since the invariance of S implies that any solution of (1.1)

$$\{ x_1(t + \tau_k, t_0, x_1(t_0)), x_2(t + \tau_k, t_0, x_2(t_0)), \ldots, x_n(t + \tau_k, t_0, x_n(t_0)) \} \in S$$

for all $t \geqslant \beta$ and $k \geqslant k_0(\beta)$ whenever $\{x_1(t_0), \ldots, n_n(t_0)\} \in S$, our first task in the proof is to show that the sequence

$$\{ x_1(t + \tau_k, t_0, x_1(t_0)), \ldots, x_n(t + \tau_k, t_0, x_n(t_0)) \}, \qquad k \geqslant k_0(\beta),$$

converges to a continuous bounded function say $w(t) = \{w_1(t), \ldots, w_n(t)\}$ defined for $t \in [\beta, \infty)$ such that $w_i(t) \in [x_*, x^*]$, $t \in [\beta, \infty)$, $i = 1, 2, \ldots, n$ and the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since x_* and x^* are independent of β, it is enough to show that

$$\{ x_1(t + \tau_k, t_0, x_1(t_0)), \ldots, x_n(t + \tau_k, t_0, x_n(t_0)) \}, \qquad k \geqslant k_0(\beta),$$

is a Cauchy sequence on compact subsets of $[\beta, \infty)$.

Let U be any compact subset of $[\beta, \infty)$, and let ε be any arbitrary positive number. Choose an integer $n_0(\varepsilon, \beta) \geqslant k_0(\beta)$ so large that for $m \geqslant k \geqslant n_0(\varepsilon, \beta)$

and all $t \in (-\infty, \infty)$

$$\sum_{i=1}^{n} |b_i(t+\tau_m) - b_i(t+\tau_k)| < (\varepsilon\mu x_*)/(4x^*), \tag{2.6}$$

$$x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t+\tau_m) - a_{ij}(t+\tau_k)| < (\mu\varepsilon x_*)/(4x^*), \tag{2.7}$$

$$(2nx^*/x_*)\exp[-\mu x_*(\beta + \tau_k - t_0)] < (\varepsilon/2x^*). \tag{2.8}$$

Denote $x_i(t, t_0, x_i(t_0))$ by $x_i(t)$ for brevity, and consider a function $v(s)$ defined by

$$\begin{aligned} v(s) &= v(x(s), x(s+\tau_m+\tau_k)) \\ &= \sum_{i=1}^{n} |\log x_i(s) - \log x_i(s+\tau_m-\tau_k)|, \qquad x \geqslant t_0, \end{aligned} \tag{2.9}$$

where $x(t) = \{x_1(t), \ldots, x_n(t)\}$ denotes a solution of (1.1) with $\{x_1(t_0), \ldots, x_n(t_0)\} \in S$. The invariance of S for (1.1), together with the elementary mean value theorem of differential calculus, implies

$$\begin{aligned} (1/x^*) \sum_{i=1}^{n} |x_i(s) - x_i(s+\tau_m+\tau_k)| &\leqslant v(s) \\ &\leqslant (1/x_*) \sum_{i=1}^{n} |x_i(s) - x_i(s+\tau_m-\tau_k)|. \end{aligned} \tag{2.10}$$

Calculating the right derivative D^+v of v along the solutions of (1.1) and simplifying

$$\begin{aligned} D^+v(s) &= \sum_{i=1}^{n} \{\operatorname{sign}[x_i(s) - x_i(s+\tau_m-\tau_k)]\} \\ &\quad \cdot \left\{ \frac{d[\log x_i(s)]}{ds} - \frac{d[\log x_i(s+\tau_m-\tau_k)]}{ds} \right\} \\ &\leqslant \sum_{i=1}^{n} |b_i(s) - b_i(s+\tau_m-\tau_k)| \\ &\quad + \sum_{i=1}^{n} \Bigg(-a_{ii}(s)|x_i(s) - x_i(s+\tau_m-\tau_k)| \\ &\qquad\qquad + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(s)|x_j(s) - x_j(s+\tau_m-\tau_k)| \Bigg) \end{aligned}$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(s)-a_{ij}(s+\tau_m-\tau_k)\right|x_j(s+\tau_m-\tau_k) \tag{2.11}$$

$$\leqslant -\mu x_* v\big(x(s), x(s+\tau_m-\tau_k)\big)+\sum_{i=1}^{n}\left|b_i(s)-b_i(s+\tau_m-\tau_k)\right|$$
$$+x^*\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(s)-a_{ij}(s+\tau_m-\tau_k)\right|. \tag{2.12}$$

Let us set $s-\tau_k=\sigma$; then we have from (2.6)–(2.7) that

$$\sum_{i=1}^{n}\left|b_i(s)-b_i(s+\tau_m-\tau_k)\right|+x^*\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(s)-a_{ij}(s+\tau_m-\tau_k)\right|$$
$$=\sum_{i=1}^{n}\left|b_i(\sigma+\tau_k)-b_i(\sigma+\tau_m)\right|$$
$$+x^*\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(\sigma+\tau_k)-a_{ij}(\sigma+\tau_m)\right|$$
$$<(\mu\varepsilon x_*)/(2x^*). \tag{2.13}$$

Now an integration of (2.12) over $(t_0, t-\tau_k)$ together with (2.13) leads to

$$v\big(x(t+\tau_k), x(t+\tau_m)\big)\leqslant v\big(x(t_0), x(t_0+\tau_m-\tau_k)\big)\exp[-\mu x_*(t+t_k-t_0)]$$
$$+\left[(\mu\varepsilon x_*)/(2x^*)\right]/(\mu x_*)$$
$$\leqslant\left[(2x_n^*)/x_*\right]\exp[-\mu x_*(t+t_k-t_0)]+(\varepsilon/2x^*)$$
$$<(\varepsilon/x^*)\quad\text{on using (2.8), (2.10)} \tag{2.14}$$

for $m\geqslant k\geqslant n_0(\varepsilon,\beta)$ and for all $t\in U\subset[\beta,\infty)$. Thus we have from the definition of v and (2.14) that

$$\sum_{i=1}^{n}\left|\log x_i(t+\tau_k)-\log x_i(t+\tau_m)\right|<(\varepsilon/x^*) \tag{2.15}$$

and hence using $x_*\leqslant x_i(t)\leqslant x^*$, $i=1,2,\ldots,n$; $t\geqslant t_0$,

$$\sum_{i=1}^{n}\left|x_i(t+\tau_k)-x_i(t+\tau_m)\right|<\varepsilon\quad\text{for all } t\in U\subset[\beta,\infty),$$
$$m\geqslant k\geqslant n_0(\varepsilon,\beta). \tag{2.16}$$

The existence of a function $w(t)=\{w_1(t),\ldots,w_n(t)\}$ defined on $[\beta,\infty)$ such that $w_i(t)\in[x_*,x^*]$, $i=1,2,\ldots,n$; $t\in[\beta,\infty)$ follows from (2.16) and the fact that $x_i(t)\in[x_*,x^*]$, $i=1,2,\ldots,n$ for $t\geqslant t_0$. Since β is arbitrary, w is in fact defined on $(-\infty, \infty)$ and hence we have

$$x_i(t + \tau_k) \to w_i(t) \quad \text{as } k \to \infty \text{ uniformly on compact subsets of } (-\infty, \infty), \quad i = 1, 2, \ldots, n. \tag{2.17}$$

Our next task is to show that w: $\mathbf{R} \to S$ is differentiable and is a solution of (1.1). Since $x(t) = \{x_1(t), \ldots, x_n(t)\}$ satisfies (1.1) we have

$$\begin{aligned} \sum_{i=1}^{n} \left| \frac{d}{dt} x_i(t + \tau_k) - \frac{d}{dt} x_i(t + \tau_m) \right| & \\ \leqslant \sum_{i=1}^{n} | f_i(t + \tau_k, x_1(t + \tau_k), \ldots, x_n(t + \tau_k)) & \\ - f_i(t + \tau_m, x_1(t + \tau_k), \ldots, x_n(t + \tau_k))| & \\ + \sum_{i=1}^{n} | f_i(t + \tau_m, x_1(t + \tau_k), \ldots, x_n(t + \tau_k)) & \\ - f_i(t + \tau_m, x_1(t + \tau_m), \ldots, x_n(t + \tau_m))| & \end{aligned} \tag{2.18}$$

where f_i $(i = 1, 2, \ldots, n)$ are as in (2.4). Since $x(t + \tau_k) = x_1(t + \tau_k), \ldots, x_n(t + \tau_k) \in S$ for large τ_k for t in each compact subset of $(-\infty, \infty)$ there exists an integer $n_1(\varepsilon) > 0$ such that if $m \geqslant k \geqslant n_1(\varepsilon)$ then for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{i=1}^{n} | f_i(t + \tau_k, x_1(t + \tau_k), \ldots, x_n(t + \tau_k)) & \\ - f_i(t + \tau_m, x_1(t + \tau_k), \ldots, x_n(t + \tau_k))| < (\varepsilon/2). & \end{aligned} \tag{2.19}$$

Similarly since $x_i(t + \tau_k) \in S$ and $x_i(t + \tau_m) \in S$ for $i = 1, 2, \ldots, n$ there is an integer $n_2(\varepsilon) > 0$ such that for large enough m, k, such that $m \geqslant k \geqslant n_2(\varepsilon)$, we have

$$\begin{aligned} \sum_{i=1}^{n} | f_i(t + \tau_m, x_1(t + \tau_k), \ldots, x_n(t + \tau_k)) & \\ - f_i(t + \tau_m, x(t + \tau_m), \ldots, x_n(t + \tau_m))| < (\varepsilon/2). & \end{aligned} \tag{2.20}$$

Thus if $m \geqslant k \geqslant n_0(\varepsilon) = \max\{n_1, n_2\}$, we have

$$\sum_{i=1}^{n} \left| \frac{d}{dt} x_i(t + \tau_k) - \frac{d}{dt} x_i(t + \tau_m) \right| < \varepsilon \tag{2.21}$$

which shows that

$$\lim_{k \to \infty} \frac{d}{dt} x_i(t + \tau_k) \quad \text{exists for } i = 1, 2, \ldots, n$$

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uniformly on all compact subsets of $(-\infty, \infty)$. Thus we have

$$\begin{aligned}\lim_{k\to\infty}\lim_{h\to\infty} & \frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h} \\ &= \lim_{h\to 0}\lim_{k\to 0}\frac{x_i(t+\tau_k+h)-x_i(t+\tau_k)}{h} \\ &= \lim_{h\to 0}\frac{w_i(t+h)-w_i(t)}{h}, \qquad i=1,2,\ldots,n,\end{aligned} \tag{2.22}$$

showing that (since the left side of (2.22) exists) w_i $(i = 1, 2, \ldots, n)$ is differentiable. Furthermore we have from (2.22) that

$$\begin{aligned}\frac{dw_i(t)}{dt} = \lim_{k\to\infty} & \left[f_i(t+\tau_k, x_1(t+\tau_k),\ldots,x_n(t+\tau_k))\right. \\ & -f_i(t+\tau_k, w_1(t),\ldots,w_n(t)) \\ & \left.+f_i(t+\tau_k, w_1(t),\ldots,w_n(t))\right] \\ = f_i(t, w_1(t),\ldots,w_n(t)); & \qquad i=1,2,\ldots,n,\end{aligned} \tag{2.23}$$

by the uniform almost periodicity of $f_1, f_2, \ldots, f_n$ on $\mathbf{R} \times S$ and hence $w(t) = (w_1(t), \ldots, w_n(t))$ is a solution of (1.1).

In order to complete the proof, we have to show that $w(t) = \{w_1(t), \ldots, w_n(t)\}$ is an almost-periodic function whose module is contained in that of $\{f_1(t, x_1, \ldots, x_n), f_2(t, x_1, x_2, \ldots, x_n), \ldots, f_n(t, x_1, \ldots, x_n)\}$ for $(x_1, x_2, \ldots, x_n) \in S$. For this purpose it is in fact sufficient (by the module containment theorem) to show that for any sequence of real numbers $\{\tau_k\}$ for which

$$\{f_1(t+\tau_k, x_1,\ldots,x_n),\ f_2(t+\tau_k, x_1,\ldots,x_n),\ldots, f_n(t+\tau_k, x_1,\ldots,x_n)\}$$

converges uniformly on $\mathbf{R}$ for all $(x_1, \ldots, x_n) \in S$, the sequence $\{w_1(t+\tau_k), \ldots, w_n(t+\tau_k)\}$ converges uniformly on $\mathbf{R}$ where τ_k tends to a finite number or $\pm\infty$ as $k \to \infty$.

In the case where $\tau_k \to \tau \neq \pm\infty$ as $k \to \infty$, we have $w_i(t+\tau_k) \to w_i(t+\tau)$, $i = 1, 2, \ldots, n$ uniformly on $\mathbf{R}$ as $k \to \infty$. Let us suppose $\tau_k \to \infty$ as $k \to \infty$. For any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that if $m \geqslant k \geqslant n_0(\varepsilon)$ then

$$\sum_{i=1}^{n} |b_i(t+\tau_k) - b_i(t+\tau_m)| < (\varepsilon/4x^*)x_*, \tag{2.24}$$

$$x^* \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(t+\tau_k) - a_{ij}(t+\tau_m)| < (\varepsilon/4x^*)\mu x_*, \tag{2.25}$$

$$n(2x^*/x_*)\exp[-\mu x_* \tau_k] < (\varepsilon/2x^*). \tag{2.26}$$

Consider the function $v(w(s), w(s+\tau_m-\tau_k))$ for $s \in [t, t+\tau_k]$ defined by

$$v(w(s), w(s+\tau_m-\tau_k)) = \sum_{i=1}^{n} |\log w_i(s) - \log w_i(s+\tau_m-\tau_k)|. \tag{2.27}$$

Using the fact w: $\mathbf{R} \to S$ and w is a solution of (1.1), calculate the right derivative D^+v of v;

$$\begin{aligned}
D^+v(w(s), w(s+\tau_m-\tau_k)) &= \sum_{i=1}^{n} \{\operatorname{sign}[w_i(s) - w_i(s+\tau_m-\tau_k)]\} \\
&\quad \cdot \left\{ \frac{dw_i(s)}{ds} \cdot \frac{1}{w_i(s)} - \frac{dw_i(s+\tau_m-\tau_k)}{ds} \frac{1}{w_i(s+\tau_m-\tau_k)} \right\} \\
&\leqslant -\mu \sum_{i=1}^{n} |w_i(s) - w_i(s+\tau_m-\tau_k)| + \sum_{i=1}^{n} |b_i(s) - b_i(s+\tau_m-\tau_k)| \\
&\quad + x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(s) - a_{ij}(s+\tau_m-\tau_k)| \qquad (2.28) \\
&\leqslant -\mu x_* v(w(s), w(s+\tau_m-\tau_k)) \\
&\quad + \sum_{i=1}^{n} |b_i(s) - b_i(s+\tau_m-\tau_k)| \\
&\quad + x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(s) - a_{ij}(s+\tau_m-\tau_k)|. \qquad (2.29)
\end{aligned}$$

If we set $s - \tau_k = \sigma$ then we have from (2.29),

$$\begin{aligned}
&D^+v(w(s), w(s+\tau_m-\tau_k)) \\
&\quad \leqslant -\mu x_* v(w(s), w(s+\tau_m-\tau_k)) \\
&\qquad + \sum_{i=1}^{n} |b_i(\sigma+\tau_k) - b_i(s+\tau_m)| \\
&\qquad + x^* \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(\sigma+\tau_k) - a_{ij}(\sigma+\tau_m)| \qquad (2.30) \\
&\quad \leqslant -\mu x_* v(w(s), w(s+\tau_m-\tau_k)) \\
&\qquad + [(\varepsilon\mu x_*)/(4x^*)] + [(\varepsilon\mu x_*)/(4x^*)]. \qquad (2.31)
\end{aligned}$$

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Integrating (2.31) over $(t, t+\tau_k)$

$$\begin{aligned} v(w(t+\tau_k), w(t+\tau_m)) & \\ &\leqslant e^{-\mu x_* \tau_k} v(w(t), w(t+\tau_m-\tau_k)) + (\varepsilon/2x^*) \qquad (2.32)\\ &\leqslant e^{-\mu x_* \tau_k} \sum_{i=1}^{n} |\log w_i(t) - \log w_i(t+\tau_m-\tau_k)| + (\varepsilon/2x^*) \\ &\leqslant e^{-\mu x_* \tau_k} \sum_{i=1}^{n} |w_i(t) - w_i(t+\tau_m-\tau_k)|(1/x_*) + (\varepsilon/2x^*) \\ &\leqslant n(2x^*/x_*)\exp[-\mu x_* \tau_k] + (\varepsilon/2x^*) \\ &< (\varepsilon/x^*). \qquad (2.33) \end{aligned}$$

Since

$$v(w(t+\tau_k), w(t+\tau_m)) \geqslant \left(\sum_{i=1}^{n} |w_i(t+\tau_k) - w_i(t+\tau_m)|\right)/x^* \qquad (2.34)$$

we have from (2.33) that

$$\sum_{i=1}^{n} |w_i(t+\tau_k) - w_i(t+\tau_m)| < \varepsilon \quad \text{for } m \geqslant k \geqslant n_0(\varepsilon)$$

which show that the sequence $\{w_1(t+\tau_k), \ldots, w_n(t+\tau_k)\}$ converges uniformly on $\mathbf{R}$. By a similar argument the above conclusion is also valid when $\tau_k \to -\infty$ as $k \to \infty$. Thus the uniform convergence of the sequence $\{w_1(t+\tau_k), \ldots, w_n(t+\tau_k)\}$ as $k \to \infty$ on $\mathbf{R}$ follows and this completes the proof of the theorem.

COROLLARY. *Suppose* b_i, a_{ij} $(i, j = 1, 2, \ldots, n)$ *in* (1.1) *are periodic with a common period say* $T > 0$. *Then if* (1.2), (1.3) *and* (2.3) *hold, the system* (1.1) *will have a periodic solution* $w(t) = \{w_1(t), \ldots, w_n(t)\}$ *of period* T *such that*

$$w_i(t+T) = w_i(t), \qquad t \in \mathbf{R};\ i = 1, 2, \ldots, n.$$

PROOF. Proof follows from that of Theorem 2.1 if we choose $\tau_k = KT$ $(K = 0, 1, 2, \ldots)$ and note that

$$\begin{aligned} &x_i(t+KT) - w_i(t) \to 0 \quad \text{as } k \to \infty;\ i = 1, 2, \ldots, n;\ t \in \mathbf{R}, \\ &x_i(t+(K+1)t) - w_i(t) \to 0 \quad \text{as } K \to \infty;\ i = 1, 2, \ldots, n;\ t \in \mathbf{R}, \\ &x_i(t+T+KT) - w_i(t+T) \to 0 \quad \text{as } k \to \infty;\ i = 1, 2, \ldots, n;\ t \in \mathbf{R}, \end{aligned}$$

implying that

$$w_i(t+T) = w_i(t); \qquad i = 1, 2, \ldots, n;\ t \in \mathbf{R}.$$

3. Global asymptotic stability

By definition we say that an almost-periodic solution, say $w(t) = \{w_1(t), \ldots, w_n(t)\}$ of (1.1) is globally asymptotically stable (or attractive) if and only if every other solution $x(t) = \{x_1(t), \ldots, x_n(t)\}$ with $x_i(0) > 0$, $i = 1, 2, \ldots, n$ is defined for all $t \geqslant 0$ and satisfies

$$\lim_{t \to \infty} |w_i(t) - x_i(t)| = 0; \quad i = 1, 2, \ldots, n. \tag{3.1}$$

A consequence of such a global asymptotic stability is that there cannot be another strictly positive almost-periodic solution of (1.1).

THEOREM 3.1. *Suppose the almost-periodic coefficients b_i, a_{ij} $(i, j = 1, 2, \ldots, n)$ satisfy* (1.2) *and* (1.3) *and the following*

$$\inf_{t \in \mathbf{R}} a_{ii}(t) > \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{n} \sup a_{ji}(t) \right\} + \mu; \qquad i = 1, 2, \ldots, n. \tag{3.2}$$

for some positive number μ. Then (1.1) *has an almost-periodic solution* (*with strictly positive components*) *which is globally asymptotically stable.*

We will omit the proof since the proof is identical with that of Theorem 3.1 of [3].

Under condition (3.2), the almost-periodic solution $w(t) = \{w_1(t), \ldots, w_n(t)\}$ of (1.1) enjoys a somewhat strong form of stability, even if the coefficients of (1.1) are subjected to a class of perturbations. We note (see Yoshizawa [7], page 17) that if a function $b\colon \mathbf{R} \to \mathbf{R}$ is almost periodic, then $H(b)$, the "hull of b" is defined as follows:

$$H(b) = \begin{cases} g \mid g\colon \mathbf{R} \to \mathbf{R} \text{ and } g(t) = \lim\limits_{k \to \infty} b(t + \tau_k) \text{ for some sequence} \\ \tau_k \text{ of real numbers, the limit being uniform in } t \in \mathbf{R}. \end{cases} \tag{3.3}$$

By Theorem 2.2 of Yoshizawa [7, page 10], members of the hull of almost-periodic functions are themselves almost periodic.

A system of the form

$$\frac{dy_i(t)}{dt} = y_i(t)\left\{B_i(t) - \sum_{j=1}^{n} A_{ij}(t)y_j(t)\right\},$$
$$i = 1,2,\dots,n;\ t > t_0 \in \mathbf{R}, \qquad (3.4)$$

where

$$B_i \in H(b_i);\quad A_{ij} \in H(A_{ij}),\qquad i,j = 1,2,\dots,n, \qquad (3.5)$$

is known as a system from the hull of (1.1). As a consequence of the definition of the hull in (3.3) and (3.5) we shall have

$$\inf_{t\in\mathbf{R}} B_i(t) = b_i^l;\qquad \sup_{t\in\mathbf{R}} B_i(t) = b_i^l,$$
$$\inf_{t\in\mathbf{R}} A_{ij}(t) = a_{ij}^l;\qquad \sup_{t\in\mathbf{R}} A_{ij}(t) = a_{ij}^u,$$

and hence the coefficients of (3.4)–(3.5) satisfy the same conditions as (1.2), (1.3) and (3.2). It will then follow from theorems (2.1) and (3.1) that solutions of (3.4)–(3.5) with $y_i(t_0) > 0$, $i = 1,2,\dots,n$ will have the following property:

$$y_i(t) > 0 \quad \text{for } t \geqslant t_0;\ i = 1,2,\dots,n,$$
$$\sum_{i=1}^{n} y_i(t) \text{ remains bounded for } t \geqslant t_0.$$

DEFINITION. A solution $x(t) = \{x_1(t),\dots,x_n(t)\}$, $t \in t_0 \in \mathbf{R}$ of (1.1) is said to be stable under perturbations from the hull of (1.1) if the following holds: let $y(t) = \{y_1(t),\dots,y_n(t)\}$ be any solution of (3.4)–(3.5) such that $y_i(t_0) > 0$, $i = 1,2,\dots,n$, and for some positive constants β_1, β_2, $\beta_1 \leqslant y_i(t) \leqslant \beta_2$, $t \geqslant t_0$, $i = 1,2,\dots,n$. Then, given any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$\left.\begin{aligned} &\sum_{j=1}^{n} |x_i(t_0) - y_i(t_0)| < \delta \\ &\sum_{i=1}^{n} \sup_{t\in\mathbf{R}} |b_i(t) - B_i(t)| < \delta \\ &\sum_{i=1}^{n}\sum_{j=1}^{n} \sup_{t\in\mathbf{R}} |a_{ij}(t) - A_{ij}(t)| < \delta \end{aligned}\right\} \Rightarrow \sum_{i=1}^{n} |x_i(t) - y_i(t)| < \varepsilon$$
$$\text{for } t \geqslant t_0. \qquad (3.6)$$

We can now formulate the following result which shows that all solutions of (1.1) are stable under perturbations from the hull of (1.1).

THEOREM 3.2. *Assume that the conditions of Theorem* (3.1) *hold. Then every strictly positive solution of* (1.1) (*including its unique almost-periodic solution*) *is stable under perturbations from the hull of* (1.1).

PROOF. Let $x(t) = \{x_1(t), \ldots, x_n(t)\}$ and $y(t) = \{y_1(t), \ldots, y_n(t)\}$ be any two strictly positive solutions of (1.1) and (3.4)–(3.5) respectively for $t \geqslant t_0 \in \mathbf{R}$. Let β_1, β_2 be constants such that

$$\begin{aligned} \beta_1 \leqslant x_i(t) \leqslant \beta_2, \\ \beta_1 \leqslant y_i(t) \leqslant \beta_2, \end{aligned} \qquad i = 1, 2, \ldots, n;\ t \geqslant t_0. \tag{3.7}$$

Consider now a function $V(t) = V(x(t), y(t))$ defined by

$$V(x(t), y(t)) = \sum_{i=1}^{n} |\log x_i(t) - \log y_i(t)|, \qquad t \geqslant t_0 \in \mathbf{R}. \tag{3.8}$$

It is easy to see that

$$\begin{aligned} \frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| &\leqslant V(x(t), y(t)) \\ &\leqslant \frac{1}{\beta_1} \sum_{i=1}^{n} |x_i(t) - y_i(t)|, \qquad t \geqslant t_0 \in \mathbf{R}. \end{aligned} \tag{3.9}$$

Calculating the right derivative D^+V of V and using (1.1) and (3.4) we derive (after simplification) that

$$\begin{aligned} D^+V(x(t), y(t)) &\leqslant -\mu \sum_{i=1}^{n} |x_i(t) - y_i(t)| + \sum_{i=1}^{n} |b_i(t) - B_i(t)| \\ &\quad + \beta_2 \sum_{j=1}^{n} \sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \qquad t \geqslant t_0, \\ &\leqslant -\mu\beta_1 V(x(t), y(t)) + \sum_{i=1}^{n} |b_i(t) - B_i(t)| \\ &\quad + \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t) - A_{ij}(t)|; \qquad t \geqslant t_0. \end{aligned} \tag{3.10}$$

An integration of (3.9) over $[t_0, t]$ with an application of (3.8) leads to

$$\begin{aligned} \frac{1}{\beta_2} \sum_{i=1}^{n} |x_i(t) - y_i(t)| &\leqslant V(x(t), y(t)) \leqslant V(x(t_0), y(t_0)) \\ &\quad + \left[\sum_{i=1}^{n} \sup_{t \in \mathbf{R}} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sup_{t \in \mathbf{R}} |a_{ij}(t) - A_{ij}(t)| \right] \frac{1}{\mu\beta_1}, \\ &\qquad t \geqslant t_0, \end{aligned}$$

and hence

$$\sum_{i=1}^{n} |x_i(t) - y_i(t)| \leqslant \left(\frac{\beta_2}{\beta_1}\right) \sum_{i=1}^{n} |x_i(t_0) - y_i(t_0)| + \frac{\beta_2}{\mu\beta_1}\left[\sum_{i=1}^{n} \sup_{t\in\mathbf{R}} |b_i(t) - B_i(t)| + \beta_2 \sum_{i=1}^{n}\sum_{j=1}^{n} \sup_{t\in\mathbf{R}} \left|a_{ij}(t) - A_{ij}(t)\right|\right]. \tag{3.11}$$

Now for any $\varepsilon > 0$ if one chooses a $\delta > 0$ such that

$$\delta < \min\left\{\frac{\varepsilon}{3}\frac{\beta_1}{\beta_2}, \frac{\varepsilon}{3}\frac{\beta_1}{\beta^2}\mu\right\} \tag{3.12}$$

then (3.6) will be satisfied and this completes the proof.

4. Some comments

The sufficient conditions obtained here provide a significant generalisation of the author's results in [2], [3]. Periodic Lotka-Volterra systems have been considered by several authors; however almost-periodic Lotka-Volterra systems seem to have been not considered in the literature so far. Our conditions of Theorem 3.1 (although sufficient) are easy to verify and offer global asymptotic stability. For some remarks regarding the evolutionary and ecological significance of the variability of environment we refer to [4]. For a discussion of the ecological implication of the condition (3.2) we refer to [5], [6]. In a forthcoming analysis we will consider almost periodic systems of the type (1.1) including infinite time delays.

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