A NEW CRITERION FOR BOREL SUMMABILITY OF FOURIER SERIES

B.N. Sahney and P.D. Kathal

(received March 26, 1968)

1. The application of Borel summability to Fourier series has been discussed by Takahashi and Wang [8] and Sahney [5]. Sahney [6] and Sinvhal [7] obtained sufficient conditions for the Borel summability of the derived Fourier series and its conjugate series, respectively. Kathal [3] obtained different conditions in the case of the conjugate series. In this paper we give a new criterion for Borel summability of Fourier series.

The authors are thankful to Dr. P.L. Sharma for his helpful criticisms and suggestions during the preparation of this paper. The authors are also thankful to the referee for many suggestions which improved the paper to the present form.

2. A sequence $\{S_m\}$ is said to be summable by Borel means [2] or summable (B) if

$$\lim_{p\to\infty} e^{-p} \sum_{m=0}^{\infty} \frac{p^m S_m}{m!}$$

exists.

Let

$$a_o/2 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

https://doi.org/10.4153/CMB-1969-074-x Published online by Cambridge University Press

be the Fourier series of the function $f(x)$ which is integrable in the sense of Lebesque in the interval $(-\pi,\pi)$ and is periodic with period 2π outside this range. Also let

$$\phi(t) = \frac{f(x+t) + f(x-t) - 2s}{2}$$

where s is a constant, and let

$$\Phi(t) = \int_0^t |\phi(u)|du.$$

3. We shall require the following lemmas in the sequel.

LEMMA 1. _If_ $\frac{1}{3} \leq \alpha < 1$ _and_ $\Phi(t) = 0(t)$ as $t \to 0+$ then

$$\lim_{p\to\infty} \int_{(\pi/p)}^{(\pi/p)^\alpha} \frac{\phi(t)}{t} \frac{[\sin(p \sin t) - \sin pt]}{\exp\{p(1 - \cos t)\}} dt = 0.$$

Proof. Considering the last integral, we have

$$\int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t)}{t} \frac{[\sin(p \sin t) - \sin pt]}{\exp\{p(1 - \cos t)\}} dt$$

$$= \int_{\pi/p}^{(\pi/p)^\alpha} \frac{\frac{\phi(t)}{t} [\sin(p \sin t) - \sin pt]}{\exp\{2p \sin^2 t/2\}} dt$$

$$= 0 \left[\frac{1}{\exp\{2p \sin^2 \frac{\pi}{2p}\}}\right] \int_{\pi/p}^{(\pi/p)} \frac{\phi(t)}{t} [\sin(p \sin t) - \sin pt] dt$$

https://doi.org/10.4153/CMB-1969-074-x Published online by Cambridge University Press

$$= \int_{\pi/p}^{(\pi/p)^\alpha} \frac{|\phi(t)|}{t} \, 0(pt^3)dt$$

$$= 0(p^{1-2\alpha}) \int_{\pi/p}^{(\pi/p)^\alpha} |\phi(t)|dt = 0(p^{1-3\alpha}) = 0(1), \quad \text{as} \quad p \to \infty.$$

This completes the proof.

LEMMA 2. *If* $0 \le \alpha < 1$ *and* $\Phi(t) = 0(t)$ as $t \to 0+$ *then*

$$\lim_{p\to\infty} \int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t+\pi/p)}{t} \left[\frac{1}{\exp\{p(1 - \cos t)\}} - \frac{1}{\exp\{p(1 - \cos[t+ \frac{\pi}{p}])\}}\right] \sin pt \, dt$$

$$= 0$$

Proof. Now

$$\int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t+\pi/p)}{t} \left[\frac{1}{\exp\{p(1 - \cos t)\}} - \frac{1}{\exp\{p(1 - \cos(t+ \frac{\pi}{p}))\}}\right] \sin pt \, dt$$

$$= \int_{\pi/p}^{(\pi/p)} \frac{|\phi(t+\pi/p)|}{t} \, 0\left[\frac{t}{\exp(2p \sin^2 \frac{t}{2})}\right] dt$$

$$= 0\left[\frac{1}{\exp(2p \sin^2 \frac{\pi}{2p})}\right] \int_{\pi/p}^{(\pi/p)^\alpha} |\phi(t+\pi/p)| \, dt$$

$$= 0[(\pi/p)^\alpha + \frac{\pi}{p}]$$

$$= 0(1) \quad \text{as} \quad p \to \infty.$$

Thus the proof is complete.

4. THEOREM If $\Phi(t) = O(t)$ as $t \to 0+$ and

(1) $$\lim_{p\to\infty} \int_{\pi/p}^{\eta} \frac{|\phi(t) - \phi(t+\pi/p)|}{t} \exp\{-p(1 - \cos t)\}\, dt = 0$$

where η is constant, then the Fourier series is summable (B) to s at the point x.

Proof. Following Zygmund [1], the mth partial sum of the Fourier series is given by

$$S_m - S = \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \sin mt\, dt + O(1).$$

The Borel transform σ_p, of S_m, is given by

$$\sigma_p - S + O(1) = \frac{2e^{-p}}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \sum_{m=0}^{\infty} \frac{p^m \sin mt}{m!}\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \frac{\sin (p \sin t)}{\exp\{p(1 - \cos t)\}}\, dt$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/p} + \int_{\pi/p}^{(\pi/p)^{\alpha}} + \int_{(\pi/p)^{\alpha}}^{\pi} \right] \frac{\phi(t)}{t} \frac{\sin(p \sin t)}{\exp\{p(1 - \cos t)\}}\, dt$$

$$= \frac{2}{\pi}[\rho_1 + \rho_2 + \rho_3], \text{ say}$$

https://doi.org/10.4153/CMB-1969-074-x Published online by Cambridge University Press

where $\frac{1}{3} \leq \alpha < \frac{1}{2}$.

Considering first of these integrals, we get

$$\rho_1 = \int_0^{\pi/p} \frac{\phi(t)}{t} \frac{\sin(p \sin t)}{\exp\{p(1 - \cos t)\}} dt$$

$$= \int_0^{\pi/p} \frac{|\phi(t)|}{t} 0(pt)dt = 0(1) \quad \text{as} \quad p \to \infty.$$

Next

$$\rho_3 = \int_{(\pi/p)^\alpha}^{\pi} \frac{\phi(t)}{t} \frac{\sin(p \sin t)}{\exp\{p(1 - \cos t)\}} dt$$

$$= \int_{(\pi/p)^\alpha}^{\pi} \frac{\phi(t)}{t} \frac{\sin(p \sin t)}{\exp\{2p \sin^2 \frac{t}{2}\}} dt$$

$$= 0 \left[\frac{p^\alpha}{\exp\{2p \sin^2 \frac{\pi^\alpha}{2p^\alpha}\}}\right] \int_{(\pi/p)^\alpha}^{\pi} |\phi(t)| dt$$

$$= 0 \left[\frac{p^\alpha}{\exp(p^{1-2\alpha})}\right] = 0(1) \quad \text{as} \quad p \to \infty, \quad \text{since} \quad \alpha < \frac{1}{2} .$$

By Lemma 1, we get

https://doi.org/10.4153/CMB-1969-074-x Published online by Cambridge University Press

$$2\rho_2 = 2\int_{(\pi/p)}^{(\pi/p)^\alpha} \frac{\phi(t)}{t}\,\frac{\sin(p\sin t)}{\exp\{p(1-\cos t)\}}\,dt$$

$$= 2\int_{(\pi/p)}^{(\pi/p)^\alpha} \frac{\phi(t)}{t}\,\frac{\sin pt}{\exp\{p(1-\cos t)\}}\,dt + 0(1)$$

$$= \int_{(\pi/p)}^{(\pi/p)^\alpha} \frac{\phi(t)}{t}\,\frac{\sin pt}{\exp(p(1-\cos t))}\,dt$$

$$- \int_0^{(\pi/p)^\alpha-\pi/p} \frac{\phi(t+\pi/p)\sin pt}{(t+\pi/p)\exp\{p(1-\cos(t+\pi/p))\}}\,dt + 0(1)$$

$$= \int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t)-\phi(t+\pi/p)}{t}\,\frac{\sin pt}{\exp\{p(1-\cos t)\}}\,dt$$

$$+ \int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t+\pi/p)}{t}\left[\frac{1}{\exp\{p(1-\cos t)\}} - \frac{1}{\exp\{p(1-\cos(t+\pi/p))\}}\right]$$

$$\times \sin pt\,dt$$

$$+ \int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t+\pi/p)}{\exp\{p(1-\cos(t+\pi/p))\}}\left[\frac{1}{t} - \frac{1}{t+\pi/p}\right]\sin pt\,dt + 0(1)$$

https://doi.org/10.4153/CMB-1969-074-x Published online by Cambridge University Press

Therefore, by Lemma 2 and (1) and on integration by parts, we get

$$\rho_2 = 0(1) + 0(\frac{1}{p}) \int_{\pi/p}^{(\pi/p)^\alpha} \frac{|\phi(t + \pi/p)|}{t(t + \pi/p)} dt$$

$$= 0(1) + 0 \left[\frac{1}{p^{1-\alpha}}\right]$$

$$= 0(1) \quad \text{as} \quad p \to \infty.$$

Hence

$$\sigma_p = 0(1) \quad \text{as} \quad p \to \infty.$$

This completes the proof of the theorem.

The proof by Sahney [4] is valid only in the case $\Delta = 1$.

REFERENCES

1. A. Zygmund, Trigonometric series, Vol. 1. (Cambridge University Press, 1959) 55.

2. G.H. Hardy, Divergent series. (Oxford University Press, 1956).

3. P.D. Kathal, Borel summability of derived Fourier series and its conjugate series. Madhya Bharti Jour. of University of Saugar (to appear).

4. P.D. Kathal and B.N. Sahney, Borel summability of Fourier series. Bolletino Unione Matematica Italiana 3 (1967) 179-182.

5. B.H. Sahney, A note on the Borel summability of Fourier series. Bolletino Unione Matematica Italiana 16 (1961) 44-47.

6. B.N. Sahney, On the Borel summability of the derived Fourier series. Bull. Cal. Math. Soc. 54 (1962) 211-218.

7. S.D. Sinvhal, Borel summability of the conjugate series of a derived series. Duke Math. J. 22 (1955) 445-450.

8. M. Takahashi and F.T. Wang, On the Borel summability of Fourier series. Pro. Phy. Math. Soc. Japan 18 (1936) 153-156.

University of Calgary
Alberta

University of Saugar
Saugar
India