TILTING MODULES AND A THEOREM OF HOSHINO

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(Received 29 July, 1990)

Introduction. Let k be an algebraically closed field, and A a finite dimensional k-algebra, which we shall assume, without loss of generality, to be basic and connected. By module is meant throughout a finitely generated right A-module. Following Happel and Ringel [**10**], we shall say that a module T_A is a tilting (respectively, cotilting) module if it satisfies the following three conditions:

(1) $\mathrm{Ext}^2_A(T, —) = 0$ (respectively, $\mathrm{Ext}^2_A(—, T) = 0$);

(2) $\mathrm{Ext}^1_A(T, T) = 0$;

(3) the number of non-isomorphic indecomposable summands of T equals the rank of the Grothendieck group $K_0(A)$ of A.

These modules have found, since their introduction, numerous applications in the representation theory of algebras (see [**2**, **9**]); hence the interest in constructing them. One method of construction uses torsion theories. Given a tilting (respectively, cotilting) module T_A, the full subcategories

$$\mathcal{T}_*(T) = \{M_A \mid \mathrm{Ext}^1_A(T, M) = 0\}$$

and

$$\mathcal{F}_*(T) = \{M_A \mid \mathrm{Hom}_A(T, M) = 0\}$$

respectively,

$$\mathcal{T}^*(T) = \{M_A \mid \mathrm{Hom}_A(M, T) = 0\}$$

and

$$\mathcal{F}^*(T) = \{M_A \mid \mathrm{Ext}^1_A(M, T) = 0\})$$

of the category mod A of A-modules are respectively the torsion class and the torsion-free class of a torsion theory. Such a torsion theory is said to be a tilting (respectively, cotilting) torsion theory. Conversely, Hoshino has shown in [**12**] that, if $(\mathcal{T}, \mathcal{F})$ is a torsion theory on mod A such that $\mathcal{T}$ contains all injective modules, and $\mathcal{T}$ or $\mathcal{F}$ contains only finitely many non-isomorphic indecomposable modules (we then say, by abuse of language, that $\mathcal{T}$ or $\mathcal{F}$ is finite), then $(\mathcal{T}, \mathcal{F})$ is a tilting torsion theory (see also [**1**], [**18**]). While the first condition is obviously necessary, the second is not as is shown by the following example. Let A be a wild hereditary algebra with at least three non-isomorphic simple modules; it was proved by Ringel ([**16**], see also [**6**]) that A has a regular tilting (and cotilting) module. For such a module, all the indecomposable preprojective modules belong to the torsion-free class, while all the indecomposable preinjective modules belong to the torsion class, so that both are infinite.

In this paper, we consider classes of algebras having the property that, for every tilting or cotilting torsion theory $(\mathcal{T}, \mathcal{F})$, $\mathcal{T}$ or $\mathcal{F}$ is finite. This is quite useful in practice, since in many applications, it is easier to start by constructing the torsion theory, then finding the corresponding module (this method was heavily applied, for instance, in [**9**, (IV, 7)]). Since all representation-finite algebras obviously satisfy this property, we shall be mainly interested in representation-infinite algebras. It was shown in [**11**, (3)] that the tame hereditary algebras satisfy the stated property. We shall prove the following generalisation of this result.

THEOREM A. *Let A be an iterated tilted algebra of euclidean type, and T_A a tilting (respectively, cotilting) module. Then $\mathcal{T}_*(T)$ or $\mathcal{F}_*(T)$ (respectively, $\mathcal{T}^*(T)$ or $\mathcal{F}^*(T)$) is finite.*

On the other hand, this property is not satisfied by the tubular algebras of [15] (because of the existence of regular tilting modules). It follows directly from the theorem that the stated property is satisfied by the tilted algebras of euclidean type. It is well-known that such an algebra is either representation-finite or one-parametric (in the sense of [17, (2.1)]). Also, this property is easily seen to be satisfied by a hereditary algebra with two non-isomorphic simple modules. Our second theorem shows that, up to finite enlargements, these are the only classes of tilted algebras which satisfy this property.

THEOREM B. *Let A be a tilted algebra. The following conditions are equivalent*:

(1) *for every tilting module T_A, $\mathcal{T}_*(T)$ or $\mathcal{F}_*(T)$ is finite, and for every cotilting module T'_A, $\mathcal{T}^*(T')$ or $\mathcal{F}^*(T')$ is finite*;

(2) *if A is tame, then A is representation-finite or one-parametric and, if A is wild, then one of the end algebras of A is zero, and the other is hereditary with two non-isomorphic simple modules.*

Clearly, if T_A is a regular tilting module, the equivalent conditions of the theorem are not satisfied, but the converse is not true, as will be shown in the course of the proof. The paper is organised as follows. In Section 1, we quote the definitions and results that will be used in the sequel, prove Theorem A, then consider the case of the tubular algebras. Section 2 is devoted to the proof of Theorem B.

1. Iterated tilted algebras of euclidean type.

1.1. Let A be a finite dimensional algebra. It is well-known that for any (basic) algebra A, there exists a bound quiver (Q, I) such that we have an isomorphism $A \xrightarrow{\sim} kQ/I$. A bound quiver algebra $A = kQ/I$ can equivalently be considered as a locally bounded k-linear category with object class the set of points in Q, and set of morphisms from x to y the vector space $kQ(x, y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x, y) = I \cap kQ(x, y)$, see [8]. A full subcategory C of A is called convex if any path in A with source and target in C lies entirely in C. We shall use freely and without further reference properties of the category $\operatorname{mod} A$, the Auslander–Reiten translations $\tau = DTr$ and $\tau^{-1} = TrD$, and the Auslander–Reiten quiver Γ_A of A, as can be found, for instance in [5], [15]. Recall that a component of Γ_A is called preprojective (respectively, preinjective) if it contains no oriented cycle, and each indecomposable module in this component is of the form $\tau^{-t}P$, with $t \geq 0$ and P projective (respectively, $\tau^s I$, with $s \geq 0$ and I injective). We shall denote by $P(a)$ (respectively, $I(a)$) the indecomposable projective (respectively, injective) module corresponding to a point a in the ordinary quiver of A. The support of a module M_A is the full subcategory of A consisting of all objects a such that $\operatorname{Hom}_A(P(a), M) \neq 0$.

For tilting theory and iterated tilted algebras, we refer to [2], [10]. By the Brenner–Butler theorem, if A is an algebra, T_A a tilting module and $B = \operatorname{End} T_A$, then the functors $\operatorname{Hom}_A(T, —)$ and $— \otimes_B T$ induce mutually inverse equivalences between the full subcategories $\mathcal{T}_*(T)$ of $\operatorname{mod} A$ and $\mathcal{Y}_*(T) = \{N_B \mid \operatorname{Tor}_1^B(N, T) = 0\}$ of $\operatorname{mod} B$, while the functors $\operatorname{Ext}_A^1(T, —)$ and $\operatorname{Tor}_1^B(—, T)$ induce mutually inverse equivalences between the full subcategories $\mathcal{F}_*(T)$ of $\operatorname{mod} A$ and $\mathcal{X}_*(T) = \{N_B \mid N \otimes_B T = 0\}$ of $\operatorname{mod} B$. Further, $(\mathcal{X}_*(T), \mathcal{Y}_*(T))$ is a torsion theory on $\operatorname{mod} B$, which is splitting if A is hereditary. Similar results hold for cotilting modules.

If Δ is a finite connected quiver without oriented cycles, an algebra A is called iterated tilted of type Δ if there exist a sequence of algebras $A = A_0, A_1, \ldots, A_m = k\Delta$ and a sequence $T_{A_i}^{(i)}$ $(0 \leq i < m)$ of tilting or cotilting modules such that $A_{i+1} = \text{End } T_{A_i}^{(i)}$ for each i (for equivalent definitions, see [**2**], [**9**]). If Δ is a Dynkin (respectively, euclidean, wild) quiver, then A is said to be of Dynkin (respectively, euclidean, wild) type. If $m \leq 1$, then A is called tilted of type Δ. Tilted algebras are characterised by the existence of complete slices [**15**, (4.2)(3)]. The structure of the representation-infinite iterated tilted algebras of euclidean type is described in [**3**, (2)]. In particular, each such algebra contains a unique full convex subcategory which is tame concealed (that is, which is the endomorphism algebra of a preprojective tilting module over a tame hereditary algebra). Further, the Auslander–Reiten quiver of such an algebra always has a unique preprojective and a unique preinjective component. All other components are called regular.

1.2. LEMMA. *Let A be a representation-infinite iterated tilted algebra of euclidean type, and T_A a tilting module without preprojective (respectively, preinjective) direct summands. Then all the homogeneous tubes in Γ_A belong to $\mathcal{F}_*(T)$ (respectively, $\mathcal{T}_*(T)$).*

Proof. If T_A has no preprojective summands, its summands are regular or preinjective, but certainly not homogeneous (because indecomposable homogeneous modules have self-extensions). Thus it follows from the orthogonality of the tubes that, for every indecomposable homogeneous module M, we have $\text{Hom}_A(T, M) = 0$. The dual statement follows from the fact that, if all summands of T are regular or proprojective, then, by the Auslander–Reiten formula, $\text{Ext}_A^1(T, M) = D\,\text{Hom}_A(M, \tau T) = 0$ for every indecomposable homogeneous module M_A.

1.3. COROLLARY. *Let A be a representation-infinite iterated tilted algebra of euclidean type. Then A has no regular tilting module and no regular cotilting module.*

Proof. By (1.2), A has no regular tilting module. That it has no regular cotilting module follows from the fact that by [**4**, (2.5)] or [**9**, (IV, 5.6)], its opposite algebra A^{op} is also a representation-infinite iterated tilted algebra of euclidean type.

1.4. *Proof of Theorem A.* We may certainly assume that A is representation-infinite. By duality, it suffices to prove the statement in case T_A is a tilting module.

We first claim that, if $\mathcal{F}_*(T)$ is infinite, then infinitely many simple homogeneous A-modules belong to $\mathcal{F}_*(T)$.

This claim will be shown by induction on the rank n of $K_0(A)$. If $n = 2$, then A is the Kronecker algebra (that is, is tame hereditary of type $\tilde{A}_{1,1}$) and the statement is trivial. Assume that the rank of $K_0(A)$ equals n. If T_A has no preprojective direct summand, the statement follows from (1.2). Otherwise, observe that the preprojective direct summands of T are partially ordered by the successor relation. We claim that T may be assumed to have as direct summand a simple projective module.

Indeed, if this is not the case, let $P(a)$ be a simple projective module, and

$$T[a] = \tau^{-1}P(a) \oplus \left(\bigoplus_{b \neq a} P(b)\right)$$

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the corresponding APR-tilting module [**2**, (1.6)]). Clearly, all direct summands of T belong to $\mathcal{T}_*(T[a])$, and hence $\mathrm{Hom}_A(T[a], T)$ is a tilting B-module, where $B = \mathrm{End}\, T[a]$. On the other hand, the functor

$$- \otimes_B T[a] : \mathrm{mod}\, B \to \mathrm{mod}\, A$$

induces an equivalence $\mathcal{Y}_*(T[a]) \xrightarrow{\sim} \mathcal{T}_*(T[a])$ and its restriction to a functor $\mathcal{F}_*(\mathrm{Hom}_A(T[a], T)) \to \mathcal{F}_*(T)$ defines a full exact embedding of the category $\mathcal{F}_*(\mathrm{Hom}_A(T[a], T))$ as a cofinite subcategory of $\mathcal{F}_*(T)$, which is closed under extensions. Furthermore, the functor

$$\mathrm{Hom}_A(T[a], —) : \mathrm{mod}\, A \to \mathrm{mod}\, B$$

induces an equivalence between the full subcategories consisting of the homogeneous modules in $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$, respectively. Inductively, applying a sequence of APR-tilts corresponding to simple projectives in the preprojective component, we reach a situation where T has as direct summand a simple projective module. As we have seen, in each APR-tilting step, the torsion-free part induced by the image of T differs by at most one indecomposable from the original one.

Assume thus that T_A has as direct summand the simple projective eA, and let $A' = A/AeA$. Then A' is a full convex subcategory of A and consequently, by [**4**, (5.2)], each of its connected components is iterated tilted of Dynkin or euclidean type. Also, by [**14**, (3.2)], all the summands of T except eA define a tilting A'-module $T' = T/TeA$ such that the canonical embedding $\mathrm{mod}\, A' \hookrightarrow \mathrm{mod}\, A$ induces an identification $\mathcal{F}_*(T') = \mathcal{F}_*(T)$. In particular, $\mathcal{F}_*(T')$ is infinite and consequently at least one of the connected components of A' is a representation-infinite iterated tilted algebra of euclidean type. We may thus assume that A' is connected.

By the induction hypothesis, infinitely many simple homogeneous A'-modules belong to $\mathcal{F}_*(T') = \mathcal{F}_*(T)$. Since $\mathrm{mod}\, A$ contains only finitely many non-isomorphic self-extending bricks which are not simple homogeneous, we infer that infinitely many simple homogeneous A-modules belong to $\mathcal{F}_*(T)$, thus establishing our claim.

A dual argument, using [**4**, (5.2)] and [**14**, (3.1)], shows that, if $\mathcal{T}_*(T)$ is infinite, then infinitely many simple homogeneous A-modules belong to $\mathcal{T}_*(T)$.

Assume now that T_A is a tilting module such that both $\mathcal{T}_*(T)$ and $\mathcal{F}_*(T)$ are infinite. The above reasoning implies that infinitely many non-isomorphic simple homogeneous A-modules belong to each of these subcategories. Let $M \in \mathcal{F}_*(T)$ and $N \in \mathcal{T}_*(T)$ be two simple homogeneous A-modules. It follows from [**4**, (2.5)] that the supports of M and N are equal, and actually equal to the unique tame concealed full convex subcategory of A. Consequently, $\mathbf{dim}\, M = \mathbf{dim}\, N$. On the other hand, if $\langle -, - \rangle$ denotes the (homological) bilinear form of A (see [**10**, (1.5)] or [**2**, (2.5)]), then

$$\langle \mathbf{dim}\, T, \mathbf{dim}\, M \rangle = -\dim_k \mathrm{Ext}^1_A(T, M) < 0$$

because $\mathrm{Ext}^2_A(T, -) = 0$, $\mathrm{Hom}_A(T, M) = 0$ and $\mathrm{Ext}^1_A(T, M) \neq 0$. Similarly

$$\langle \mathbf{dim}\, T, \mathbf{dim}\, N \rangle > 0,$$

a contradiction which completes the proof of the theorem.

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1.5. COROLLARY. *Let A be a tilted algebra of euclidean type, and T_A a tilting (respectively, cotilting) module. Then $\mathcal{T}_*(T)$ or $\mathcal{F}_*(T)$ (respectively, $\mathcal{T}^*(T)$ or $\mathcal{F}^*(T)$) is finite.*

1.6. We end this section by considering another important class of tame algebras, namely that of the tubular algebras [**15**, (5)]. Such an algebra also has a unique preprojective and a unique preinjective component, all other components being called regular. The above results are not true for tubular algebras, as they have regular tilting modules, and regular cotilting modules, which induce infinite torsion and torsion-free classes.

LEMMA. *Let A be a tubular algebra. Then A has a regular tilting module and a regular cotilting module.*

Proof. It follows from the definition of tubular algebras that A contains a full convex subcategory B which is a tilted algebra of euclidean type having a complete slice $\mathcal{S}$ in its preinjective component such that A is a branch extension of B by a branch rooted at a regular indecomposable B-module. Let P_A be the direct sum of the indecomposable projective A-modulus which belong to the regular component, and U_A denote the slice module of $\mathcal{S}$, considered as an A-module. Then modulo some τ-shift $\mathcal{S}$, U_A may be assumed to be a regular A-module (which, in the notation of [**15**, (5)], lies in $\mathcal{P}_\infty$). The module

$$T_A = P_A \oplus \tau_A^{-1} U$$

(where τ_A denotes the Auslander–Reiten translation in $\operatorname{mod} A$) is clearly a regular tilting A-module. Observe that, by [**15**, (5.2) (5)], $\operatorname{End} T$ is again a tubular algebra. On the other hand, both $\mathcal{T}_*(T)$ and $\mathcal{F}_*(T)$ are infinite, as they respectively contain all the preinjective and all the preprojective A-modules.

The statement about cotilting modules follows from the fact that A^{op} is cotubular and hence, by [**15**, (5.2)(3)], also tubular.

2. Tilted algebras.

2.1. LEMMA. *Let A be a representation-infinite tilted algebra of euclidean type.*

(i) *Assume that A has a complete slice $\mathcal{S}$ in its preinjective component, and let M_A be a predecessor of $\mathcal{S}$ such that $\operatorname{Ext}_A^1(M, M) = 0$. Then the class $\mathcal{F}_*(M) = \{X_A \mid \operatorname{Hom}_A(M, X) = 0\}$ or the class $\operatorname{Gen}(M_A)$ of all A-modules generated by M is finite.*

(ii) *Assume that A has a complete slice $\mathcal{S}$ in its preprojective component, and let M_A be a successor of $\mathcal{S}$ such that $\operatorname{Ext}_A^1(M, M) = 0$. Then the class $\mathcal{T}_*(M) = \{X_A \mid \operatorname{Ext}_A^1(M, X) = 0\}$ or the class $\operatorname{Cogen}(\tau M)$ of all A-modules cogenerated by τM is finite.*

Proof. We shall first prove (i). It follows from the hypothesis that there exists a tame hereditary algebra B, and a tilting module T_B without preinjective direct summands such that $A = \operatorname{End} T_B$ and $\mathcal{Y}_*(T)$ consists of the predecessors of $\mathcal{S}$.

Assume first that M has an indecomposable preprojective direct summand M_0. We claim that in this case $\mathcal{F}_*(M)$ is finite. Indeed, since $\mathcal{F}_*(M) \subseteq \mathcal{F}_*(M_0)$ and $\mathcal{X}_*(T)$ is finite, the finiteness of $\mathcal{F}_*(M)$ would follow from the finiteness of the class

$$\mathcal{F}_*(M_0) \cap \mathcal{Y}_*(T) = \{X_A \in \mathcal{Y}_*(T) \mid \operatorname{Hom}_A(M_0, X) = 0\}.$$

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Now, the image of any $X \in \mathscr{F}_*(M_0) \cap \mathscr{Y}_*(T)$ under the equivalence $- \otimes_A T : \mathscr{Y}_*(T) \simeq \mathscr{T}_*(T)$ belongs to the class

$$\mathscr{F}_*(M_0 \otimes T) = \{Y_B \mid \mathrm{Hom}_B(M_0 \otimes_A T, Y) = 0\}$$

that is, $[\mathscr{F}_*(M_0) \cap \mathscr{Y}_*(T)] \otimes_A T \subseteq \mathscr{F}_*(M_0 \otimes_A T)$. However the latter class is finite, since $M_0 \otimes_A T$ is an indecomposable preprojective B-module and B is tame hereditary. This completes the proof of our claim.

Next, assume that M has no preprojective direct summand. We shall show that in this case $\mathrm{Gen}(M_A)$ is finite. Again, the finiteness of $\mathrm{Gen}(M_A)$ would follow from the finiteness of the class

$$\mathrm{Gen}(M) \cap \mathscr{Y}_*(T) = \{X_A \in \mathscr{Y}_*(T) \mid X \text{ is generated by } M\}.$$

Now, the image of any $X \in \mathrm{Gen}(M) \cap \mathscr{Y}_*(T)$ under the right exact functor $- \otimes_A T$ clearly belongs to the class

$$\mathrm{Gen}(M \otimes_A T) = \{Y_B \mid Y \text{ is generated by } M \otimes_A T\}$$

that is, $[\mathrm{Gen}(M) \cup \mathscr{Y}_*(T)] \otimes_A T \subseteq \mathrm{Gen}(M \otimes_A T)$. Applying the construction dual to that of Bongartz in [**7**, (2.1)], let

$$0 \to (M \otimes_A T)^d \to E_B \to (DM)_B \to 0$$

be the universal exact sequence, where $d = \dim_k \mathrm{Ext}^1_B(DB, M \otimes_A T)$. It follows from [**7**, (2.1)] that $U_B = (M \otimes_A T) \oplus E$ is a tilting module. Since, as observed in [**15**, (4.1)(1)], the indecomposable summands of E are injective or successors of $M \otimes_A T$, then U has no preprojective direct summand. By [**11**, (3.2)], $\mathscr{T}_*(U) = \mathrm{Gen}(U)$ is finite. Consequently so is $\mathrm{Gen}(M) \cap \mathscr{Y}_*(T)$. This completes the proof of (i).

The proof of (ii) is dual: if M has a preinjective direct summand, we prove as above that $\mathscr{T}_*(M)$ is finite, while if M has no preinjective direct summand, the finiteness of $\mathrm{Cogen}(\tau M)$ is proved using the fact that, if T_B is a tilting module over a hereditary algebra B such that $\mathrm{End}\, T_B = A$, then the functor $\mathrm{Tor}^A_1(-, T)$ is left exact, because $\mathrm{Ext}^2_A(-, T) = 0$.

Remark. Under the stated hypothesis, M is a partial tilting module. The above lemma may then be used to give an alternative proof of (1.5).

2.2. Recall that an algebra A is called a finite enlargement of a full convex subcategory B if all but at most finitely many non-isomorphic indecomposable A-modules have their support entirely contained in B.

Corollary. *Let A be a finite enlargement in the preinjective (respectively, preprojective) component of a tilted algebra B of euclidean type having a complete slice in that component. Then, for every tilting module T_A, $\mathscr{T}_*(T)$ or $\mathscr{F}_*(T)$ is finite.*

Proof. We shall assume that A is a finite enlargement in the preinjective component of B (the other case is treated similarly). Let $\mathscr{S}$ be a complete slice in the preinjective component such that all predecessors of $\mathscr{S}$ are B-modules. If T is a successor of $\mathscr{S}$, clearly $\mathscr{T}_*(T)$ is finite. Otherwise, let T_0 denote the direct sum of all indecomposable summands of T which precede $\mathscr{S}$. By (2.1), $\mathscr{F}_*(T_0) = \{X_B \mid \mathrm{Hom}_B(T_0, X) = 0\}$ or $\mathrm{Gen}(T_0) = \{X_B \mid X \text{ is generated by } T_0\}$ is finite. Since A is a finite enlargement of B, also $\mathscr{T}_*(T)$ or $\mathscr{F}_*(T)$ is finite.

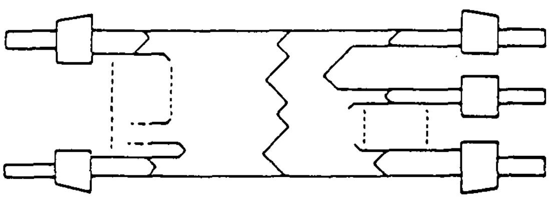
2.3. For tilted algebras of wild type, we shall use the notations and results of **[14]**. In particular, let Δ be a wild quiver, $B = k\Delta$ and $T = \bigoplus_{i=1}^{n} T_i$ a tilting B-module such that $A = \operatorname{End} T_B$ (where T_i is indecomposable for all i). We let I (respectively, J) denote the set of all i ($1 \le i \le n$) such that $\operatorname{Ext}_A^1(T_i, M) \neq 0$ (respectively, $\operatorname{Hom}_A(T_i, M) \neq 0$) for only finitely many indecomposables $M \in \mathcal{F}_*(T)$ (respectively, $M \in \mathcal{T}_*(T)$). The algebras $A_\infty = \operatorname{End}(\bigoplus_{i \in I} T_i)$ and ${}_\infty A = \operatorname{End}(\bigoplus_{i \in J} T_i)$ are respectively called the right end algebra and the left end algebra of A. Then

(i) A is an iterated one-point extension of ${}_\infty A$, and

(ii) there is a hereditary quotient algebra ${}_\infty B$ of B and a tilting ${}_\infty B$-module ${}_\infty T'$ without preinjective direct summands such that $\operatorname{End}({}_\infty T') = {}_\infty A$. Moreover, ${}_\infty B = B_1 \times \ldots \times B_t$, with B_i connected for all i, ${}_\infty T' = T'_1 \oplus \ldots \oplus T'_t$, where T'_i is a tilting B_i-module without preinjective direct summands and ${}_\infty A = A_1 \times \ldots \times A_t$ for $A_i = \operatorname{End} T'_i$.

The dual statements holds for A_∞.

The Auslander–Reiten quiver Γ_A has the following shape.



Again, it has preprojective and preinjective components, all the other being called regular. A component containing a complete slice is called a connecting component.

2.4. *Proof of Theorem B.* We shall first show that (i) implies (ii). The statement is trivial if A is representation-finite, and follows from (1.5) if A is of euclidean type. We may thus assume that A is a representation-infinite tilted algebra of wild type.

First, one of the end algebras ${}_\infty A$ or A_∞ is zero. Indeed, if this is not the case, the slice module T_A of a complete slice in the connecting component is a tilting (and also a cotilting) module with both $\mathcal{T}_*(T)$ and $\mathcal{F}_*(T)$ (and also $\mathcal{T}^*(T)$ and $\mathcal{F}^*(T)$) infinite.

Without loss of generality, we may supppose that A_∞ is zero. We claim that ${}_\infty A$ is connected. Indeed, if this is not the case, then we can write ${}_\infty A = A_1 \times A_2$ with A_1 connected, and both A_1 and A_2 representation-infinite. By (2.3)(ii), A_1 is a tilted algebra having a complete slice $\mathcal{S}$ in the preinjective component, and $\mathcal{S}$ may be chosen so that any predecessor of $\mathcal{S}$ in $\operatorname{mod} A$ is an A_1-module. Let T_1 be the slice module of $\mathcal{S}$, considered as an A-module, let e_1 denote the identity of A_1, and 1 the identity of A; then $T = T_1 \oplus (1 - e_1)A$ is a tilting A-module. Since all indecomposable regular A_1-modules belong to $\mathcal{F}_*(T)$, then $\mathcal{F}_*(T)$ is infinite, and since all A_2-modules belong to $\mathcal{T}_*(T)$, it is infinite as well. This shows that ${}_\infty A$ is connected.

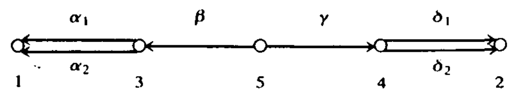
We shall now show that ${}_\infty A$ is tame or has only two non-isomorphic simple modules (it is then necessarily hereditary). In order to prove it, we shall assume that this is not the case and construct a tilting module T_A such that both $\mathcal{T}_*(T)$ and $\mathcal{F}_*(T)$ are infinite. Now, there exists a wild hereditary algebra B and a tilting module U_B such that ${}_\infty A = \operatorname{End} U_B$. Then U_B has no preinjective direct summands, and $\mathcal{Y}_*(U)$ is a cofinite full subcategory of the category $\operatorname{mod} A$, closed under predecessors. By **[14**, (2.1)**]**, each regular component $\mathcal{C}$

of Γ_B contains a complete cone $\Gamma_{\mathscr{C}}$, closed under predecessors and entirely contained in $\mathscr{T}_*(U)$. By [**16**, **6**], B has a regular tilting module V. We clearly may assume that all the summands of V belong to the cones $\Gamma_{\mathscr{C}}$. Consequently, $T_0 = \mathrm{Hom}_B(U, V)$ is a regular partial tilting A-module. Let P_A denote the direct sum of all non-isomorphic indecomposable projectives which belong to the preinjective component of Γ_A. Then the module $T_A = T_0 \oplus P$ satisfies our assertion.

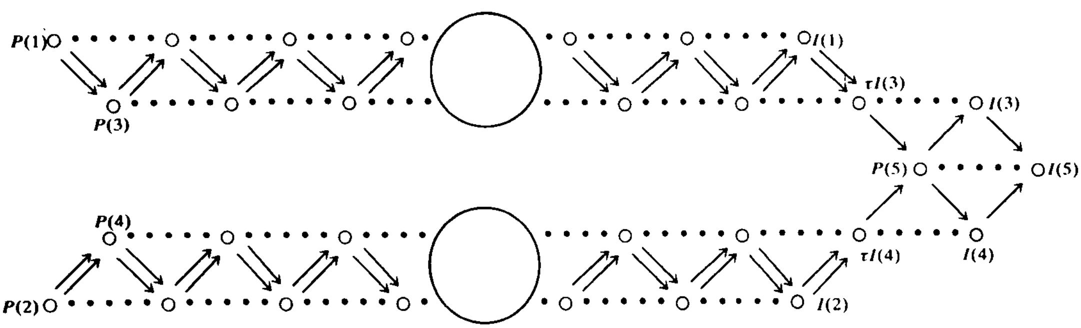
If ${}_\infty A$ is zero, the proof is entirely analogous using cotilting modules.

We shall finally prove that, conversely, (ii) implies (i). If A is tame and one-parametric then, by (2.3) or [**14**, (4.2b)], A is a finite enlargement of a tilted algebra of euclidean type and the statement follows from (2.2). If A is wild, one of its end algebras is zero and the other is hereditary with two simple modules. By [**13**, (2.6)], all regular indecomposable modules have self-extensions and consequently all the summands of a tilting (or cotilting module are preprojective or preinjective. Assume that A_∞ is zero. If a tilting module T_A has a preprojective summand T_0, then $\mathscr{F}_*(T) \subseteq \mathscr{F}_*(T_0) = \{M_A \mid \mathrm{Hom}_A(T_0, M) = 0\}$, which consists of predecessors of T_0 (and T_0 has only finitely many non-isomorphic predecessors) and possibly modules whose support is not contained in ${}_\infty A$ (and A is a finite enlargement of ${}_\infty A$). If T_A has no preprojective summand, it is preinjective, $\mathscr{T}_*(T) = \mathrm{Gen}(T)$ consists then of successors of T, and T has only finitely many non-isomorphic successors. The proofs of the cases where ${}_\infty A$ is zero, or where T is a cotilting module are dual.

2.5. EXAMPLE. Let A be the algebra given by the quiver



bound by $\beta\alpha_1 = 0$, $\beta\alpha_2 = 0$, $\gamma\delta_1 = 0$, $\gamma\delta_2 = 0$. Then A is a tilted algebra having a complete slice in its preinjective component and ${}_\infty A$ is the direct product of two copies of the Kronecker algebra. The Auslander–Reiten quiver Γ_A is given by



The indecomposable A-modules with injective dimension at most one are precisely $I(1)$, $I(2)$, $\tau I(3)$, $\tau I(4)$, $I(3)$, $I(4)$, $I(5)$ and $P(5)$. Any cotilting module T_A must then have these as summands and hence $\mathscr{T}^*(T)$ is always finite. On the other hand,

$$T'_A = P(2) \oplus P(4) \oplus P(5) \oplus I(1) \oplus \tau I(3)$$

is a tilting module with both $\mathscr{T}_*(T')$ and $\mathscr{F}_*(T')$ infinite.

ACKNOWLEDGEMENTS. This paper was written while the second author was visiting the first at the Université de Sherbrooke. Both gratefully acknowledge support from the Natural Sciences and Engineering Research Council of Canada.

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https://doi.org/10.1017/S0017089500009575 Published online by Cambridge University Press