

FINITE ELEMENT ERROR ESTIMATES FOR SUBSONIC FLOW

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Abstract

Error estimates are derived for a finite element analysis of plane steady subsonic flows described by the full potential equation. The analysis is based on the use of the theory of variational inequalities to accommodate the subsonic flow constraint and leads to a suboptimal estimate relative to that obtained for linear potential flow. We then consider an alternative dual formulation of the problem and obtain an optimal estimate subject to reasonable regularity assumptions.

Introduction

Plane steady compressible flow problems are of practical interest in analysis of nozzle flows, cascade flows in turbomachines and, particularly, exterior flow past airfoils. Let us consider a profile such as an airfoil with uniform incident flow in the far field. If the incident velocity is sufficiently moderate, the flow is everywhere subsonic (subcritical) and the governing equation is elliptic. Above a critical incident velocity, the flow becomes transonic with a local imbedded supersonic “pocket” adjacent to the airfoil and is of mixed type—elliptic in the subsonic flow and hyperbolic in the local supersonic flow region. At sufficiently high incident velocities, the supersonic region may be terminated by an imbedded shock discontinuity. For the subsonic flow case, and even for slightly supercritical

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flows including weak shocks, the flow may be assumed to remain irrotational and governed by the full potential equation [17, 19].

The mathematical theory for subsonic flows is relatively well developed with existence and uniqueness results established for sufficiently regular data using both classical analysis based on quasiconformal mapping and also functional analytic techniques [1, 10, 11, 16]. Analytic solutions for the mixed elliptic-hyperbolic problem have been constructed via hodograph transformations, but the analysis of these flows in general is more difficult [2, 13] and the standard-function analytic techniques associated with elliptic problems clearly are not applicable.

Several finite element methods have been developed to compute approximate solutions to the full potential equation [3, 4, 5, 14, 18]. A viable finite element theory should include error estimates for the approximations and conditions on the regularity of the solution or data for convergence. While these estimates are easy to obtain for the standard linear potential flow problem, the nonlinear problem is more difficult. Very restrictive estimates have been obtained for the special case of a fictitious (non-physical) gas using techniques based on those for analysis of the minimal surface equation and applied for shock-free redesign [6, 7]. A viable error analysis for the subsonic flow of a real gas has not previously been developed. A major complication for the error analysis of subsonic flow of a real gas stems from the mathematical restriction that, for ellipticity, the flow should be everywhere less than the speed of sound in the gas. The constraint is violated for supercritical flow, but the established error analysis techniques are then no longer applicable. This factor was overlooked in an earlier attempt [15] to provide error estimates for this problem.

In the present study, we derive error estimates for the full potential equation using the theory of variational inequalities to include the mathematical restriction that admissible velocities be subsonic. In Section 1, we define the subsonic compressible flow problem, constraint subset and variational inequality formulation for the potential solution. This is followed in Section 2.1 by the finite element approximation to the variational inequality and error analysis. Since the approximation subset is not contained in the constraint subset, the analysis leads to a suboptimal estimate. This difficulty has led us to explore an alternative dual formulation involving velocity components in Section 2.2. The central idea is that the subsonic constraint is easier to accommodate directly in terms of the velocity components, rather than as a condition on the gradient of the potential. We can thereby avoid the difficulty in the previous formulation which led to a suboptimal estimate. However, the far-field boundary condition now enters in a more complex manner. We consider two formulations in which the far-field condition is embedded in the space and using a Lagrange multiplier, respectively. Using the latter formulation, an optimal error estimate is obtained for the dual formulation.

1. Analysis

1.1 Formulation

Consider the steady flow of a compressible gas having density ρ and velocity \mathbf{q} . Conservation of mass implies

$$\nabla \cdot (\rho \mathbf{q}) = 0 \quad \text{in } \Omega, \tag{1}$$

where $\rho = \rho(q^2)$ with $q = |\mathbf{q}|$.

The adiabatic equation of state for pressure $p = k\rho^\gamma$ and the equation for the speed of sound, $dp/d\rho = a^2$, yield

$$\rho(q^2) = \rho_0 \left(1 - \frac{\gamma - 1}{2a_0^2} q^2 \right)^{1/(\gamma - 1)}, \tag{2}$$

where ρ_0 is the stagnation density, a_0 is the stagnation speed of sound, and γ is the ratio of specific heats for the gas. Typically, γ lies in the interval (1, 2) for a gas.

For irrotational flow $\mathbf{q} = \nabla u$ and, using this with (1) and (2), we have the full potential equation

$$\nabla \cdot \left(\rho (|\nabla u|^2) \nabla u \right) = 0 \quad \text{in } \Omega. \tag{3}$$

The classical form of the subsonic flow problem is then to solve (3) subject to prescribed boundary data on the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ of the flow domain Ω (Γ_0 and Γ_1 are disjoint). In particular, one frequently deals with data of the form

$$u = g_0 \quad \text{on } \Gamma_0, \tag{4}$$

$$\rho \frac{\partial u}{\partial n} = g_1 \quad \text{on } \Gamma_1, \tag{5}$$

representing a prescribed asymptotic far-field boundary condition on a remote boundary Γ_0 in the uniform flow region and a mass flux boundary condition on a profile such as an airfoil. We remark that in practice $g_1 = 0$ and we shall restrict our analysis to this case. In addition, for lifting airfoils, the circulation must be specified by means of the Kutta condition (using, for instance, a branch cut with circulation determined as a potential jump). We shall restrict our attention to flows without circulation. The flow domain Ω is a simply-connected bounded domain in \mathbb{R}^N or a multiply-connected domain for which the flow problem can be posed on a simply-connected subdomain by appealing to physical and geometric symmetry, as in uniform flow past a symmetric airfoil at zero angle of incidence to the flow. The above problem statement is complete for subsonic flows, i.e., $q < a_*$, where a_* is the critical velocity and is equal to $\sqrt{2/(\gamma + 1)} a_0$. For transonic flows, (3) is no longer elliptic, and one must include a further entropy condition to prevent non-physical rarefaction discontinuities.

Rather than working with the classical statement of the subsonic full potential problem, one can instead consider a functional minimization problem on a closed convex subset K_δ specifying the subsonic flow constraint. We first introduce the following function spaces and notations. Let

$$H_{\Gamma_i} = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_i\}, \quad i = 0, 1$$

be equipped with the H_0^1 norm $\|\cdot\|$. Then provided that $\text{meas}(\Gamma_i) \neq 0$, H_{Γ_i} is a Hilbert space for $i = 0, 1$. Let H'_{Γ_i} denote the dual space of H_{Γ_i} and let $\langle \cdot, \cdot \rangle$ denote the duality pairing on $H'_{\Gamma_0} \times H_{\Gamma_0}$. The norm on $W^{m,r}(\Omega)$ is denoted by $\|\cdot\|_{m,r}$ and the norm on $L^r(\Omega)$ by $\|\cdot\|_{L^r}$. To consider the variational formulation of (3)–(5), we first note that (3) is the Euler-Lagrange equation of the variational functional

$$J(u) = \int_{\Omega} \int_0^{|\nabla u|^2} \frac{1}{2} \rho(t) dt dx \quad (6)$$

We see from (2) that the functional J is well defined for functions u in $H^1(\Omega)$ that satisfy the constraint $|\nabla u| < a_*$ a.e. in Ω . Thus, we consider the variational minimization problem:

Find $u \in K_\delta$ such that

$$J(u) = \min J(v); \quad v \in K_\delta, \quad (7)$$

where

$$K_\delta = \{v \in H^1(\Omega); v = g_0 \text{ on } \Gamma_0 \text{ and } |\nabla v| \leq \delta < a_* \text{ a.e. in } \Omega\}. \quad (8)$$

The introduction of the constant δ enables us to obtain positive, finite bounds for the density and its derivatives and to write the subsonic constraint as a non-strict inequality to facilitate the analysis. We assume δ is close to a_* . It is clear that K_δ is a bounded, closed and convex subset of $g_0 + H_{\Gamma_0}$. In fact, K_δ is compact in $H^1(\Omega)$. We shall show later that J is continuous and strictly convex on K_δ . These results enable us to deduce the unique solvability of (7).

Taking the first variation of J in (7), the variational statement for subsonic flow can be expressed as the equivalent variational inequality problem:

Find $u \in K_\delta$ such that

$$\langle J'u, v - u \rangle \geq 0 \quad \text{for all } v \text{ in } K_\delta, \quad (9)$$

where J' is the Gâteaux derivative of J . Thus, J' corresponds to the full potential operator in the classical statement of problem (3) and is a non-linear operator defined according to

$$\langle J'u, v \rangle = \int_{\Omega} \rho \nabla u \cdot \nabla v dx \quad (10)$$

for all u in K_δ and v in H_{Γ_0} .

The properties of J' depend on the form of the nonlinearity—that is, on the properties of the density $\rho = \rho(q^2)$ in (2). We have:

(i) for $0 \leq q \leq \delta < a_*$, ρ is positive bounded below and above. In fact,

$$\rho_0 \geq \rho(q^2) \geq \rho(\delta^2) > \rho_0 \left(1 - \frac{\gamma - 1}{\gamma + 1}\right)^{1/(\gamma-1)} = \rho_0 \left(\frac{2}{\gamma + 1}\right)^{1/\gamma} \equiv \rho_{\min} > 0,$$

since $\gamma > 1$ for a real gas. Moreover, as $\rho' = -\rho_0^{\gamma-1} \rho^{2-\gamma} / 2a_0^2$, $|\rho'|$ is bounded above by $\rho_0 / 2a_0^2$.

(ii) Since

$$\begin{aligned} (\rho(q^2)q)' &= \rho(q^2) + 2\rho'(q^2)q^2 \\ &= \rho^{2-\gamma}(q^2)(1 - q^2/a_*^2)\rho_0^{\gamma-1}, \end{aligned}$$

then

$$\begin{aligned} (\rho(q^2)q)' &> 0 && \text{if } q \in [0, a_*); \\ (\rho(q^2)q)' &= 0 && \text{if } q = a_*; \\ (\rho(q^2)q)' &< 0 && \text{if } q \in \left(a_*, \left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2} a_*\right). \end{aligned}$$

Furthermore, if $0 \leq q \leq \delta < a_*$, for δ arbitrarily close to a_*

$$(\rho(q^2)q)' \geq \rho^{2-\gamma}(\delta^2)(1 - \delta^2/a_*^2)\rho_0^{\gamma-1} \geq C(\delta),$$

where

$$C(\delta) = \left(\frac{2}{\gamma + 1}\right)^{(2-\gamma)/(\gamma+1)} \rho_0 \left(1 - \frac{\delta^2}{a_*^2}\right) > 0.$$

(iii) For $q \in [0, a_*)$, $\rho(q^2)q$ is a Lipschitz function of q with Lipschitz constant ρ_0 , as one may easily derive from (ii). Now, suppose $y, z \in \mathbb{R}^n$, $n \geq 2$, and $|y|, |z| < a_*$; then

$$\begin{aligned} \left| \rho(|y|^2)y - \rho(|z|^2)z \right|^2 &= \left(\rho(|y|^2)|y| - \rho(|z|^2)|z| \right)^2 \\ &\quad + 2\rho(|z|^2)\rho(|y|^2)(|z||y| - y \cdot z) \\ &\leq \rho_0^2 |y| - |z| \|^2 + 2\rho_0^2 (|z||y| - y \cdot z) \\ &= \rho_0^2 |y - z|^2. \end{aligned}$$

Thus, $\rho(|y|^2)y$ is a Lipschitz function of y in \mathbb{R}^n provided that $|y| \leq a_*$.

We now show that J' is Lipschitz continuous and monotone in K_δ . For any u, v , in K_δ and w in H_{Γ_0} ,

$$\begin{aligned} \langle J'u - J'v, w \rangle &= \int_\Omega \left[\rho(|\nabla u|^2) \nabla u - \rho(|\nabla v|^2) \nabla v \right] \cdot \nabla w \, dx \\ &\leq \int_\Omega \rho_0 |\nabla u - \nabla v| |\nabla w| \, dx \\ &\leq \rho_0 \|u - v\| \|w\|. \end{aligned}$$

Thus

$$\|J'u - J'v\|^* \leq \rho_0 \|u - v\|, \tag{11}$$

where $\|\cdot\|^*$ is the norm on H'_{Γ_0} . As a consequence, the functional J is continuously differentiable in K_δ .

From (ii) and the mean value theorem, we see that for any $u, v \in K_\delta, u \neq v$,

$$\begin{aligned} \langle J'u - J'v, u - v \rangle &= \int_\Omega \left(\rho(|\nabla u|^2) |\nabla u| - \rho(|\nabla v|^2) |\nabla v| \right) (|\nabla u| - |\nabla v|) \, dx \\ &\quad + \int_\Omega \left[\rho(|\nabla u|^2) + \rho(|\nabla v|^2) \right] (|\nabla u| |\nabla v| - \nabla u \cdot \nabla v) \, dx \\ &= \int_\Omega \int_0^1 (\rho(t)t)' \Big|_{t=s|\nabla u|+(1-s)|\nabla v|} ds (|\nabla u| - |\nabla v|)^2 \, dx \\ &\quad + 2\rho_{\min} \int_\Omega (|\nabla u| |\nabla v| - \nabla u \cdot \nabla v) \, dx \end{aligned}$$

and hence

$$\langle J'u - J'v, u - v \rangle > 0.$$

Thus, J' is strictly monotone on K_δ and consequently J is strictly convex. As K_δ is bounded, the existence and uniqueness of solutions of (7) and, equivalently, of (9) may be established by appealing to standard theory. Furthermore, we may deduce from (i) and (ii) that

$$\langle J'u - J'v, u - v \rangle \geq C(\delta) \|u - v\|^2 \tag{12}$$

for all u, v in K_δ .

2. Estimates

2.1 Standard Finite Element Method

Let T_h be a triangulation of Ω and $P_k(\Omega)$ be the space of piecewise polynomials of degree $\leq k$ in Ω . Let $V_h \subset H^1(\Omega)$ be the space of linear finite elements with respect to T_h . Let u be the weak solution of the variational inequality

problem (9), and let u_h be the corresponding finite element solution, that is,

$$\langle J'u_h, v_h - u_h \rangle \geq 0 \quad \text{for all } v_h \in K_\delta^h, \tag{13}$$

where $K_\delta^h = V_h \cap K_\delta$ is a finite element subset of K_δ . Setting $v = u_h$ in (9) and adding it to (13), we get

$$\begin{aligned} 0 &\leq \langle J'u_h, v_h - u_h \rangle + \langle J'u, u_h - u \rangle \\ &= \langle J'u_h - J'u, v_h - u_h \rangle + \langle J'u, v_h - u \rangle. \end{aligned}$$

Thus

$$\langle J'u - J'u_h, v_h - u_h \rangle \leq \langle J'u, v_h - u \rangle. \tag{14}$$

Now, by the monotonicity of J' in K_δ given in (12), we have

$$\begin{aligned} C(\delta)\|u - u_h\|^2 &\leq \langle J'u - J'u_h, u - u_h \rangle \\ &= \langle J'u - J'u_h, u - v_h \rangle + \langle J'u - J'u_h, v_h - u_h \rangle \end{aligned}$$

for all v_h in K_δ^h . Thus, using (11) and (14),

$$C(\delta)\|u - u_h\|^2 \leq \rho_0\|u - u_h\| \|u - v_h\| + \langle J'u, v_h - u \rangle. \tag{15}$$

Integrating the term $\langle J'u, v_h - u \rangle$ in (15) by parts and applying the Hölder inequality, we obtain

$$\begin{aligned} C(\delta)\|u - u_h\|^2 &\leq \rho_0\|u - u_h\| \|u - v_h\| \\ &\quad + \left\| \nabla \cdot (\rho|\nabla u|^2 \nabla u) \right\|_{L^{p'}(\Omega)} \|u - v_h\|_{L^p(\Omega)}, \end{aligned} \tag{16}$$

where $p \geq 2$ and $1/p + 1/p' = 1$. To see that the term $\|\nabla \cdot (\rho \nabla u)\|_{L^{p'}}$ is bounded when $u \in W^{2,p}(\Omega)$, first note that

$$|\nabla \cdot (\rho \nabla u)| \leq |\rho \nabla^2 u| + 2|\rho' |\nabla u|^2| |\nabla u|^2 \left| \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \right|.$$

Recalling that $|\nabla u|^2 \leq \delta$ a.e. in Ω , we have from (i) and (ii)

$$\begin{aligned} \|\nabla \cdot (\rho \nabla u)\|_{L^{p'}} &\leq C \left(\rho_0 \|u\|_{2,p'} + \frac{\rho_0}{a_0^2} \delta^2 \|u\|_{2,p'} \right) \\ &\leq C \|u\|_{2,p'} \leq C \|u\|_{2,p}, \end{aligned} \tag{17}$$

since $p \geq 2$. Here C denotes a generic constant. Now from (16)

$$\|u - u_h\|^2 \leq C^{-1}(\delta) \left[\rho_0 \|u - u_h\| \|u - v_h\| + \|\nabla \cdot (\rho \nabla u)\|_{L^{p'}} \|u - v_h\|_{L^p} \right],$$

so

$$\begin{aligned} \|u - u_h\|^2 &\leq \frac{1}{2} \|u - u_h\|^2 + \frac{1}{2} C^{-2}(\delta) \rho_0^2 \|u - v_h\|^2 \\ &\quad + C^{-1}(\delta) \|u - v_h\|_{L^p} C \|u\|_{2,p}, \end{aligned}$$

whence

$$\|u - u_h\|^2 \leq C^{-2}(\delta)\rho_0^2\|u - v_h\|^2 + C^{-1}(\delta)C\|u\|_{2,p}\|u - v_h\|_{L^p}. \tag{18}$$

Note that as $\delta \rightarrow a_*$, $C^{-1}(\delta) \rightarrow \infty$ and thus the error estimate becomes invalid when the flow is no longer subsonic.

Let Π_h denote the linear interpolation operator on $C^0(\bar{\Omega})$, with respect to the triangulation T_h , i.e., for $v \in C^0(\bar{\Omega})$,

$$\Pi_h v|_T \in P_1(T) \quad \text{for all } T \text{ in } T_h$$

and $\Pi_h v(P) = v(P)$ at all nodal points P in T_h .

As $u \in K_\delta$ and $W^{1,\infty}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, $\Pi_h u$ is well defined. However, as $\Pi_h u \notin K_\delta^h$ in general, we cannot set v_h in (18) as $\Pi_h u$ to obtain an optimal estimate. Instead, we shall construct an element in K_δ^h with which approximation results may be obtained easily.

From standard approximation theory, we have, for v in $W^{2,p}(\Omega)$, $p > 2$,

$$|\nabla(\Pi_h v - v)(x)| \leq Ch^{1-2/p}\|v\|_{2,p}$$

for a.e. x in Ω , where C is a constant independent of v and h . If we restrict v to K_δ , then

$$|\nabla(\Pi_h v)(x)| \leq \delta + Ch^{1-2/p}\|v\|_{2,p}$$

for a.e. x in Ω . Based on this inequality, and motivated by the analysis of [8, 12], we introduce the element

$$r_h u = \frac{\delta \Pi_h u}{\delta + Ch^{1-2/p}\|u\|_{2,p}}, \tag{19}$$

where $p > 2$. Note that as $h \rightarrow 0$, $r_h u \rightarrow \Pi_h u$. Also,

$$|r_h u| \leq \delta \quad \text{a.e. in } \Omega.$$

Thus, $r_h u \in K_\delta^h$. Moreover,

$$\|u - r_h u\| = O(h + h^{1-2/p}) \tag{20}$$

and

$$\|u - r_h u\|_{L^p} = O(h^2 + h^{1-2/p}). \tag{21}$$

Thus, using (20) and (21) in (18), we have the estimate

$$\|u - u_h\| = O(h^{1/2-1/p}) \quad \text{for } u \in W^{2,p}(\Omega), \quad p > 2. \tag{22}$$

REMARKS. 1. The suboptimality of this result relative to the $O(h)$ result for linear theory originates from the ‘‘incomplete’’ approximation theory for the subset K_δ rather than the nonoptimality of the abstract error estimate.

2. One can obtain an estimate without making the above regularity assumption on u as follows: The second term on the right-hand side of (15) is bounded by $C\|J'u\|*\|v_h - u\|$ and the first term on the right-hand side is, as before, bounded

by

$$\frac{1}{2}C(\delta)\|u - u_h\|^2 + \frac{1}{2}\rho_0^2C^{-1}(\delta)\|u - v_h\|^2.$$

Thus,

$$\|u - u_h\|^2 \leq \rho_0^2C^{-2}(\delta)\|u - v_h\|^2 + CC^{-1}(\delta)\|v_h - u\|. \tag{23}$$

With this approach, we do not need to impose the regularity condition $u \in W^{2,p}(\Omega)$ for $p \geq 2$. However, the error estimate is necessarily suboptimal because of the presence of the second term on the right-hand side of (23).

3. With the establishment of the error estimates, it is easy to prove the convergence of the finite element approximation u_h to the solution u .

2.2 Dual Finite Element Method

The error estimate (22) derived in the previous section is useful in proving the convergence of the finite element approximation. However, even if we assume u to be in $W^{2,\infty}(\Omega)$, this error estimate would still be suboptimal due to the inadequacy of K_δ^h . More specifically, with this strong smoothness assumption, the bound on $\|u - u_h\|$ would be of $O(h^{1/2})$, which is inferior to the standard $O(h)$ estimates obtained for many other problems. The presence of the nonlinearity and critical flow degeneracy at $q = a^*$ may lead one to admit the possible suboptimality, but numerical evidence suggests that the energy error for $q < a^*$ is actually $O(h)$ [4]. In this section, we follow a successful approach used for elastoplastic torsion problems [8, 9, 12] to derive an optimal estimate for the dual finite element approximation for subsonic flow. (By working with the velocity directly rather than the potential, the previous difficulty—Remark 1, Section 2.1—can be circumvented.)

Let $u \in K_\delta$ be the unique solution of the variational inequality problem (9) and let v be an element of K_δ . We introduce as new variables for a dual formulation

$$\xi = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right) \quad \text{and} \quad \eta = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right).$$

Clearly, the subsonic constraint becomes

$$\eta \equiv |\eta| = |\nabla v| \leq \delta, \quad \text{so } |\eta| \in L^\infty(\Omega),$$

$$\eta \in L_\delta \equiv \{ \zeta \in L^2(\Omega) \times L^2(\Omega); \zeta \leq \delta \}.$$

The subsonic constraint in this formulation appears as a constraint on the magnitude of the elements in L_δ . This seemingly trivial change is actually an important step in establishing the optimal error estimate. While we cannot guarantee that the linear interpolant of elements of K_δ be in K_δ^h , a fact leading us to a suboptimal estimate, the constraint in L_δ involves no derivative and the linear interpolant of any element in L_δ will belong to the approximating space L_h

of L_δ defined in the next section. This then allows us to utilize the optimal approximation result in our error analysis. The far-field boundary condition $\phi = g_0$ on Γ_0 may be restated in terms of $\partial g_0/\partial\tau$ where $\tau = (n_2, -n_1)$ is the tangent vector. We first embed this condition in the space for the dual problem and then subsequently consider using a Lagrange multiplier for the boundary constraint. An error estimate is finally derived for the finite element approximation of the dual formulation with multiplier treatment of the boundary condition.

We first show that the vector ξ defined above is a solution of the following dual variational inequality problem. Find $\xi \in L_\delta \cap H$ such that

$$\int_\Omega \rho(\xi^2)\xi \cdot (\eta - \xi) \, dx \geq 0 \quad \text{for all } \eta \text{ in } L_\delta \cap H, \tag{24}$$

where

$$H = \left\{ \eta \in L^2(\Omega) \times L^2(\Omega); \int_\Omega \eta \cdot \nabla w \, dx = \int_{\Gamma_0} -\frac{\partial g_0}{\partial\tau} w \, ds \text{ for all } w \in H_{\Gamma_1} \right\}.$$

For notational convenience, henceforth we shall write H_Γ for H_{Γ_1} . Now

$$\int_\Omega \rho(|\nabla u|^2)\nabla u \cdot (\nabla v - \nabla u) \, dx = \int_\Omega \rho(\xi^2)\xi \cdot (\eta - \xi) \, dx \tag{25}$$

follows directly from the definition of ξ and η and the property

$$\xi \cdot \eta = \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(-\frac{\partial u}{\partial x}\right)\left(-\frac{\partial v}{\partial x}\right) = \nabla u \cdot \nabla v.$$

Furthermore, since $\eta = (\partial v/\partial y, -\partial v/\partial x)$ and $v = g_0$ on $\Gamma_0 (v \in K_\delta)$, then for any w in H_Γ ,

$$\begin{aligned} \int_\Omega \eta \cdot \nabla w \, dx &= \int_{\Gamma_0} w\eta \cdot \mathbf{n} \, ds - \int_\Omega (\nabla \cdot \eta)w \, dx \\ &= \int_{\Gamma_0} -w \frac{\partial g_0}{\partial\tau} \, ds. \end{aligned} \tag{26}$$

Thus, $\eta \in H$ and so ξ is a solution of the dual variational inequality (24).

In fact, the converse is also true, i.e., if ξ is a solution of (24), then one may find function u satisfying (9) and such that $\text{curl } u = \xi$ (see (27) below). To show this, we take $\eta \in L_\delta \cap H$. Then, for all $w \in H_\Gamma$, we again have (26). Now, for w in $H_0^1(\Omega) \subset H_\Gamma$, (26) is reduced to $\int_\Omega (\nabla \cdot \eta)w \, dx = 0$ which implies $\nabla \cdot \eta = 0$ a.e. in Ω . Consequently, we have $\int_{\Gamma_0} w\eta \cdot \mathbf{n} \, ds = \int_{\Gamma_0} -w\partial g_0/\partial\tau \, ds$ for all $w \in H_\Gamma$; therefore,

$$\eta \cdot \mathbf{n} = -\partial g_0/\partial\tau \text{ a.e. on } \Gamma_0. \tag{27}$$

Next, observe that as Ω is simply connected, there exists a function v in $H^1(\Omega)$, unique up to an additive constant, such that

$$\eta = \text{curl } v \equiv \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right)$$

(so that $\nabla \cdot \eta = 0$ is satisfied automatically) and that by (27)

$$-\frac{\partial g_0}{\partial \tau} = \eta \cdot \mathbf{n} = \frac{\partial v}{\partial y} n_1 - \frac{\partial v}{\partial x} n_2 = -\nabla v \cdot \tau = -\frac{\partial v}{\partial \tau}. \tag{28}$$

To fix v , we may take $v = g_0$ on Γ_0 to obtain $v \in K_\delta$. In particular, we may set η to ξ , and the corresponding v to u . Thus, in view of equation (25), we have shown the converse.

Formulation (24) is, in some sense, an unconventional mixed problem, as most mixed problems involve a Lagrange multiplier and may be related to a saddle-point problem. This situation is clarified by noting that the space H involves a constraint arising from the far-field boundary condition and may be relaxed with the aid of a Lagrange multiplier.

In order to relax the constraint $\int_\Omega \eta \cdot \nabla w \, dx = \int_{\Gamma_0} -(\partial g_0 / \partial \tau) w \, ds$ for w in H_Γ , we introduce the multiplier $\chi \in H_\Gamma$ and consider the variational problem using Lagrange multipliers: find the pair $\langle \xi, \chi \rangle$ in $(L_\delta \cap H) \times H_\Gamma$ such that

$$\int_\Omega \rho(\xi^2) \xi \cdot (\eta - \xi) \, dx \geq \int_\Omega \nabla \chi \cdot (\eta - \xi) \, dx \quad \text{for all } \eta \text{ in } L_\delta. \tag{29}$$

This formulation (29) will be useful later when we estimate the error of the finite element approximation.

We have the following theorem:

THEOREM. *The mixed variational inequality problem (24) is equivalent to the variational problem (29).*

PROOF. Let (ξ, χ) be a solution of (29). By restricting η in (29) to the set $L_\delta \cap H$, we have

$$\int_\Omega \nabla \chi \cdot (\eta - \xi) \, dx = 0,$$

since $\chi \in H_\Gamma$. Thus, ξ is a solution of (24). Next, let ξ be a solution of the mixed problem (24). To prove the existence of χ in H_Γ such that (29) holds for all η in L_δ , first observe that ξ is the solution of the minimization problem: find $\xi \in L_\delta \cap H$ such that

$$J_1(\xi) = \min J_1(\eta), \quad \eta \in L_\delta \cap H, \tag{30}$$

where

$$J_1(\eta) = \int_\Omega \int_0^{\eta^2} \frac{1}{2} \rho(t) \, dt \, dx.$$

We shall prove the desired inequality by contradiction. Suppose that for all χ in H_Γ , there exists η in L_δ such that

$$J_1(\eta) - \int_\Omega \nabla \chi \cdot \eta \, dx < J_1(\xi). \tag{31}$$

Clearly, $\eta \neq \xi$, otherwise we have an immediate contradiction. This and the uniqueness of ξ imply that $J_1(\xi) < J_1(\eta)$ for all χ . Now, let $\{\chi_k\}$ be a sequence of functions in H_Γ converging to 0, and let $\{\eta_k\}$ be the corresponding sequence in L_δ . As L_δ is a compact set in $L^2(\Omega) \times L^2(\Omega)$, there is a convergent subsequence $\{\eta_k\}$ converging to some η in L_δ . From (31), we have

$$J_1(\eta_k) - \|\nabla \chi\| \delta < J_1(\xi).$$

Taking the limit $k \rightarrow \infty$, we have $J_1(\eta) < J_1(\xi)$ which contradicts the fact that ξ is a minimum point of (30). Thus, we have shown the existence of χ in H_Γ such that (29) holds for all η in L_δ , since $\int_\Omega \xi \cdot \nabla \chi \, dx = 0$.

REMARK. The variational problem (29) may be shown to be equivalent to the saddle point problem: find $\langle \xi, \chi \rangle \in L_\delta \times H_\Gamma$ such that

$$L(\xi, w) \leq L(\xi, \chi) \leq L(\eta, \chi) \tag{32}$$

for all (η, w) in $L_\delta \times H_\Gamma$, where

$$L(\eta, w) = \int_\Omega \int_0^{\eta^2} \frac{1}{2} \rho(t) \, dt \, dx - \int_\Omega \eta \cdot \nabla w \, dx + \int_{\Gamma_0} \frac{\partial g_0}{\partial \tau} w \, ds.$$

2.3 Error Estimate for Dual Method

Let us now consider the approximate dual method. Given a triangulation T_h of finite elements of Ω , we define the following approximation spaces:

$$\begin{aligned} \Lambda_h &= \{ \xi_h \in L^2(\Omega) \times L^2(\Omega); \xi_h|_T \in P_0(T) \text{ for } T \text{ in } T_h \}; \\ L_h &= \{ \xi_h \in \Lambda_h; |\xi_h| \leq \delta \text{ a.e. in } \Omega \}; \\ V_h &= \{ w_h \in C^0(\bar{\Omega}); w_h|_T \in P_1(T) \text{ for } T \text{ in } T_h, w_h = 0 \text{ on } \Gamma \} \end{aligned}$$

and

$$H_h = \left\{ \xi_h \in \Lambda_h; \int_\Omega \xi_h \cdot \nabla w_h \, dx = 0 \text{ for all } w_h \text{ in } V_h \right\}.$$

The approximation to the dual problem (24) takes the form: find $\xi_h \in L_h \cap H_h$ such that

$$\int_\Omega \rho(\xi_h^2) \xi_h \cdot (\eta_h - \xi_h) \, dx \geq 0 \tag{33}$$

for all η_h in $L_h \cap H_h$. If (ξ, χ) is a solution of the variational problem (29), then setting $\eta = \xi_h (\in L_h \cap H_h \cap L_\delta)$ in (29), we have

$$\int_\Omega \rho(\xi^2) \xi \cdot (\xi_h - \xi) \, dx \geq \int_\Omega \nabla \chi \cdot (\xi_h - \xi) \, dx. \tag{34}$$

Adding the inequalities (33) and (34), we have

$$\begin{aligned} & \int_{\Omega} \rho(\xi_h^2) \xi_h \cdot (\eta_h - \xi_h) \, dx + \int_{\Omega} \rho(\xi^2) \xi \cdot (\xi_h - \xi) \, dx \\ & \geq \int \nabla \chi \cdot (\xi_h - \xi) \, dx. \end{aligned} \tag{35}$$

Using the monotonicity property (iii) of the function $\rho(t^2)t$ and (35), we have

$$\begin{aligned} C(\delta) \|\xi_h - \xi\|_{L^2 \times L^2}^2 & \leq \int_{\Omega} (\rho(\xi_h^2) \xi_h - \rho(\xi^2) \xi) \cdot (\xi_h - \xi) \, dx \\ & \leq \int_{\Omega} \rho(\xi_h^2) \xi_h \cdot (\xi_h - \xi) \, dx + \int_{\Omega} \rho(\xi_h^2) \xi_h \cdot (\eta_h - \xi_h) \, dx \\ & \quad - \int_{\Omega} \nabla \chi \cdot (\xi_h - \xi) \, dx \\ & \leq \int_{\Omega} \rho(\xi_h^2) \xi_h \cdot (\eta_h - \xi_h) \, dx - \int_{\Omega} \nabla(\chi - w_h + w_h) \cdot (\xi_h - \xi) \, dx \\ & \leq \int_{\Omega} \rho(\xi_h^2) \xi_h \cdot (\eta_h - \xi_h) \, dx - \int_{\Omega} \nabla w_h \cdot (\xi_h - \xi) \, dx \\ & \quad + \frac{1}{2} C(\delta) \|\xi_h - \xi\|_{L^2 \times L^2}^2 + \frac{1}{2} C^{-1}(\delta) \int_{\Omega} |\nabla(\chi - w_h)|^2 \, dx, \end{aligned} \tag{36}$$

where w_h is an arbitrary element in V_h .

Now, $\xi_h \in H_h$ and $\xi \in H$, so $\int_{\Omega} \nabla w_h \cdot (\xi_h - \xi) \, dx = 0$. Moreover, as $\xi_h|_T \in P_0(T) \times P_0(T)$, so

$$\rho(\xi_h^2)|_T \in P_0(T).$$

Thus

$$\rho(\xi_h^2) \xi_h|_T \in P_0(T) \times P_0(T) \quad \text{for all } T \text{ in } T_h$$

and hence $\rho(\xi_h^2) \xi_h \in L_h$. Now, L_h is a closed, convex subset of $L^2(\Omega) \times L^2(\Omega)$. Thus one may define an orthogonal projection R_h from $L^2(\Omega) \times L^2(\Omega)$ onto L_h . Setting η_h in (36) as $R_h \xi$, we clearly have

$$\int \rho(\xi_h^2) \xi_h \cdot (R_h \xi - \xi) \, dx = 0.$$

Finally, let $\Pi_h: C^0(\bar{\Omega}) \rightarrow V_h$ be the V_h -interpolation operator on T_h . The regularity assumption $\chi \in H^2(\Omega) \rightarrow C^0(\bar{\Omega})$ allows us to conclude $\Pi_h \chi \in V_h$ and

$$\|\Pi_h \chi - \chi\| \leq Ch \|\chi\|_{2,2}.$$

Setting $w_h = \Pi_h \chi$ in (36), we then have

$$\|\xi_h - \xi\|_{L^2 \times L^2}^2 \leq CC^{-2}(\delta) h^2 \|\chi\|_{2,2}^2,$$

or

$$\|\xi_h - \xi\|_{L^2 \times L^2} = O(h/C(\delta)).$$

We summarize this result in the following theorem:

THEOREM. *Let ξ be the solution of the mixed variational inequality problem (24) and ξ_h be the finite element solution of the approximation problem (33). Let χ be the Lagrange multiplier in (29) corresponding to (24). Under the regularity assumption $\chi \in H^2(\Omega)$, we have $\|\xi - \xi_h\|_{L^2 \times L^2} = O(h/C(\delta))$.*

REMARKS. 1. In the proof above, the multiplier formulation rather than the original dual formulation is used because ξ_h is not necessarily in H . The multiplier relaxes the restriction for ξ_h .

2. Notice that, in the above proof, the regularity of ξ is not needed. Instead, the regularity of the multiplier χ is required.

3. In practice, it is easier to implement and numerically solve the saddle point problem (32) than the variational problem (29). Since these problems are equivalent, the error estimate established is also applicable to the numerical solution of (32).

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