BULL. AUSTRAL. MATH. SOC. 55R35, 55P35, 55N91
VOL. 58 (1998) [387–391]

FINITE GENERATION OF EQUIVARIANT COHOMOLOGY FOR A p-COMPACT GROUP G

HYANG - SOOK LEE

For a p-compact group G and G-space X, we prove the finite generation of the equivariant cohomology $H_G^*(X)$ and give the form of the Poincaré series of $H_G^*(X)$.

0. INTRODUCTION

For a compact Lie group G, the algebra $H_G^* = H^*(BG, \mathbb{F}_p)$ is finitely generated. This result is extended to finite loop spaces and other loop spaces called p-compact groups [**2**]. Dwyer and Wilkerson introduced p-compact groups and proved lots of their properties in detail [**2**]. Their work shows that a p-compact group has much of the rich internal structure of a compact Lie group. In [**3**], Quillen proved finite generation of $H_G^*(X) = H^*(EG \times_G X)$ for the general equivariant cohomology ring of a G-space X, where G is a compact Lie group. In this paper, we generalise theorems on finite generation of $H_G^*(X)$ for a p-compact group G and G-space X. We also give the form of the Poincaré series of $H_G^*(X)$. In the first section, we give brief definitions and properties as background. In Section 2, we give the main results.

1. PRELIMINARIES

A graded vector space H^* over a field F is finite dimensional if each H^i is finite dimensional over F and $H^i = 0$ for all but finite number of i. A space X is $\mathbb{F}_p$-finite if H^*X is finte dimensional over the finite field $\mathbb{F}_p$. Let $\varepsilon_X : X \to X_p\hat{}$ be a natural map for any space X where $(\_)_p\hat{}$ is the $\mathbb{F}_p$-completion functor constructed by Bousfield and Kan [**1**]. If ε_X is a homotopy equivalence, we say X is $\mathbb{F}_p$-complete.

DEFINITION 1.1: A p-compact group is a loop space G satisfying the following equivalent conditions.

(1) G is $\mathbb{F}_p$-finite, $\mathbb{F}_p$-complete and $\pi_0 G$ is a finite p-group.
(2) G is $\mathbb{F}_p$-finite and BG is $\mathbb{F}_p$-complete.

Received 19th February, 1998
The author was supported by Ministry of Education, BSRI - 96 - 1424, partially supported by KOSEF 97-07-01-02-01-5.

Theorem 1.2. [2] *If G is a p-compact group, then $H^*(BG, \mathbb{F}_p)$ is finitely generated as an algebra.*

From classical algebra, if X is connected then H^*X is finitely generated as an algebra if and only if H^*X is Noetherian as a graded ring if and only if every graded ideal in H^*X has a finite number of homogeneous generators if and only if every graded submodule of a graded finitely generated H^*X-module is itself finitely generated. Also a H^*X-module satisfies the ascending chain condition on submodules if and only if every submodule of H^*X-module is finitely generated.

A homomorphism $f : G \to K$ of p-compact groups is a pointed map $Bf : BG \to BK$. The homogeneous space K/G is defined to be the homotopy fibre of Bf over the basepoint of BK. The homomorphism f is said to be a monomorphism if K/G is $\mathbb{F}_p$-finite and an epimorphism if $\Omega(K/G)$ is a p-compact group. A short exact sequence $G \xrightarrow{f} H \xrightarrow{g} K$ of p-compact groups is a sequence such that $BG \to BH \to BK$ is a fibration sequence where f is a monomorphism and g is an epimorphism.

Let G be a p-compact group and X be a G-space defined to be a fibration $EG \times_G X \to BG$ with X as the fibre. Here EG is the universal bundle over BG. The equivariant cohomology of the G-space X is defined by the formula

$$H^*_G(X) = H^*(EG \times_G X)$$

where $H^*(\_)$ means ordinary cohomology.

A morphism $(f, u) : (G, X) \to (G', X')$ is a homomorphism of p-compact groups $f : G \to G'$ and for a G-space X and G'-space X', a map $u : X \to X'$ which is f-equivariant, that is, $u(gx) = f(g)u(x)$. Let $EG \to BG$ and $EG' \to BG'$ be the principal G and G'-bundle respectively, and consider the following diagram

where p_1 and p_2 are induced by the projections of $EG \times EG'$ onto its factors, $\overline{u}$ is induced by u and $v : EG \to EG'$ is an f-equivariant map, $\overline{v \times u}$ is the map on orbit spaces induced by $v \times u$. This diagram yields a canonical homomorphism

$$(p_1^*)^{-1} p_2^* \overline{u}^* : H^*(EG' \times_{G'} X') \to H^*(EG \times_G X).$$

This homomorphism will be denoted by

$$(f, u)^* : H^*_{G'}(X') \to H^*_G(X).$$

If X is a point, we write $H^*_G(pt) = H^*(BG) = H^*_G$.

2. The finiteness theorem

Let G be a p-compact group and X be a G-space. In this section we study the finite generation of $H_G^*(X)$. The coefficient ring is assumed to be a finite field $\mathbb{F}_p$.

Lemma 2.1. [2] *If G is a p-compact group and M is a $\mathbb{F}_p$-vector space which is a module over $\pi_1 BG$, then $H^*(BG, M)$ is finitely generated as a module over $H^* BG$.*

Theorem 2.2. [2] *A homomorphism $f : G \to K$ of p-compact groups is a monomorphism if and only if the ring H_G^* is a finitely generated module over H_K^*.*

Let $f : G \to K$ be a monomorphism of p-compact groups. We consider $H_G^*(X)$ as an algebra over H_K^* by means of the homomorphism $(f, \rho)^*$ where ρ is the map from X to a point. Then we have the following theorem.

Theorem 2.3. *If X is $\mathbb{F}_p$-finite, then $H_G^*(X)$ is a finitely generated H_K^*-module.*

Proof: We consider the Serre spectral sequence for the E_2-term of the fibre space $EG \times_G X \to BG$

$$E_2^{s,t} = H^s\big(BG, H^t(X)\big) \Longrightarrow H_G^{s+t}(X).$$

Now X is $\mathbb{F}_p$-finite, hence the E_2-term is finitely generated as a module over H_G^* by Lemma 2.1. Since H_G^* is a Noetherian ring, E_r is also a finitely generated H_G^*-module by induction on r. $E_r = E_\infty$ for sufficiently large r, and hence $H_G^*(X)$ is a finitely generated module over H_G^*. But H_G^* is a finitely generated H_K^*-module (Theorem 2.2). Therefore $H_G^*(X)$ is a finitely generated module over H_K^*. ∎

Corollary 2.4. *$H_G^*(X)$ is a finitely generated algebra over $\mathbb{F}_p$.*

A homomorphism $\zeta : R \to S$ of graded commutative rings is finite if S is a finitely generated module over R via ζ.

Corollary 2.5. *If $(f, u) : (G, X) \to (G', X')$ is a morphism such that f is a monomorphism and X is $\mathbb{F}_p$-finite, then $(f, u)^* : H_{G'}^*(X') \to H_G^*(X)$ is finite.*

Proof: If we choose a monomorphism $G' \to K$, then $H_G^*(X)$ is a finitely generated module over H_K^*, hence also a finitely generated module over $H_{G'}^*(X)$. ∎

Now we recall the Euler-Poincaré function φ which maps certain modules to elements of an Abelian group and satisfies the following condition;

If $0 \to M' \to M \to M'' \to 0$ is exact, then $\varphi(M)$ is defined if and only if $\varphi(M')$ and $\varphi(M'')$ are defined, and $\varphi(M) = \varphi(M') + \varphi(M'')$.

Remark. Assume φ is defined on finite dimensional vector spaces over $\mathbb{F}_p$, and is equal to the dimension. Then the values of φ are in the additive group of integers.

If X is $\mathbb{F}_p$-finite, the Poincaré series of $H_G^*(X)$ is defined by

$$P_t\big(H_G^*(X)\big) = \sum_{i=0}^{\infty}\big(\dim_{\mathbb{F}_p} H_G^i(X)\big)t^i.$$

If $H_G^*(X)$ is finite dimensional over $\mathbb{F}_p$, then $P_t\big(H_G^*(X)\big)$ is a polynomial.

https://doi.org/10.1017/S0004972700032366 Published online by Cambridge University Press

If X is $\mathbb{F}_p$-finite, we showed $H_G^*(X)$ is a finitely generated H_K^*-module in Theorem 2.3. Let m be the number of generators of H_K^* as an algebra over $\mathbb{F}_p$. Then we give the following type of Poincaré series of $H_G^*(X)$.

PROPOSITION 2.6. *The Poincaré series of $H_G^*(X)$ is a rational function of the form*

$$P_t\left(H_G^*(X)\right) = \frac{f(t)}{\prod_{i=0}^{m}(1-t^{d_i})}$$

where d_i's are the corresponding degrees of generators of H_K^* and $f(t)$ is a polynomial with integer coefficients.

PROOF: We use induction on m. For $m = 0$, $H_G^*(X)$ is a finitely generated $\mathbb{F}_p$-module, and hence $P_t\left(H_G^*(X)\right) = f(t)$ is a polynomial in $\mathbb{Z}[t]$. Assume $m \geqslant 1$. Since H_K^* is a finitely generated algebra over $\mathbb{F}_p$, we set $H_K^* = \mathbb{F}_p\langle x_1, x_2, \cdots, x_m\rangle$ where $\deg x_i = d_i \geqslant 1$. We consider the following exact sequence by multiplying x_m on $H_G^*(X)$

$$0 \to C_n \to H_G^n(X) \xrightarrow{x_m} H_G^{n+d_m}(X) \to L_{n+d_m} \to 0.$$

Let $C = \oplus_n C_n$ and $L = \oplus_n L_n$. Then C and L are finitely generated H_K^*-modules (as a submodule and factor module respectively) and annihilated by x_m, hence are graded $\mathbb{F}_p\langle x_1, x_2, \cdots, x_{m-1}\rangle$-modules. By the Remark,

$$\dim C_n - \dim H_G^n(X) + \dim H_G^{n+d_m}(X) - \dim L_{n+d_m} = 0.$$

Multiplying by t^{n+d_m} and summing over n,

$$\begin{aligned}\sum_{n=0} t^{n+d_m}\dim C_n - \sum_{n=0} t^{n+d_m}\dim H_G^n(X)& \\ + \sum_{n=0} t^{n+d_m}\dim H_G^{n+d_m}(X) - \sum_{n=0} t^{n+d_m}\dim L_{n+d_m}& \\ = 0.&\end{aligned}$$

Hence

$$\begin{aligned} t^{d_m}\cdot\left(\sum_{n=0}\dim C_n\cdot t^n\right) - t^{d_m}\cdot\left(\sum_{n=0}\dim H_G^n(X)\cdot t^n\right)& \\ + \sum_{n=0}\dim H_G^{n+d_m}(X)\cdot t^{n+d_m} - \sum_{n=0}\dim L_{n+d_m}\cdot t^{n+d_m}& \\ = t^{d_m}\cdot P_t(C) - t^{d_m}\cdot P_t\left(H_G^*(X)\right) + P_t\left(H_G^*(X)\right) - P_t(L) - g(t)& \\ = 0& \end{aligned}$$

where

$$g(t) = \sum_{i=0}^{d_m-1}\dim H_G^i(X)\cdot t^i - \sum_{i=0}^{d_m-1}\dim L_i\cdot t^i \in \mathbb{Z}[t].$$

https://doi.org/10.1017/S0004972700032366 Published online by Cambridge University Press

Then

$$\begin{aligned}\left(1-t^{d_m}\right)P_t\left(H_G^*(X)\right) &= P_t(L)-t^{d_m}P_t(C)+g(t)\\ &= \frac{f_1(t)}{\prod_{i=1}^{m-1}(1-t^{d_i})}-\frac{t^{d_m}\cdot f_2(t)}{\prod_{i=1}^{m-1}(1-t^{d_i})}+g(t)\end{aligned}$$

by induction. Therefore

$$\begin{aligned}P_t\left(H_G^*(X)\right) &= \frac{f_1(t)-t^{d_m}\cdot f_2(t)+g(t)\prod_{i=1}^{m-1}\left(1-t^{d_i}\right)}{\prod_{i=1}^{m}(1-t^{d_i})}\\ &= \frac{f(t)}{\prod_{i=1}^{m}(1-t^{d_i})}\end{aligned}$$

where $f(t)\in\mathbb{Z}[t]$. □

References

[1] A.K. Bousfield and D.K. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. **304** (Springer-Verlag, Berlin, 1972).

[2] W.G. Dwyer and C.W. Wilkerson, 'Homotopy fixed point methods for Lie groups and finite loop spaces', *Ann. of Math.* **139** (1994), 395–442.

[3] D. Quillen, 'The spectrum of an equivariant cohomology ring I', *Ann. of Math.* **94** (1991), 549–572.

Department of Mathematics
Ewha Women's University
Seoul 120-750
Korea
e-mail: hsl@mm.ewha.ac.kr