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AN EXTENDED LOOMIS–WHITNEY INEQUALITY FOR POSITIVE DOUBLE JOHN BASES

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Abstract. In this paper, we establish an extended Loomis–Whitney inequality for positive double John bases, which generalises Ball's result [**1**]. Moreover, a different extension of the Loomis–Whitney inequality is deduced.

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1. Introduction. A convex body K (i.e. compact, convex sets with non-empty interior) in $\mathbb{R}^n$ is in John's position if the maximal volume ellipsoid of K is the Euclidean unit ball. John [**3**, **11**] proved that a convex body K is in John's position if and only if there exist contact points $\bar{u}_1, \ldots, \bar{u}_m$ of K and B_2^n (common points of their boundaries) and positive real numbers $c_1, \ldots, c_m$ such that

$$\sum_{i=1}^{m} c_i\bar{u}_i = 0 \quad \text{and} \quad I_n = \sum_{i=1}^{m} c_i\bar{u}_i \otimes \bar{u}_i,$$

where $\bar{u}_i \otimes \bar{u}_i$ is the usual rank one orthogonal projection onto the span of $\bar{u}_i$ and I_n is the identity on $\mathbb{R}^n$. The first condition guarantees that the $\{\bar{u}_i\}_1^m$ do not all lie on one side of the sphere. The second condition guarantees that the $\{\bar{u}_i\}_1^m$ do not all lie close to a proper subspace of $\mathbb{R}^n$. In the case of a symmetric convex body, the first condition is redundant, since we can take any sequence $\{\bar{u}_i\}$ of contact points satisfying the second condition and replace each $\bar{u}_i$ by the pair $\pm\bar{u}_i$, each with half the weight of the original.

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The above identity states that the $\bar{u}_i$'s are distributed rather like an orthonormal basis in the sense that for each $x \in \mathbb{R}^n$,

$$|x|^2 = \sum_{i=1}^{m} c_i \langle \bar{u}_i, x\rangle^2, \quad \sum_{i=1}^{m} c_i = n,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product.

DEFINITION 1. Let $\{\bar{u}_i\}_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$. We call $\{\bar{u}_i\}_1^m$ a *John basis* with weights $c_1, \ldots, c_m > 0$, if

$$I_n = \sum_{i=1}^{m} c_i \bar{u}_i \otimes \bar{u}_i. \tag{1.1}$$

Note that the condition (1.1) guarantees $m \geq n$.

In fact, John's decomposition of the identity holds in a much more general context. We refer to [**3**, **4**, **7**, **8**, **9**, **12**, **15**, **17**] and references therein for an extensive survey of John's decomposition.

In particular, Giannopoulos et al. [**7**] provided a generalisation of John's representation of the identity for the maximal volume position of two arbitrary smooth convex bodies. This remarkable work can be stated as follows. Let K, L be two (not necessarily symmetric) smooth convex bodies in $\mathbb{R}^n$. We say that L is of maximal volume in K if $L \subseteq K$ and, for every $w \in \mathbb{R}^n$ and $T \in SL(n)$, the affine image $w + T(L)$ of L is not contained in the interior of K.

THEOREM. *If L is of maximal volume in K, then for every z belonging to the interior of L, we can find contact points $v_1, \ldots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \ldots, u_m$ of $(K - z)^\circ$ and $(L - z)^\circ$, and positive real numbers $c_1, \ldots, c_m$, such that $\sum c_i u_i = 0$, $\langle u_i, v_i \rangle = 1$, and*

$$I_n = \sum_{i=1}^{m} c_i u_i \otimes v_i. \tag{1.2}$$

Here K° is the polar body of K, defined by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

As usual, $u_i \otimes v_i$ denotes the rank one projection defined by $u_i \otimes v_i(x) = \langle u_i, x\rangle v_i$. Moreover, there exists a choice of z such that we simultaneously have $\sum c_i u_i = \sum c_i v_i = 0$. This automatically holds for two symmetric convex bodies.

It is easy to verify from (1.2) that for each $x \in \mathbb{R}^n$

$$|x|^2 = \sum_{i=1}^{m} c_i \langle u_i, x\rangle \langle v_i, x\rangle \tag{1.3}$$

and

$$\sum_{i=1}^{m} c_i = n. \tag{1.4}$$

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Motivated by the result of Giannopoulos et al. [7], we give the following definition.

DEFINITION 2. Let $u_i, v_i \in \mathbb{R}^n$, $i = 1, \ldots, m$. We call the sequence of pairs $\{(u_i, v_i)\}_1^m$ a *double John basis* with weights $c_1, \ldots, c_m > 0$ if

(i) $\langle u_i, v_i \rangle = 1$,
(ii) $I_n = \sum_{i=1}^m c_i u_i \otimes v_i$.

Let $I \subseteq \{1, 2, \ldots, m\}$. Denote by $|I|$ its cardinality.

DEFINITION 3. A double John basis $\{(u_i, v_i)\}_1^m$ is said to be a *positive double John basis* if it satisfies

$$\det\left(\sum_{i \in I, |I| = n} u_i \otimes v_i\right) \geq 0. \tag{1.5}$$

In this paper, by using positive double John bases as defined above, we will establish an extension of the well-known Loomis–Whitney inequality.

The well-known Loomis–Whitney inequality (see [**13**] and [**5**, p. 95]) states that for a convex body K in $\mathbb{R}^n$ and a canonical orthonormal basis $\{e_i\}_1^n$, we have

$$V(K)^{n-1} \leq \prod_{i=1}^n V_{n-1}(P_{e_i}K),$$

where $P_{e_i}K$ is the projection of K onto the 1-codimensional subspace $e_i^\perp$ orthogonal to e_i.

The remarkable fact that the orthonormal basis in the above inequality can be replaced by any John basis was established by Ball [**1**]. Using induction, Ball gave an elegant proof of the following result: If K is a convex body in $\mathbb{R}^n$, and $(\bar{u}_1, \ldots, \bar{u}_m)$ is a John basis with weights $c_1, \ldots, c_m > 0$, then

$$V(K)^{n-1} \leq \prod_{i=1}^m V_{n-1}(P_{\bar{u}_i}K)^{c_i}. \tag{1.6}$$

Using a slightly different method than Ball, we establish the following generalisation of inequality (1.6).

THEOREM 1.1. *Let K be a convex body in $\mathbb{R}^n$. If $\{(u_i, v_i)\}_1^m$ is a positive double John basis with weights $c_1, \ldots, c_m > 0$, then*

$$V(K)^{2(n-1)} \leq \prod_{i=1}^m (|u_i||v_i| V_{n-1}(P_{u_i}K) V_{n-1}(P_{v_i}K))^{c_i}. \tag{1.7}$$

From an application of Theorem 1.1, we give a different extension of the Loomis–Whitney inequality, which generalises Zhang's result [**18**].

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THEOREM 1.2. *Let K be a convex body in $\mathbb{R}^n$. If $\{(u_i, v_i)\}_1^m$ is a sequence of pair of non-zero vectors in $\mathbb{R}^n$ such that*

$$\det\left(\sum_{i\in I,|I|=n} u_i\otimes v_i\right)\geq 0$$

and $\sum_{i=1}^m u_i\otimes v_i := A$ is a positive definite matrix, then

$$V(K)^{2(n-1)}\leq \det\left(\frac{A+A^T}{2}\right)^{-1}\prod_{i=1}^m\left(\frac{|u_i||v_i|V_{n-1}(P_{u_i}K)V_{n-1}(P_{v_i}K)}{c_i}\right)^{c_i}, \tag{1.8}$$

where $c_i = \langle(\frac{A+A^T}{2})^{-1}u_i, v_i\rangle$.

The rest of this paper is organised as follows: In Section 2 some of the basic notations and preliminaries are established. Section 3 contains the proofs of the main results.

2. Notations and Preliminaries. For K and L convex bodies in $\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, the Minkowski sum $K+L$ of K and L is defined by

$$K+L=\{x+y; x\in K, y\in L\},$$

and the scalar multiplication λK is defined by

$$\lambda K=\{\lambda x; x\in K\}.$$

The Minkowski sum of finitely many line segments is called a *zonotope*.

As a consequence of Minkowski's theorem (see [**6**, **16**]), the volume of $K+\lambda L$ can be represented by a polynomial in λ,

$$V(K+\lambda L)=\sum_{i=0}^m\binom{n}{i}V_i(K,L)\lambda^i, \tag{2.1}$$

where

$$V_i(K,L)=V\left(\underbrace{K,\dots,K}_{n-i},\underbrace{L,\dots,L}_{i}\right)$$

is called the ith mixed volume of K and L, where K appears $n-i$ times and L appears i times. The Brunn–Minkowski inequality states that $V(K+\lambda L)^{1/n}$ is a concave function of λ in $[0,\infty)$. Differentiation of (2.1) at $\lambda=0$ gives Minkowski's first inequality

$$V_1(K,L)\geq V(K)^{(n-1)/n}V(L)^{1/n} \tag{2.2}$$

with equality if and only if K and L are homothetic. By Cauchy's projection formula [**6**], we can easily obtain that if $\bar{u}$ is a unit vector then

$$V_1(K,[-\bar{u},\bar{u}])=\frac{2}{n}V_{n-1}(P_{\bar{u}_i}K)$$

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for any convex body K. Let $[-u_1, u_1], \dots, [-u_m, u_m]$ be m line segments, their Minkowski sum is

$$Z = [-u_1, u_1] + \cdots + [-u_m, u_m].$$

So

$$\begin{aligned} V_1(K, Z) &= V_1\left(K, \sum_{i=1}^{m} [-u_i, u_i]\right) = V_1\left(K, \sum_{i=1}^{m} |u_i|[-\bar{u}_i, \bar{u}_i]\right) \\ &= \frac{2}{n} \sum_{i=1}^{m} |u_i| V_{n-1}(P_{\bar{u}_i} K) = \frac{2}{n} \sum_{i=1}^{m} |u_i| V_{n-1}(P_{u_i} K), \end{aligned} \tag{2.3}$$

where $\bar{u}_i = u_i/|u_i|$. If $u_1, \dots, u_m \in \mathbb{R}^n$, $m \geq n$, we have (see [**14**, p. 73])

$$V\left(\sum_{i=1}^{m} [-u_i, u_i]\right) = 2^n \sum_{1 \leq i_1 < \cdots < i_n \leq m} |\det(u_{i_1}, \dots, u_{i_n})|. \tag{2.4}$$

In order to prove Theorem 1.1, it should be noted that if $p > 0$, $r_i, a_i > 0$, the weighted pth means $\sum_i (r_i a_i^p)^{1/p}$ decrease with p for all $a_i > 0$ if and only if $r_i \geq 1$. See [**10** (2.10.5), p. 29]. In particular, the inequality

$$\sum_i r_i a_i^2 \leq \left(\sum_i r_i a_i\right)^2 \tag{2.5}$$

is true for all $a_i > 0$ when $r_i \geq 1$.

3. Proof of the main results.

The Cauchy–Binet formula can be stated as follows.

LEMMA 3.1. *Let $m \geq n$ be integers and $I \subseteq \{1, 2, \dots, m\}$. Let A be a $n \times m$ matrix and B a $m \times n$ matrix. If A_I denotes the square matrix obtained from A by keeping only the columns with indices in I, and B_I denotes the square matrix obtained from B by keeping the rows with indies in I, then we have the formula*

$$\det(AB) = \sum_{|I|=n} \det(A_I) \det(B_I).$$

Using the Cauchy–Binet formula and the fact that $\langle u_i, v_i \rangle = 1$, the following critical lemma was proved by Giannopoulos et al. [7].

LEMMA 3.2. *If a sequence of pairs $\{(u_i, v_i)\}_1^m$ is a positive double John basis, with weights $c_1, \dots, c_m > 0$, then for $\lambda_i, \delta_i > 0$, $i = 1, \dots, m$, we have*

$$\det\left(\sum_{i=1}^{m} c_i \lambda_i u_i \otimes u_i\right) \det\left(\sum_{i=1}^{m} c_i \delta_i v_i \otimes v_i\right) \geq \prod_{i=1}^{m} (\lambda_i \delta_i)^{c_i}. \tag{3.1}$$

Proof. Let $I \subseteq \{1, 2, \dots, m\}$. Write $\lambda_I = \prod_{i \in I} \lambda_i$, $\delta_I = \prod_{i \in I} \delta_i$ and use the notations $U_I = \det(u_i; i \in I)$, $V_I = \det(v_i; i \in I)^T$, where T is the notation of transpose. Moreover, we write $(\sqrt{c}U)_I$ for $\det(\sqrt{c_i}u_i; i \in I)$ and $(\sqrt{c}V)_I$ for $\det(\sqrt{c_i}v_i; i \in I)^T$.

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Applying the Cauchy–Binet formula, we obtain

$$1 = \det I_n = \det\left(\sum_{i=1}^{m} c_i u_i \otimes v_i\right) = \sum_{\substack{|I|=n \\ I\subseteq\{1,2,\ldots,m\}}} (\sqrt{c}U)_I(\sqrt{c}V)_I \tag{3.2}$$

and

$$\det\left(\sum_{i=1}^{m} c_i \lambda_i u_i \otimes v_i\right) = \sum_{\substack{|I|=n \\ I\subseteq\{1,2,\ldots,m\}}} \lambda_I(\sqrt{c}U)_I(\sqrt{c}V)_I.$$

Being a positive double John basis $\{(u_i, v_i)\}_1^m$, it is easy to verify that

$$\det\left(\sum_{i\in I, |I|=n} u_i \otimes v_i\right) = U_I V_I \geq 0.$$

This guarantees that the coefficients $(\sqrt{c}U)_I(\sqrt{c}V)_I$ are all non-negative. Then applying the arithmetic–geometric means inequality [**10**] with coefficients $(\sqrt{c}U)_I(\sqrt{c}V)_I$, we get

$$\begin{aligned}\sum_{\substack{|I|=n \\ I\subseteq\{1,2,\ldots,m\}}} \lambda_I(\sqrt{c}U)_I(\sqrt{c}V)_I &\geq \prod_{\substack{|I|=n \\ I\subseteq\{1,2,\ldots,m\}}} \lambda_I^{(\sqrt{c}U)_I(\sqrt{c}V)_I} \\ &= \prod_{j=1}^{m} \lambda_j^{\sum_{j\in I,|I|=n}(\sqrt{c}U)_I(\sqrt{c}V)_I}.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{j\in I,|I|=n} (\sqrt{c}U)_I(\sqrt{c}V)_I &= \sum_{|I|=n}(\sqrt{c}U)_I(\sqrt{c}V)_I - \sum_{j\notin I,|I|=n} (\sqrt{c}U)_I(\sqrt{c}V)_I \\ &= \det\left(\sum_{i=1}^{m} c_i u_i \otimes v_i\right) - \det(I_n - c_j u_j \otimes v_j) \\ &= 1 - \det(I_n)\det(1 - c_j\langle u_j, v_j\rangle) \\ &= c_j,\end{aligned}$$

since $\langle u_j, v_j\rangle = 1$.

Thus, we obtain that

$$\det\left(\sum_{i=1}^{m} c_i \lambda_i u_i \otimes v_i\right) \geq \prod_{i=1}^{m} \lambda_i^{c_i}. \tag{3.3}$$

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By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\det\left(\sum_{i=1}^{m} c_i\lambda_i u_i \otimes u_i\right)\det\left(\sum_{i=1}^{m} c_i\delta_i v_i \otimes v_i\right)\\
&= \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\dots,m\}}} \lambda_I(\sqrt{c}U)_I^2 \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\dots,m\}}} \delta_I(\sqrt{c}V)_I^2\\
&\geq \left(\sum_{\substack{|I|=n\\ I\subseteq\{1,2,\dots,m\}}} c_I\sqrt{\lambda_I\delta_I}\,U_I V_I\right)^2 .
\end{aligned}$$

Then applying (3.3), we deduce (3.1), and complete the proof. □

REMARK 1. Lemma 3.2 is a revisional version of the original Proposition 4.4 in [7]. In the original proof, the coefficients $(\sqrt{c}U)_I(\sqrt{c}V)_I$ are not necessarily positive and we can find a counter example. For example, in $\mathbb{R}^2$, suppose $m = 3$. Let $u_1 = (1, 2)$, $u_2 = (\frac{3}{2}, \frac{1}{2})$, $u_3 = (-1, \frac{18}{161})$, $v_1 = (\frac{1}{2}, \frac{1}{4})$, $v_2 = (-\frac{1}{3}, 3)$, $v_3 = (-\frac{161}{194}, \frac{1771}{1164})$ and $c_1 = \frac{1}{5}$, $c_2 = \frac{96}{215}$, $c_3 = \frac{291}{215}$. It is easy to check that they satisfy $\sum_{i=1}^{3} c_i u_i \otimes v_i = I_n$, $c_1 + c_2 + c_3 = 2$ and $\langle u_i, v_i\rangle = 1$, $i = 1, 2, 3$. But $\det(u_1, u_2)\cdot\det(v_1, v_2) < 0$. If we put $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 100$ and $\delta_i = 1$, $i = 1, 2, 3$, then the inequality of Proposition 4.4 in [7] does not hold. We realise that the condition of the positive coefficients is necessary. In order to complete the proof of Lemma 3.2, the condition (1.5) has to be added to the definition of double John basis. Since

$$\sum_{|I|=n}(\sqrt{c}U)_I(\sqrt{c}V)_I = 1,$$

the double John basis with the restricted condition $U_I V_I \geq 0$ always exists. For example, let $u_1 = (1, \frac{1}{2})$, $u_2 = (-1, \frac{1}{2})$, $u_3 = (-1, -\frac{1}{2})$, $v_1 = (\frac{1}{2}, 1)$, $v_2 = (-\frac{1}{2}, 1)$, $v_3 = (-\frac{1}{2}, -1)$ and $c_1 = \frac{1}{2}$, $c_2 = 1$, $c_3 = \frac{1}{2}$. It is easy to check that $\{(u_i, v_i)\}_1^3$ is a positive double John basis of $\mathbb{R}^2$, with weights c_1, c_2, c_3.

Proof of Theorem 1.1. Let $I \subseteq \{1, 2, \dots, m\}$. Denote by $|I|$ its cardinality. Write $\alpha_I = \prod_{i\in I}\alpha_i$, $\beta_I = \prod_{i\in I}\beta_i$ and $c_I = \prod_{i\in I} c_i$. For $\alpha_i, \beta_i > 0$, $i = 1, \dots, m$, let

$$Z_1 = \sum_{i=1}^{m}[-\alpha_i u_i, \alpha_i u_i], \quad Z_2 = \sum_{i=1}^{m}[-\beta_i v_i, \beta_i v_i].$$

Then by (2.4), we have

$$V(Z_1) = 2^n \sum_{1\leq i_1<\cdots<i_n\leq m} |\det(\alpha_{i_1}u_{i_1}, \dots, \alpha_{i_n}u_{i_n})| = 2^n \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\dots,m\}}} \alpha_I |\det_{i\in I}(u_i)| \tag{3.4}$$

and

$$V(Z_2) = 2^n \sum_{1\leq i_1<\cdots<i_n\leq m} |\det(\beta_{i_1}v_{i_1}, \dots, \beta_{i_n}v_{i_n})| = 2^n \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\dots,m\}}} \beta_I |\det_{i\in I}(v_i)|. \tag{3.5}$$

For sufficiently small $\varepsilon > 0$, we take two sequences of positive numbers λ_i, δ_i, $i = 1, 2, \ldots, m$, such that

$$c_I\lambda_I^{\varepsilon} = \prod_{i\in I} c_i\lambda_i^{\varepsilon} \geq 1, \quad c_I\delta_I^{\varepsilon} = \prod_{i\in I} c_i\delta_i^{\varepsilon} \geq 1.$$

Therefore, by inequality (2.5), we have

$$\begin{aligned}
&\det\left(\sum_{i=1}^{m} c_i\lambda_i u_i \otimes u_i\right) \det\left(\sum_{i=1}^{m} c_i\delta_i v_i \otimes v_i\right)\\
&= \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}} c_I\lambda_I\Big(\det_{i\in I}(u_i)\Big)^2 \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}} c_I\delta_I\Big(\det_{i\in I}(v_i)\Big)^2\\
&= \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}} (c_I\lambda_I^{\varepsilon})\lambda_I^{1-\varepsilon}\Big(\det_{i\in I}(u_i)\Big)^2 \sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}} (c_I\delta_I^{\varepsilon})\delta_I^{1-\varepsilon}\Big(\det_{i\in I}(v_i)\Big)^2\\
&\leq \left(\sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}} c_I\lambda_I^{\frac{1+\varepsilon}{2}}\,|\det_{i\in I}(u_i)|\right)^2 \left(\sum_{\substack{|I|=n\\ I\subseteq\{1,2,\ldots,m\}}} c_I\delta_I^{\frac{1+\varepsilon}{2}}\,|\det_{i\in I}(v_i)|\right)^2\\
&= \left(\frac{V(\sum_{i=1}^{m} c_i\lambda_i^{\frac{1+\varepsilon}{2}}[-u_i, u_i])}{2^n}\right)^2 \left(\frac{V(\sum_{i=1}^{m} c_i\delta_i^{\frac{1+\varepsilon}{2}}[-v_i, v_i])}{2^n}\right)^2.
\end{aligned}$$

Put $c_i\lambda_i^{\frac{1+\varepsilon}{2}} = \alpha_i$, $c_i\delta_i^{\frac{1+\varepsilon}{2}} = \beta_i$. By (3.4), (3.5) and Lemma 3.2, we obtain that

$$V(Z_1)V(Z_2) \geq 2^{2n}\prod_{i=1}^{m}\left(\left(\frac{\alpha_i}{c_i}\right)^{\frac{2}{1+\varepsilon}}\right)^{\frac{c_i}{2}} \prod_{i=1}^{m}\left(\left(\frac{\beta_i}{c_i}\right)^{\frac{2}{1+\varepsilon}}\right)^{\frac{c_i}{2}} = 2^{2n}\prod_{i=1}^{m}\left(\frac{\alpha_i\beta_i}{c_i^2}\right)^{\frac{c_i}{1+\varepsilon}}.$$

Since the all c_i are fixed and λ_i, δ_i were taken arbitrarily, we can let $\varepsilon \to 0$ to obtain

$$V(Z_1)V(Z_2) \geq 2^{2n}\prod_{i=1}^{m}\left(\frac{\alpha_i\beta_i}{c_i^2}\right)^{c_i}. \tag{3.6}$$

Now for each i, let

$$\alpha_i = \frac{c_i}{|u_i|V_{n-1}(P_{u_i}K)}, \quad \beta_i = \frac{c_i}{|v_i|V_{n-1}(P_{v_i}K)}.$$

From Minkowski's first inequality (2.2), (2.3) and (1.4), we obtain

$$
\begin{aligned}
V(K)^{2(n-1)} &\leq V(Z_1)^{-1}V(Z_2)^{-1}V_1(K,Z_1)^n V_1(K,Z_2)^n \\
&= V(Z_1)^{-1}V(Z_2)^{-1}\left(\frac{2}{n}\sum_{i=1}^{m}\alpha_i|u_i|V_{n-1}(P_{u_i}K)\right)^n\left(\frac{2}{n}\sum_{i=1}^{m}\beta_i|v_i|V_{n-1}(P_{v_i}K)\right)^n \\
&= 2^{2n}V(Z_1)^{-1}V(Z_2)^{-1} \\
&\leq 2^{2n}\left(2^{2n}\prod_{i=1}^{m}\left(\frac{\alpha_i\beta_i}{c_i^2}\right)^{c_i}\right)^{-1} \\
&= \prod_{i=1}^{m}(|u_i||v_i|V_{n-1}(P_{u_i}K)V_{n-1}(P_{v_i}K))^{c_i}.
\end{aligned}
$$

This completes the proof. □

REMARK 2. If $u_i = v_i$ are unit vectors for all i, then the positive double John basis will become a John basis, with weights $c_1, \ldots, c_m$ and Theorem 1.1 coincides with Ball's result (1.6). In fact, by the Cauchy–Binet formula, (1.1) implies that

$$
1 = \det I_n = \det\left(\sum_{i=1}^{m} c_i u_i \otimes v_i\right) = \sum_{|I|=n} c_I U_I^2.
$$

Note that the coefficients U_I^2 are always non-negative.

Proof of Theorem 1.2. For $\alpha_i, \beta_i > 0$, $i = 1, 2, \ldots, m$, let $Z_1 = \sum_{i=1}^{m}\alpha_i[-u_i, u_i]$ and $Z_2 = \sum_{i=1}^{m}\beta_i[-v_i, v_i]$. Since A is a positive definite matrix, there exists a non-singular matrix Q such that

$$
Q^TQ = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\sum_{i=1}^{m} u_i \otimes v_i + \sum_{i=1}^{m} v_i \otimes u_i\right).
$$

Let $y = Qx$ for $x \in \mathbb{R}^n$. Then

$$
\begin{aligned}
|y|^2 = \langle Qx, Qx\rangle &= \left\langle \frac{1}{2}(A + A^T)x, x\right\rangle = \frac{1}{2}\langle Ax, x\rangle + \frac{1}{2}\langle A^Tx, x\rangle \\
&= \frac{1}{2}\sum_{i=1}^{m}\langle u_i, x\rangle\langle v_i, x\rangle + \frac{1}{2}\sum_{i=1}^{m}\langle v_i, x\rangle\langle u_i, x\rangle \\
&= \sum_{i=1}^{m}\langle u_i, x\rangle\langle v_i, x\rangle \\
&= \sum_{i=1}^{m} c_i\langle \tilde{u}_i, y\rangle\langle \tilde{v}_i, y\rangle, \qquad (3.7)
\end{aligned}
$$

where $\tilde{u}_i = c_i^{-\frac{1}{2}}Q^{-T}u_i$, $\tilde{v}_i = c_i^{-\frac{1}{2}}Q^{-T}v_i$ and $c_i = \langle Q^{-1}Q^{-T}u_i, v_i\rangle$. It follows that

$$
\langle \tilde{u}_i, \tilde{v}_i\rangle = c_i^{-1}\langle Q^{-1}Q^{-T}u_i, v_i\rangle = 1
$$

and

$$\det\left(\sum_{i\in I,|I|=n}\tilde{u}_i\otimes\tilde{v}_i\right)=c_I^{-1}\det\left(Q^{-T}\left(\sum_{i\in I,|I|=n}u_i\otimes v_i\right)Q^{-1}\right)\geq 0.$$

By the definition of the positive double John basis, (1.3) and (3.7), it is easy to check that $\{(\tilde{u}_i,\tilde{v}_i)\}_1^m$ is a positive double John basis with weights $c_1,\ldots,c_m$.

So we have

$$Z_1=\sum_{i=1}^m\alpha_i[-u_i,u_i]=\sum_{i=1}^m\alpha_i\big[-c_i^{\frac{1}{2}}Q^T\bar{u}_i,c_i^{\frac{1}{2}}Q^T\bar{u}_i\big]=\sum_{i=1}^m\alpha_ic_i^{\frac{1}{2}}[-Q^T\tilde{u}_i,Q^T\tilde{u}_i]$$

and

$$Z_2=\sum_{i=1}^m\beta_ic_i^{\frac{1}{2}}[-Q^T\tilde{v}_i,Q^T\tilde{v}_i].$$

Multiplying Q^{-T} on both sides leads to

$$Q^{-T}Z_1=\sum_{i=1}^m\alpha_ic_i^{\frac{1}{2}}[-\tilde{u}_i,\tilde{u}_i]$$

and

$$Q^{-T}Z_2=\sum_{i=1}^m\beta_ic_i^{\frac{1}{2}}[-\tilde{v}_i,\tilde{v}_i].$$

By (3.6), we obtain

$$\det(Q^{-T})^2V(Z_1)V(Z_2)\geq 2^{2n}\prod_{i=1}^m\left(\frac{\alpha_ic_i^{\frac{1}{2}}\beta_ic_i^{\frac{1}{2}}}{c_i^2}\right)^{c_i}=2^{2n}\prod_{i=1}^m\left(\frac{\alpha_i\beta_i}{c_i}\right)^{c_i}.$$

Noticing $\det(\frac{A+A^T}{2})=(\det Q^T)^2$, it follows that

$$V(Z_1)V(Z_2)\geq 2^{2n}\det\left(\frac{A+A^T}{2}\right)\prod_{i=1}^m\left(\frac{\alpha_i\beta_i}{c_i}\right)^{c_i}.$$

For each i, let

$$\alpha_i=\frac{c_i}{|u_i|V_{n-1}(P_{u_i}K)},\quad \beta_i=\frac{c_i}{|v_i|V_{n-1}(P_{v_i}K)}.$$

From (2.2), (2.3) and (1.4), we have

$$\begin{aligned} V(K)^{2(n-1)} &\le V(Z_1)^{-1} V_1(K, Z_1)^n V(Z_2)^{-1} V_1(K, Z_2)^n \\ &= V(Z_1)^{-1} V(Z_2)^{-1} \left(\frac{2}{n} \sum_{i=1}^{m} \alpha_i |u_i| V_{n-1}(P_{u_i} K) \right)^n \left(\frac{2}{n} \sum_{i=1}^{m} \beta_i |v_i| V_{n-1}(P_{v_i} K) \right)^n \\ &= 2^{2n} V(Z_1)^{-1} V(Z_2)^{-1} \\ &\le 2^n \left(2^{2n} \det\left(\frac{A + A^T}{2} \right) \prod_{i=1}^{m} \left(\frac{\alpha_i \beta_i}{c_i} \right)^{c_i} \right)^{-1} \\ &= \det\left(\frac{A + A^T}{2} \right)^{-1} \prod_{i=1}^{m} \left(\frac{|u_i||v_i| V_{n-1}(P_{u_i} K) V_{n-1}(P_{v_i} K)}{c_i} \right)^{c_i}. \end{aligned}$$

This gives the result. □

REMARK 3. The case of $u_i = v_i$ for each i was proved by Zhang [**18**]. If $m = n$ and $u_i = v_i = e_i$, we obtain the classical Loomis–Whitney inequality.

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