TAUBERIAN ESTIMATES CONCERNING THE REGULAR HAUSDORFF AND [J, f(x)] TRANSFORMATIONS

A. MEIR

1. Introduction. Denote by $\{t(x)\}$ some linear transform of the sequence

$$\{s_n\} \qquad (n \geqslant 0,\, s_n = a_0 + a_1 + \ldots + a_n),$$

of the form

$$t(x) = \sum_{k=0}^{\infty} c_k(x) s_k \qquad (x \geqslant x_0 \geqslant 0),$$

where x attains continuous or only integer values. The problem of estimating $|t(x) - s_m|$ as x and m tend to ∞ with some connection between them was considered first by H. Hadwiger **(3)** assuming the Tauberian condition $na_n = O(1)$ on the sequence $\{s_k\}$, specifying the transform $t(x)$ to be the usual Abel transform and $x = 1 - n^{-1}$. Papers of Agnew **(1)**, Garten **(2)**, and Jakimovski **(5, 6)** deal with similar problems concerning other transformation methods.

The same problem but replacing the Tauberian condition $na_n = O(1)$ by $b_n = O(1)$, where

$$b_n = (n+1)^{-1}(a_1 + 2a_2 + \ldots + na_n), \tag{1.1}$$

was solved for special transformation methods by V. Garten **(2)** and P. Hartman **(4)**.

In this paper our aim is to state and prove the corresponding results under the condition (1.1) for a class of regular Hausdorff and $[J, f(x)]$ transforms. The Abel transform and the Cesàro-transform of order $\alpha \geqslant 1$ are included in our theorem as special cases.

2. Definitions, notations, and lemmas. The regular Hausdorff-transformation is defined as follows: Let $\beta(t)$ be a function of bounded variation on $[0, 1]$, satisfying

$$\beta(0+) = \beta(0) = 0, \qquad \beta(1) = 1. \tag{2.1}$$

The Hausdorff-transform $H_n(\beta)$ of a sequence $\{s_k\}$ is

$$H_n(\beta) = \sum_{k=0}^{n} \binom{n}{k} s_k \int_0^1 u^k (1-u)^{n-k} d\beta(u), \qquad n \geqslant 0. \tag{2.2}$$

The $[J, f(x)]$-transformation was defined in **(7)** by Jakimovski as follows: Let

Received December 6, 1963.

$$f(x) = \int_0^1 t^x d\beta(t),$$

where $\beta(t)$ is a function of bounded variation on [0, 1] satisfying

(2.3) $$\beta(0+) = \beta(0) = 0, \qquad \beta(1-) = \beta(1) = 1,$$

and let the $[J, f(x)]$-transform, say $J_x(\beta)$ of a sequence $\{s_k\}$, be defined by

(2.4) $$J_x(\beta) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} f^{(k)}(x) s_k = \sum_{k=0}^{\infty} \frac{x^k}{k!} s_k \int_0^1 t^x \left(\log \frac{1}{t}\right)^k d\beta(t)$$

for $x \geqslant 0$. The regularity of this transformation has been proved in **(7)**.

We shall use the following notations:

(2.5) $$d_k \equiv d_k(x) = \frac{x^k}{k!} \int_0^1 t^x \left(\log \frac{1}{t}\right)^k d\beta(t), \qquad k \geqslant 0;$$

(2.6) $$D_k \equiv D_k(x) = \sum_{j=k}^{\infty} d_j, \qquad k \geqslant 0;$$

(2.7) $$\delta_k \equiv \delta_k(n) = \binom{n}{k} \int_0^1 u^k (1-u)^{n-k} d\beta(u), \qquad k \geqslant 0.$$

(2.8) $$\Delta_k \equiv \Delta_k(n) = \begin{cases} \sum_{j=k}^{n} \delta_j & \text{if} \quad 0 \leqslant k \leqslant n, \\ 0 & \phantom{\text{if}} \quad k \geqslant n+1. \end{cases}$$

We shall use in our proofs the following lemmas:

LEMMA 1. *If the $[J, f(x)]$-transformation is regular and $0 \leqslant \beta(t) \leqslant 1$ on* [0, 1], *then*

(2.9) $$0 \leqslant D_k(x) \leqslant 1, \qquad k \geqslant 0, x > 0.$$

If the $H_n(\beta)$-transform is regular and $0 \leqslant \beta(t) \leqslant 1$ on [0, 1], *then*

(2.10) $$0 \leqslant \Delta_k(n) \leqslant 1, \qquad k \geqslant 0, n \geqslant 0.$$

Proof. By the regularity of the transformation, $D_0(x) = 1$ **(7**, p. 142**)**. Let $k \geqslant 1$. By (2.5) and (2.6) and changing the variable of integration to $u = x \log(1/t)$, we obtain

(2.11) $$1 - D_k = \frac{1}{(k-1)!} \int_0^{\infty} e^{-u} u^{k-1} (1 - \beta(e^{-u/x})) du$$

and since $0 \leqslant \beta(t) \leqslant 1$, we have at once $0 \leqslant 1 - D_k \leqslant 1$. So (2.9) is proved. The proof of (2.10) is completely analogous.

https://doi.org/10.4153/CJM-1965-029-0 Published online by Cambridge University Press

LEMMA 2. *For every* ϵ $(0 < \epsilon < 1)$ *and every* $u \geqslant 0$

$$\sum_{|k-u|\geqslant \epsilon u} \frac{u^k}{k!} e^{-u} \leqslant \frac{K_\epsilon}{1+u^2}, \tag{2.12}$$

K_ϵ *being dependent only on* ϵ; *and for every* ϵ $(0 < \epsilon < 1)$, *every* $n \geqslant 0$ *and every* u, $0 \leqslant u \leqslant 1$,

$$\sum_{|k-nu|\geqslant \epsilon n} \binom{n}{k} u^k (1-u)^{n-k} \leqslant \frac{K'_\epsilon}{1+n^2}, \tag{2.13}$$

K'_ϵ *being dependent only on* ϵ.

Proof. Let $u \geqslant 1$. Then

$$\sum_{|k-u|\geqslant \epsilon u} \frac{u^k}{k!} e^{-u} \leqslant \frac{1}{\epsilon^4 u^4} \sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u} (k-u)^4.$$

By easy calculation, this is equal to

$$\frac{1}{\epsilon^4 u^4}(3u^2 + u) \leqslant \frac{4}{\epsilon^4} \cdot \frac{1}{u^2} \leqslant \frac{8}{\epsilon^4} \frac{1}{u^2+1}.$$

For $0 \leqslant u < 1$ the sum on the left-hand side of (2.12) is clearly $\leqslant 1$. So $K_\epsilon = 8\epsilon^{-4}$ is suitable for all $u \geqslant 0$. Thus (2.12) is proved. The proof of (2.13) is completely analogous. K'_ϵ may be chosen as ϵ^{-4}.

LEMMA 3. *Suppose the* $[J, f(x)]$*-transformation is regular,*

$$0 \leqslant \beta(t) \leqslant 1, \tag{2.14}$$

and

$$\begin{cases} \text{(i)} & \int_0 u^{-1}(1-\beta(e^{-u}))du < +\infty, \\ \text{(ii)} & \int_q^\infty u^{-1} V(e^{-u})du < +\infty, \end{cases} \tag{2.15}$$

where $V(t)$ *is the variation of* $\beta(u)$ *from* 0 *to* t. *Then for every* $q > 0$

$$\lim_{m\to\infty, x\to\infty, m/x \to q} \left\{ \sum_{k=1}^{m} k^{-1}(1-D_k) + \sum_{K=m+1}^{\infty} k^{-1} D_k \right\} = A_q, \tag{2.16}$$

where

$$A_q = \int_0^q t^{-1}(1-\beta(e^{-t}))dt + \int^\infty t^{-1}\beta(e^{-t})dt. \tag{2.17}$$

Remark. In fact, Lemma 3 still remains true if we assume instead of (2.15) (ii) the weaker condition

$$\int^\infty u^{-1}\beta(e^{-u})du < +\infty.$$

If $\beta(t)$ is non-decreasing on $[0, 1]$, this condition is equivalent to (2.15) (ii). Thus Lemma 3 improves Jakimovski's result (**6**, Theorem (3.1)), where $na_n = O(1)$ was supposed.

The significance of (2.15)(ii) will appear in the proof of Theorem 1.

Proof. Let ϵ, $0 < \epsilon < 1$, be fixed. By (2.11),

$$(2.18)\quad S_1 \equiv \sum_{k=1}^{m} k^{-1}(1 - D_k) = \int_0^\infty \sum_{k=1}^{m} \frac{u^k}{k!} e^{-u} \cdot u^{-1}(1 - \beta(e^{-u/x}))du$$
$$= \int_0^{m(1-\epsilon)} + \int_{m(1-\epsilon)}^{m(1+\epsilon)} + \int_{m(1+\epsilon)}^{\infty} \equiv I_1 + I_2 + I_3.$$

Now

$$\sum_{k=1}^{m} \frac{u^k}{k!} e^{-u} = 1 - e^{-u} - \sum_{k=m+1}^{\infty} \frac{u^k}{k!} e^{-u}$$

and if $u \leqslant m(1 - \epsilon)$, clearly $m \geqslant u + \epsilon u$; thus by (2.12)

$$\sum_{k=1}^{m} \frac{u^k}{k!} e^{-u} = 1 - O((1 + u^2)^{-1}).$$

Thus

$$I_1 = \int_0^{m(1-\epsilon)/x} u^{-1}(1 - \beta(e^{-u}))\{1 - O([1 + (xu)^2]^{-1})\}du$$

and by (2.15)(i)

$$(2.19)\qquad I_1 = \int_0^{q(1-\epsilon)} u^{-1}(1 - \beta(e^{-u}))du + o(1)$$

as $m \to \infty$, $x \to \infty$, $mx^{-1} \to q$.

By (2.14)

$$(2.20)\qquad |I_2| \leqslant \int_{m(1-\epsilon)}^{m(1+\epsilon)} u^{-1}\, du = \log \frac{1 + \epsilon}{1 - \epsilon}.$$

If $u \geqslant m(1 + \epsilon)$, clearly $u - \frac{1}{2}\epsilon u \geqslant m$; thus by (2.12) and (2.14)

$$(2.21)\qquad I_3 = \int_{m(1+\epsilon)}^{\infty} u^{-1}O(u^{-2})du = o(1) \qquad \text{as } m \to \infty.$$

Now $\epsilon > 0$ was arbitrary. Let $\epsilon \to 0$. From (2.19), (2.20), and (2.21),

$$(2.22)\qquad \lim S_1 = \int_0^q u^{-1}(1 - \beta(e^{-u}))du$$

as $m \to \infty$, $x \to \infty$, $m/x \to q$.

By (2.11) and changing the order of summation and integration, as justified by (2.15)(ii),

$$(2.23)\quad S_2 \equiv \sum_{k=m+1}^{\infty} k^{-1}D_k = \int_0^\infty \sum_{k=m+1}^{\infty} \frac{u^k}{k!} e^{-u}u^{-1}\beta(e^{-u/x})du$$
$$= \int_0^{m(1-\epsilon)} + \int_{m(1-\epsilon)}^{m(1+\epsilon)} + \int_{m(1+\epsilon)}^{\infty} \equiv I'_1 + I'_2 + I'_3.$$

Now by (2.14)

$$|I'_1| \leqslant \int_0^{m(1-\epsilon)} \frac{1}{m+1} \sum_{k=m}^{\infty} \frac{u^k}{k!} e^{-u} du$$

and since $m \geqslant u + \epsilon u$, (2.12) implies that

$$(2.24) \qquad |I'_1| \leqslant \frac{1}{m+1} \int_0^m O((1+u^2)^{-1}) du = o(1) \qquad \text{as } m \to \infty,$$

$$(2.25) \qquad |I'_2| \leqslant \int_{m(1-\epsilon)}^{m(1+\epsilon)} u^{-1}\, du = \log \frac{1+\epsilon}{1-\epsilon},$$

$$I'_3 = \int_{m(1+\epsilon)}^{\infty} \left(1 - \sum_{k=0}^{m} \frac{u^k}{k!} e^{-u}\right) u^{-1} \beta(e^{-u/x}) du,$$

and since $m \leqslant u - \frac{1}{2}\epsilon u$, by (2.12),

$$(2.26) \qquad I'_3 = \int_{m(1+\epsilon)}^{\infty} (1 - O(u^{-2})) u^{-1} \beta(e^{-u/x}) du$$

$$= \int_{q(1+\epsilon)}^{\infty} u^{-1} \beta(e^{-u}) du + o(1)$$

as $m \to \infty$, $x \to \infty$, $mx^{-1} \to q$.

Since $\epsilon > 0$ was arbitrary, by (2.23)–(2.26),

$$(2.27) \qquad \lim S_2 = \int_q^{\infty} u^{-1} \beta(e^{-u}) du$$

as $m \to \infty$, $x \to \infty$, $mx^{-1} \to q$. Equations (2.22) and (2.27) prove the lemma.

LEMMA 4. *Suppose that the $H_n(\beta)$-transformation is regular,*

$$(2.28) \qquad 0 \leqslant \beta(t) \leqslant 1,$$

and

$$(2.29) \qquad \int_0^1 u^{-1} \beta(u) du < +\infty.$$

Then for every $q > 0$

$$(2.30) \qquad \lim_{m\to\infty,\ n\to\infty,\ mn^{-1}\to q} \left\{\sum_{k=1}^{m} k^{-1}(1-\Delta_k) + \sum_{k=m+1}^{m+n} k^{-1}\Delta_k\right\} = B_q,$$

where $\Delta_k = 0$ when $k \geqslant n+1$ and

$$(2.31) \qquad B_q = \begin{cases} \int_0^q t^{-1}\beta(t)dt + \int_q^1 t^{-1}(1-\beta(t))dt, & q \leqslant 1, \\ \int_0^1 t^{-1}\beta(t)dt + \log q, & q \geqslant 1. \end{cases}$$

Remark. Lemma 4 improves Jakimovski's result (5, Theorem 1) for the case $na_n = O(1)$, since only (2.28) is assumed instead of the more restrictive assumption that $\beta(t)$ is non-decreasing on $[0, 1]$.

Proof. From (2.7) we get easily, by integration by parts and summation,

$$\Delta_k = k \int_0^1 \binom{n}{k}(1-t)^{n-k}t^k t^{-1}(1-\beta(t))dt, \qquad k \geqslant 1, \tag{2.32}$$

$$1 - \Delta_k = k \int_0^1 \binom{n}{k}(1-t)^{n-k}t^k t^{-1}\beta(t)dt, \qquad 1 \leqslant k \leqslant n. \tag{2.33}$$

Now if $m \geqslant n$,

$$\sum_{k=1}^{m} k^{-1}(1-\Delta_k) = \sum_{k=1}^{n} k^{-1}(1-\Delta_k) + \sum_{k=n+1}^{m} k^{-1}. \tag{2.34}$$

By (2.33) this is equal to

$$\int_0^1 (1-(1-t)^n)t^{-1}\beta(t)dt + \sum_{k=n+1}^{m} k^{-1},$$

which by (2.29) becomes

$$\int_0^1 t^{-1}\beta(t)dt + \log q + o(1)$$

as $m \to \infty$, $n \to \infty$, $mn^{-1} \to q$.

If $m < n$ we have, by (2.32) and (2.33),

$$\begin{aligned} \sum_{k=1}^{m} k^{-1}(1-\Delta_k) + \sum_{k=m+1}^{n} k^{-1}\Delta_k &= \int_0^1 \sum_{k=1}^{m} \binom{n}{k}(1-t)^{n-k}t^k t^{-1}\beta(t)dt \\ &\quad + \int_0^1 \sum_{k=m+1}^{n} \binom{n}{k}(1-t)^{n-k}t^k t^{-1}(1-\beta(t))dt \\ &\equiv I_1 + I_2. \end{aligned} \tag{2.35}$$

Let ϵ, $0 < \epsilon < 1$, be fixed, and denote for brevity $mn^{-1}(1-\epsilon) = \theta_1$, $mn^{-1}(1+\epsilon) = \theta_2$.

$$I_1 = \int_0^{\theta_1} + \int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{1} \equiv I_{11} + I_{12} + I_{13}. \tag{2.36}$$

If $0 \leqslant t \leqslant \theta_1$, clearly $m - nt \geqslant m\epsilon$, and since $mn^{-1} \to q > 0$ we may assume that $m - nt \geqslant 2^{-1}\epsilon qn$. Thus by Lemma 2, (2.13), we have

$$\begin{aligned} I_{11} &= \int_0^{\theta_1} \{1-(1-t)^n - O(n^{-2})\}t^{-1}\beta(t)dt \\ &= \int_0^{q(1-\epsilon)} t^{-1}\beta(t)dt + o(1) \end{aligned} \tag{2.37}$$

as $n \to \infty$, $m \to \infty$, $mn^{-1} \to q$;

$$(2.38) \qquad |I_{12}| \leqslant \int_{\theta_1}^{\theta_2} t^{-1}dt = \log\frac{1+\epsilon}{1-\epsilon}.$$

If $\theta_2 \leqslant t \leqslant 1$, it is easily seen that $tn - m \geqslant m\epsilon \geqslant 2^{-1}\epsilon qn$. Thus, by (2.13),

$$(2.39) \qquad |I_{13}| \leqslant \int_{\theta_2}^{1} O(n^{-2})t^{-1}dt = o(1) \qquad \text{as } n \to \infty, m \to \infty, m/n \to q.$$

From (2.36)–(2.39) and since $\epsilon > 0$ is arbitrary,

$$(2.40) \qquad I_1 = \int_0^q t^{-1}\beta(t)dt + o(1) \qquad \text{as } m \to \infty, n \to \infty, m/n \to q.$$

Using exactly the same reasoning,

$$(2.41) \qquad I_2 = \int_q^1 t^{-1}(1 - \beta(t))dt + o(1).$$

If $q > 1$, we have for sufficiently large values of m and n that $m > n$. Thus (2.34) proves the lemma in this case. If $q < 1$, then $m < n$ for sufficiently large values of m and n. Thus (2.35), (2.40), and (2.41) prove the lemma. If $q = 1$, the expressions (2.34) and (2.35) both tend to the same limit, since for $q = 1$ both expressions defining B_q in (2.31) are equal. This completes our proof.

LEMMA 5. *For* $u \geqslant 2$

$$(2.42) \qquad S \equiv \sum_{k=2}^{\infty} \log k \frac{u^k}{k!} e^{-u} = O(\log u).$$

Proof.

$$\log u - S = \log u(1+u)e^{-u} + \sum_{k=2}^{\infty} \log\frac{u}{k}\frac{u^k}{k!}e^{-u}.$$

Thus

$$(2.43) \qquad |\log u - S| \leqslant O(\log u) + \sum_{k=2}^{\infty} \left|\log\frac{u}{k}\right| \frac{u^k}{k!} e^{-u}$$

$$= O(\log u) + \sum_{2 \leqslant k \leqslant 2u} + \sum_{k > 2u}$$

$$\equiv O(\log u) + \sigma_1 + \sigma_2.$$

Now trivially in the first sum $|\log(u/k)| \leqslant \log u$; so

$$(2.44) \qquad \sigma_1 \leqslant \log u.$$

In the second sum $|\log(k/u)| \leqslant k$; thus by (2.12)

$$(2.45) \qquad \sigma_2 \leqslant O(u^{-1}) = O(1).$$

LEMMA 6 (Agnew **6**). *Suppose* $\{b_k\}$ $(k \geqslant 1)$ *is a bounded sequence. Let* $\{c_k(x)\}$ *be a sequence of functions defined for* $x > 0$ *satisfying*

https://doi.org/10.4153/CJM-1965-029-0 Published online by Cambridge University Press

(2.46) $$\lim_{x\to\infty} c_k(x) = 0, \qquad k = 1, 2, \ldots,$$

(2.47) $$\limsup_{x\to\infty} \sum_{k=1}^{\infty} |c_k(x)| = A < \infty.$$

Then

(2.48) $$\limsup_{x\to\infty} \left| \sum_{k=1}^{\infty} c_k(x) b_k \right| \leqslant A \cdot \limsup_{k\to\infty} |b_k|.$$

The constant A *in* (2.48) *is the best possible in the sense that there exists a bounded sequence* $\{b_k\}$ *with* $0 < \limsup |b_k| < \infty$ *and such that both sides of* (2.48) *are equal.*

3. The main theorems.

THEOREM 1. *Suppose that the* $[J, f(x)]$*-transformation is regular, the function* $\beta(t)$ *occurring in* (2.4) *is continuous and satisfies* (2.14) *and* (2.15), *and the functions*

(3.1) $$\text{(i) } t^{-1}(1 - \beta(e^{-t})) \quad \textit{and} \quad \text{(ii) } t^{-1}\beta(e^{-t})$$

are non-increasing for $t > 0$. *Then for every sequence* $\{s_m\}$ *satisfying* (1.1), *for its transform* $J_x(\beta)$ *and for every* $q > 0$

(3.2) $$\limsup_{m\to\infty,\ x\to\infty,\ mx^{-1}\to q} |s_m - J_x(\beta)| \leqslant C_q \cdot \limsup_{n\to\infty} |b_n|,$$

where

(3.3) $$C_q = A_q + 2\beta(e^{-q})$$

and A_q *was defined by* (2.17).

The constant C_q *is the best possible in the sense that there exists a sequence* $\{s_m\}$ *with* $0 < \limsup |b_n| < \infty$ *such that both sides of* (3.2) *are equal.*

THEOREM 2. *Suppose that the* $H_n(\beta)$*-transformation is regular, the function* $\beta(t)$ *occurring in* (2.2) *is continuous and satisfies* (2.28) *and* (2.29), *and the functions*

(3.4) $$\text{(i) } \quad t^{-1}(1 - \beta(t)) \quad \textit{and} \quad \text{(ii) } \quad t^{-1}\beta(t)$$

are non-increasing for $0 < t \leqslant 1$. *Then for every sequence* $\{s_m\}$ *satisfying* (1.1), *for its transform* $H_n(\beta)$, *and for every* $q > 0$,

(3.5) $$\limsup_{m\to\infty,\ n\to\infty,\ mn^{-1}\to q} |s_m - H_n(\beta)| \leqslant D_q \limsup_{n\to\infty} |b_n|,$$

where

(3.6) $$D_q = \begin{cases} B_q + 2(1 - \beta(q)), & q \leqslant 1, \\ B_q, & q \geqslant 1, \end{cases}$$

and B_q *was defined by* (2.31). *The constant* D_q *in* (3.6) *is the best possible in the sense that there exists a sequence* $\{s_m\}$ *with* $0 < \limsup |b_n| < \infty$ *such that both sides of* (3.5) *are equal.*

Examples. (i) For Theorem 1: Let $\beta(t) = t$; then the $J_x(\beta)$-transform is Abel's transform and is easily seen to satisfy the conditions of Theorem 1.

(ii) For Theorem 2: Let $\beta(t) = 1 - (1 - t)^\alpha$, where $\alpha \geqslant 1$; then the $H_n(\beta)$ transform is Cesàro's transform of order α, and satisfies the conditions of Theorem 2.

Proof of Theorem 1. We define $b_0 = 0$. By (1.1) we have

$$a_\nu = \nu^{-1}b_\nu + b_\nu - b_{\nu-1}, \qquad \nu \geqslant 1; \tag{3.7}$$

thus

$$s_k = a_0 + \sum_{\nu=1}^{k} \nu^{-1}b_\nu + b_k \tag{3.8}$$

and since $b_\nu = O(1)$,

$$s_k = O(\log k), \qquad k \geqslant 2. \tag{3.9}$$

Next we show that the transform $J_x(\beta)$ exists for all $x \geqslant 2$. By (2.5) and (3.9)

$$\begin{aligned}\sum_{k=2}^{\infty} |d_k||s_k| &\leqslant \int_0^1 \sum_{k=2}^{\infty} O(\log k)\frac{x^k}{k!}t^x\left(\log\frac{1}{t}\right)^k |d\beta(t)| \\ &= \int_0^{e^{-1}} + \int_{e^{-1}}^1 \equiv J_1 + J_2.\end{aligned} \tag{3.10}$$

Since $\log k < k$ and $\log (1/t) \leqslant 1$ for $t \geqslant e^{-1}$, we have

$$J_2 = O(x)\int_{e^{-1}}^1 |d\beta(t)| = O(x), \tag{3.11}$$

and since for $t \leqslant e^{-1}$, $x \log (1/t) \geqslant x \geqslant 2$, by (2.42),

$$\begin{aligned}J_1 &= O\left\{\int_0^{e^{-1}} \log\left(x\log\frac{1}{t}\right)|d\beta(t)|\right\} \\ &= O(\log x) + O\left\{\int_0^{e^{-1}} \log\log\frac{1}{t}\,|d\beta(t)|\right\}.\end{aligned} \tag{3.12}$$

But

$$\begin{aligned}\int_0^{e^{-1}} \log\log\frac{1}{t}\,|d\beta(t)| &= \int_0^{e^{-1}} |d\beta(t)| \int_t^{e^{-1}} \frac{du}{u\log(1/u)} \\ &= \int_0^{e^{-1}} \frac{du}{u\log(1/u)}\int_0^u |d\beta(t)| = \int_0^{e^{-1}} \frac{V(u)}{u\log(1/u)}\,du,\end{aligned} \tag{3.13}$$

which by (2.15)(ii) is $O(1)$.

By (3.10)–(3.13), for every fixed $x \geqslant 2$

$$\sum_{k=0}^{\infty} |d_k||s_k| < +\infty, \tag{3.14}$$

and

$$\sum_{k=2}^{\infty} |d_k| \log k < +\infty. \tag{3.15}$$

Now for every $N \geqslant 2$, we easily see that

$$\sum_{\nu=0}^{\infty} d_\nu s_\nu - \sum_{k=0}^{N} a_k D_k = \sum_{\nu=N+1}^{\infty} d_\nu s_\nu - \sum_{\nu=N+1}^{\infty} d_\nu s_N.$$

Since by (3.9) $s_N = O(\log N)$, this yields

$$\left| \sum_{\nu=0}^{\infty} d_\nu s_\nu - \sum_{k=0}^{N} a_k D_k \right| \leqslant \sum_{\nu=N+1}^{\infty} |d_\nu||s_\nu| + O\left\{ \sum_{\nu=N+1}^{\infty} |d_\nu| \log \nu \right\}, \tag{3.16}$$

which by (3.14) and (3.15) is $o(1)$ as $N \to \infty$. Thus for every fixed $x \geqslant 2$

$$\begin{aligned} s_m - J_x(\beta) &= \sum_{k=0}^{m} a_k - \sum_{k=0}^{\infty} a_k D_k \\ &= \sum_{k=1}^{m} a_k(1 - D_k) - \sum_{k=m+1}^{\infty} a_k D_k, \end{aligned} \tag{3.17}$$

which by (3.7) is equal to

$$\sum_{k=1}^{m} (k^{-1}(k+1)b_k - b_{k-1})(1 - D_k) - \sum_{k=m+1}^{\infty} (k^{-1}(k+1)b_k - b_{k-1})D_k.$$

Now

$$\begin{aligned} &\sum_{k=m+1}^{M} (k^{-1}(k+1)b_k - b_{k-1})D_k \\ &\qquad = \sum_{k=m+1}^{M} b_k(k^{-1}(k+1)D_k - D_{k+1}) - b_m D_{m+1} + b_M D_{M+1}, \end{aligned}$$

and since by (1.1) and (3.15) $b_M D_{M+1} = o(1)$ as $M \to \infty$, we have

$$\begin{aligned} s_m - J_x(\beta) &= \sum_{k=1}^{m-1} \{k^{-1}(k+1)(1 - D_k) - (1 - D_{k+1})\}b_k \\ &\quad + \{m^{-1}(m+1)(1 - D_m) + D_{m+1}\}b_m - \sum_{k=m+1}^{\infty} \{k^{-1}(k+1)D_k - D_{k+1}\}b_k \\ &\equiv \sum_{k=1}^{\infty} c_k b_k. \end{aligned} \tag{3.18}$$

We want to apply Lemma 6 to the last expression. First by the regularity of the transformation for every fixed $k \geqslant 1$

$$\lim_{x \to \infty} D_k(x) = 1;$$

thus clearly (2.46) is satisfied. For computing the value of the constant A of (2.48) we have to evaluate the upper limit of $\sum |c_k|$ when $x \to \infty$, $m \to \infty$,

$mx^{-1} \to q$. By integration by parts and changing the variable, we obtain from (2.11)

$$c_k = -\int_0^\infty \frac{(ux)^k}{k!} e^{-xu} \cdot ud\left(\frac{1-\beta(e^{-u})}{u}\right) \qquad \text{if } k \leqslant m-1,$$

which by (3.1) (i) is non-negative. By (2.9)

$$c_m \geqslant 0,$$

and for $k \geqslant m+1$

$$c_k = (k+1)x^{-1} \int_0^\infty \frac{(xu)^{k+1}}{(k+1)!} e^{-xu}\, d\left(\frac{\beta(e^{-u})}{u}\right),$$

which by (3.1) (ii) is non-positive. Consequently

$$\text{(3.19)} \qquad \sum_{k=1}^\infty |c_k| = \sum_{k=1}^m k^{-1}(1-D_k) + \sum_{k=m+1}^\infty k^{-1}D_k + 2D_{m+1} + 1 - D_1.$$

Now the limit of the first two sums on the right-hand side is A_q (cf. (2.17)) and by our regularity assumption $1 - D_1 = o(1)$ as $x \to \infty$. By (2.11)

$$\text{(3.20)} \qquad D_{m+1} = \int_0^\infty \frac{u^m}{m!} e^{-u}\beta(e^{-u/x})du.$$

Therefore

$$\begin{aligned} D_{m+1} - \beta(e^{-q}) &= \int_0^\infty \frac{u^m}{m!} e^{-u}\{\beta(e^{-u/x}) - \beta(e^{-q})\}du \\ &= \int_0^{(m-2)(1-\epsilon)} + \int_{(m-2)(1-\epsilon)}^{m(1+2\epsilon)} + \int_{m(1+2\epsilon)}^\infty \\ &\equiv J_1^* + J_2^* + J_3^*. \end{aligned}$$

By (2.12) and (2.14),

$$\text{(3.21)} \qquad J_1^* = O\left\{m^{-1}(m-1)^{-1}\int_0^m du\right\} = o(1) \qquad \text{as } m \to \infty,$$

$$\text{(3.22)} \qquad J_3^* = O\left\{\int_m^\infty u^{-2}du\right\} = o(1) \qquad \text{as } m \to \infty,$$

and since $\beta(t)$ is continuous and $m/x \to q$, for every given $\eta > 0$, if $\epsilon > 0$ is small enough,

$$\text{(3.23)} \qquad |J_2^*| \leqslant \eta \int_0^\infty \frac{u^m}{m!} e^{-u}du = \eta$$

for $m \geqslant m_0$ and $x \geqslant x_0$. In other words, by (3.21)–(3.23)

$$\text{(3.24)} \qquad \lim_{m\to\infty,\ x\to\infty,\ mx^{-1}\to q} D_{m+1} = \beta(e^{-q}).$$

Our Theorem 1 now follows from Lemmas 3 and 6, (3.19) and (3.24).

Proof of Theorem 2. Since $\Delta_k = 0$ for $k \geqslant n+1$, for both $m \leqslant n$ and $n < m$,

$$(3.25) \qquad s_m - H_n(\beta) = \sum_{k=0}^{m} a_k - \sum_{j=0}^{n} \delta_j \sum_{k=0}^{j} a_k = \sum_{k=0}^{m} a_k(1-\Delta_k) - \sum_{k=m+1}^{m+n} a_k \Delta_k.$$

Hence by (3.7), after easy computation,

$$s_m - H_n(\beta) = \sum_{k=1}^{m} \{k^{-1}(k+1)(1-\Delta_k) - (1-\Delta_{k+1})\} b_k + b_m - \sum_{k=m+1}^{m+n} \{k^{-1}(k+1)\Delta_k - \Delta_{k+1}\} b_k \equiv \sum_{k=1}^{m+n} \gamma_k b_k.$$

Now from (2.7) and (2.8), after partial integration and summation,

$$(3.26) \qquad k^{-1}(k+1)(1-\Delta_k) - (1-\Delta_{k+1}) = -\binom{n}{k} \int_0^1 (1-t)^{n-k} t^{k+1} d\left(\frac{\beta(t)}{t}\right),$$

which is non-negative by (3.4)(ii); $k = 1, 2, \ldots$. Also

$$(3.27) \qquad k^{-1}(k+1)\Delta_k - \Delta_{k+1} = \binom{n}{k} \int_0^1 (1-t)^{n-k} t^{k+1} d\left(\frac{1-\beta(t)}{t}\right),$$

which by (3.4)(i) is non-negative; $k = 1, 2, \ldots$. Thus by easy computation

$$(3.28) \qquad \sum_{k=1}^{n+m} |\gamma_k| = \sum_{k=1}^{m} k^{-1}(1-\Delta_k) + \sum_{k=m+1}^{m+n} k^{-1}\Delta_k + 2\Delta_{m+1} + 1 - \Delta_1.$$

Next we want to apply Lemma 6 to (3.25). First, we observe that by the regularity of the transformation for every fixed k, $1 - \Delta_k = o(1)$ as $n \to \infty$; thus (2.46) is satisfied. To compute the constant A of (2.48) we have to evaluate the upper limit of $\sum|\gamma_k|$ as $m \to \infty$, $n \to \infty$, $mn^{-1} \to q$. By Lemma 4, the first two sums of (3.28) tend to B_q; cf. (2.31). By the regularity assumption,

$$(3.29) \qquad \lim_{n\to\infty} (1 - \Delta_1) = 0.$$

If $m \geqslant n$, then $\Delta_{m+1} = 0$ by definition, and thus our theorem is already proved for $q > 1$.

If $m < n$, we have by (2.32) and simple computation

$$(3.30) \quad \Delta_{m+1} - (1-\beta(q)) = \int_0^n \binom{n-1}{m} \left(1-\frac{u}{n}\right)^{n-1-m} \left(\frac{u}{n}\right)^m \left(\beta(q) - \beta\left(\frac{u}{n}\right)\right) du = \int_0^{m(1-\epsilon)} + \int_{m(1-\epsilon)}^{m(1+\epsilon)} + \int_{m(1+\epsilon)}^{n} \equiv J'_1 + J'_2 + J'_3$$

for any fixed ϵ, $0 < \epsilon < 1$. Now, since $mn^{-1} \to q$, we have for sufficiently large m and n if $u \leqslant m(1 - \epsilon)$ that

$$m - \frac{n-1}{n} u \geqslant \tfrac{1}{2}\epsilon q n.$$

Thus by (2.13)

$$J'_1 = O\left\{\frac{1}{n^2} \int_0^m du\right\} = o(1) \qquad \text{as } n \to \infty. \tag{3.32}$$

If $u \geqslant m(1 + \epsilon)$, then

$$\frac{n-1}{n} u - m \geqslant \tfrac{1}{2}\epsilon q n.$$

Thus (2.13) yields

$$J'_3 = O\left\{\frac{1}{n^2} \int_m^n du\right\} = o(1) \qquad \text{as } n \to \infty. \tag{3.33}$$

Since the function $\beta(t)$ is continuous and $mn^{-1} \to q$ for every given $\eta > 0$, if $\epsilon > 0$ is small enough,

$$|J'_2| \leqslant \eta \int_0^n \binom{n-1}{m} \left(1 - \frac{u}{n}\right)^{n-1-m} \left(\frac{u}{n}\right)^m du = \eta \tag{3.34}$$

for $m \geqslant m_0$, $n \geqslant n_0$.

By (3.30)–(3.34)

$$\lim_{m\to\infty,\ n\to\infty,\ mn^{-1}\to q} \Delta_{m+1} = 1 - \beta(q). \tag{3.35}$$

Thus by (3.28), (3.29), (3.30), and (3.35)

$$\lim \sum_{k=1}^{n} |\gamma_k| = B_q + 2(1 - \beta(q)), \qquad q < 1. \tag{3.36}$$

In the case of $q = 1$, both $m \geqslant n$ and $m \leqslant n$ are possible; but since $\beta(1) = 1$, both expressions defining D_q in (3.6) are equal.

By Lemma 6, (3.25), (3.36), and our last remark, the theorem is proved for $q \leqslant 1$ also.

We state the following theorems without proof.

THEOREM 1′. *If we replace condition* (3.1) (i) *in Theorem* 1 *by the assumption that*

(3.1) (i)′ $$t^{-1}(1 - \beta(e^{-t}))$$

is non-decreasing for $t > 0$, the conclusion (3.2) *holds with*

$$C'_q = 2 + \int_q^\infty t^{-1}\beta(e^{-t})dt - \int_0^q t^{-1}(1 - \beta(e^{-t}))dt$$

instead of C_q.

https://doi.org/10.4153/CJM-1965-029-0 Published online by Cambridge University Press

THEOREM 2′. *If we replace condition* (3.4) (ii) *in Theorem* 2 *by the assumption that*

(3.4) (ii)′ $$t^{-1}\beta(t)$$

is non-decreasing for $0 < t \leqslant 1$, *the conclusion* (3.5) *holds with*

$$D'_q = 2 + |\log q| - \int_0^1 t^{-1}\beta(t)dt$$

instead of D_q.

REFERENCES

1. R. P. Agnew, *Abel transforms and partial sums of Tauberian series*, Ann. Math., *50* (1949), 110–17.
2. V. Garten, *Über Taubersche Konstanten bei Cesàroschen Mittelbildungen*, Comm. Math. Helv., *25* (1951), 311–35.
3. H. Hadwiger, *Über ein Distanztheorem bei der* A-*Limitierung*, Comm. Math. Helv., *16* (1944), 209–14.
4. P. Hartman, *Taubers theorem and absolute constants*, Amer. J. Math., *69* (1947), 599–606.
5. A. Jakimovski, *Tauberian constants for Hausdorff transformations*, Bull. Res. Council Isr., *9F* (1961), 175–84.
6. ——— *Tauberian constants for the* $[J, f(x)]$ *transformations*, Pac. J. Math., *12* (1962), 567–76.
7. ——— *Sequence-to-function analogues to Hausdorff-transformations*. Bull. Res. Council Isr., *8F* (1960), 135–54.

The University of Alberta, Calgary