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CHARACTERISTICALLY NILPOTENT ALGEBRAS

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Introduction. Our aim in this paper is to extend (Theorem 1.7) to general algebras a classical result of Lie algebras due to Leger and Tôgô [**6**]. This extension requires, in turn, extension to general algebras of the concept of characteristically nilpotent algebras introduced by Dixmier and Lister [**3**] for Lie algebras. Based on this extended concept, we introduce in § 2 a new concept of radical (and semisimplicity) for general algebras and Lie triple systems. We study in some detail the consequences of the newly introduced concepts, furnishing necessary examples. With a stronger notion of characteristically nilpotent Mal'cev algebra arising out of these concepts, we obtain (Proposition 3.6) for such an algebra the parallel to the Leger-Tôgô result mentioned at the outset. In § 4, we deal with a further generalization of the concept of characteristic nilpotency leading to extension of very recent results of Chao [**1**] and Tôgô [**12**].

In what follows, all vector spaces considered are assumed to be finite-dimensional over the ground field.

1. In this section we introduce the notions of characteristically nilpotent algebras, characteristically solvable algebras (see [**3**; **10**]), and obtain some results relating to these concepts.

Let A be a non-associative algebra over an arbitrary field F and $D(A)$ the Lie algebra of all derivations of A. Let

$$A^{[1]} = \{\sum x_i D_i | \, x_i \in A, D_i \in D(A)\}$$

and define inductively $A^{[k+1]} = \{\sum y_j D_j | \, y_j \in A^{[k]}, D_j \in D(A)\}$.

Definition 1.1. An algebra A is said to be *characteristically nilpotent* (C-nilpotent) if there exists an integer n such that $A^{[n]} = 0$.

Remark 1. If A is a C-nilpotent algebra, then every derivation D of A is a nilpotent linear transformation on A. Conversely, if every derivation of an algebra A is a nilpotent transformation, then the associative subalgebra of linear transformations on A generated by the Lie algebra $D(A)$ is nilpotent [**4**, Theorem 2.1]; this means precisely that A is C-nilpotent. Thus, A is C-nilpotent if and only if every derivation of A is a nilpotent linear transformation.

Remark 2. Suppose that L is a C-nilpotent Lie algebra with multiplication $[x, y]$. Since the mapping ad x: $y \to [x, y]$ is a derivation for every x in L (by Engel's Theorem [**4**, p. 31]), L will then be a nilpotent Lie algebra.

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We recall that the *annihilator ideal* I of an algebra A is precisely the set of all absolute divisors of zero in A, and note the following necessary condition for C-nilpotency for an algebra.

LEMMA 1.2. *The annihilator ideal I of a* C*-nilpotent algebra A is contained in AA.*

Proof. Suppose the contrary. Let x be a non-zero element of I such that $x \notin AA$. Then A can be written as the direct sum of the subspaces:

$$A = T \oplus \{\alpha x\}_{\alpha \in F},$$

where T contains AA; T is an ideal of A. The mapping θ of A into itself defined to be zero on T and to be the identity on $\{\alpha x\}_{\alpha \in F}$ is a non-nilpotent derivation of A, contradicting the assumption. This contradiction proves the lemma.

Let now K be an extension field of the base field F. Let A_K be the algebra obtained by extending F to K. Then we assert that $D(A_K) = (D(A))_K$, the extension over K of the derivation algebra of A over F. An immediate consequence of this assertion would be the important fact that the algebra A is C-nilpotent if and only if A_K is C-nilpotent. A proof of this assertion can be modelled on [**5**, proof of Theorem 5]. However, we indicate here a proof kindly suggested by Professor K. McCrimmon. We need only prove that

$$D(A_K) \subseteq (D(A))_K.$$

For this, let $\{k_i\}_{i \in I}$ be a basis for K over F, so that $A_K = \bigoplus_I k_i A$. If D is a derivation of A_K, let $aD = \sum k_i aD_i$ for a and aD_i in A. Then the maps $a \to aD_i$ are easily verified to be derivations of A. If $\{a_1, a_2, \ldots, a_n\}$ is a basis for A over F, then it is also a basis for A_K over K, and, for any a_j, only finitely many $a_j D_i$ are non-zero. If $D_1, \ldots, D_m$ are the only non-zero terms appearing in $a_1 D, a_2 D, \ldots, a_n D$, then they are the only non-zero terms appearing in any aD for $a = \sum f_i a_i, f_i \in F$, in A, so that $D = \sum_{i=1}^m k_i D_i$ is a finite sum, i.e., D is a linear combination over K of elements of $D(A)$, when restricted to A, i.e., $D \in (D(A))_K$, proving the desired inclusion.

PROPOSITION 1.3 (cf. [**6**, Theorem 6]). *Let A be an algebra which is a direct sum of ideals A_i $(i = 1, 2, \ldots, r)$. Then A is* C*-nilpotent if and only if the A_i are* C*-nilpotent algebras.*

Proof. Let A be C-nilpotent. Any derivation D_i of A_i can be trivially extended to a derivation D of A (D coinciding with D_i on A_i and with 0 on A_j $(j \neq i)$). This observation immediately shows that A_i should be C-nilpotent.

For the converse part, it suffices to observe that the proof of the result referred to against the proposition itself works for the general case, the role of centre (cent) therein being played by the annihilator ideal, in view of the availability of Lemma 1.2 (used in that proof). The details are omitted.

Remark 3. We incidentally note that if A is a nilpotent algebra such that $AD(A) \subseteq AA$, then A is also C-nilpotent. (We recall that an algebra A is nilpotent if the series of subspaces

$$A^1 = A, A^2 = AA, \ldots, A^k = \{A^{k-1}A, AA^{k-1}\}, \ldots$$

coincides with the zero space after a finite stage.) Further, in Proposition 1.3, if A is such a C-nilpotent algebra, then so are the A_i, and conversely (cf. [**6**, end of § 4]).

Definition 1.4. An algebra A is said to be *characteristically solvable* (see [**10**, p. 201]) if $D(A)$ is solvable as a Lie algebra.

A C-nilpotent algebra is easily seen to be characteristically solvable. We have the following result.

PROPOSITION 1.5 (cf. ([**10**, Lemma 3]). *If A is a solvable associative algebra such that the centre of A is contained in AA, and if $D(A)$ is the direct sum of the radical and a semisimple ideal, then A is characteristically solvable.*

Proof. Let $D(A) = R \oplus S$, for the radical R of $D(A)$ and a semisimple ideal S of $D(A)$. Now, A being solvable is also nilpotent. Consequently, the associative multiplication algebra (see [**8**]) of A is nilpotent. Let L_x (R_x) denote the left (right) multiplication by x in A. It is known that $\operatorname{ad} x = R_x - L_x$ is a derivation of A. The set $\{\operatorname{ad} x\}_{x \in A}$ is easily seen to be an ideal (a nilpotent ideal in fact) of $D(A)$, i.e., $\{\operatorname{ad} x\}_{x \in A} \subseteq R$. For a non-zero derivation D of A belonging to S, $[R, D] = 0$; in particular, $[R_x - L_x, D] = R_{xD} - L_{xD} = 0$ for all x in A, i.e., $xD \in \operatorname{cent} A \subseteq AA$. Then $AD \subseteq A^2$, $AD^2 \subseteq A^2D \subseteq AD \cdot A + A \cdot AD \subseteq \{A^2A, AA^2\} \equiv A^3$, etc., ... ; by the nilpotency of A, we have $AD^n = 0$, where $A^{n+1} = \{0\}$. In other words, every element of S is a nilpotent derivation, so that S will be a nilpotent ideal of $D(A)$, a contradiction; $S = 0$. Thus $D(A) = R$, and the proposition is proved.

Remark 4. Proposition 1.5 holds for a solvable Lie algebra [**10**, Lemma 3] in the place of a solvable associative algebra. In the case of a nilpotent Lie algebra A, when $D(A)$ is the direct sum of the nil radical and a semisimple ideal, we can conclude that A is C-nilpotent. We require the use of [**6**, Theorem 1] to arrive at this assertion. In fact, this auxiliary result finds its generalization in Theorem 1.7 below.

We recall now that a non-associative algebra A is said to be nilpotent, if there exists a fixed integer n such that all products of n elements of A are zero, irrespective of how they are associated [**8**]. For such algebras we can prove as in [**6**, Lemma 2], the following result.

LEMMA 1.6. *If a nilpotent algebra A is a direct sum of two non-zero ideals one of which annihilates the algebra, then $D(A)$ is not nilpotent.*

Now, let A be a nilpotent algebra which is also C-nilpotent. (The assumption of nilpotency of A is superfluous in the case of a Lie algebra A.) Let L be the Lie multiplication algebra of A, i.e. the Lie algebra of linear transformations on A generated by the left (right) multiplication L_x (R_x) in A. Since A is nilpotent, each element of L is a nilpotent linear transformation [**8**, Chapter II, Theorem 2.4]. We denote by L^* the Lie algebra generated by $\{L_x, R_x\}_{x \in A}$ and the derivations of A. Then $L^* = L + D(A)$ (sum as vector spaces and not necessarily a direct sum); $[L, D(A)] \subseteq L$. The associative subalgebra of linear transformations on A generated by L $(D(A))$ is nilpotent, and there exists an integer m (n) such that all (associative) products of m (n) elements in L $(D(A))$ are zero. The inclusion $[L, D(A)] \subseteq L$ enables us to see that all products of mn elements of L^* are zero, so that L^* is a nilpotent Lie algebra. This conclusion is part of the following main theorem of the paper (see also Remark 5 below).

THEOREM 1.7. *A nilpotent algebra A is* C*-nilpotent if and only if L^* is a nilpotent Lie algebra, and A is not one-dimensional.*

To complete the proof of Theorem 1.7 we need a known lemma. This lemma (which is stated without proof, in its original source) is given below, with a proof for the sake of completeness.

LEMMA 1.8 (cf. [**6**, Theorem 2]). *Let A be an algebra over a field F such that L^* is nilpotent. Then A is a direct sum of two characteristic ideals B and C such that all derivations of A are nilpotent on B,* $\dim C \leqq 1$, *and* $CC = 0$, *i.e., C annihilates the algebra A.*

Proof. $D(A)$, being a subalgebra of L^*, is a nilpotent Lie algebra of linear transformations on A. Let $A = A_0 \oplus A_1$ be the Fitting decomposition of A relative to $D(A)$ [**4**, Theorem 2.4]. A_0 and A_1 being L^*-invariant subspaces of A (by [**4**, Lemma 2.1]) are characteristic ideals of A. Each derivation of A is nilpotent on $A_0 \equiv B$. Further,

$$C \equiv A_1 = \sum_{D \in D(A)} A_{1D},$$

where A_{1D} is the Fitting 1-component of A relative to the derivation D [**4**, p. 37]. If $(L^*)^{k+1} = 0$, then for any x in A and for $r \geqq k$,

$$[\ldots [[L_x, D], D], \ldots , D \ (r \text{ times})] = 0,$$

i.e. $L_{xD^r} = 0$. Similarly, $R_{xD^r} = 0$. But any element of A_{1D} is of the form xD^t for some x in A, where t can be chosen to be greater than k without loss of generality. Thus $L_y = 0 = R_y$ for all y in C. To complete the proof of the lemma, we have only to show that $\dim C \leqq 1$. Suppose, on the contrary, that $\dim C > 1$ and that x and y are two linearly independent elements in C. Let U be a complementary subspace of x in C containing y. Then the mappings

$$\begin{aligned} D&: \quad A_0D = 0, \qquad xD = x, \qquad UD = 0,\\ D'&: \quad A_0D' = 0, \qquad xD' = y, \qquad UD' = 0, \end{aligned}$$

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are two (distinct) derivations of A such that $[D, D'] = D'$. This means that $D(A)$ is not nilpotent, contradicting the hypothesis. Thus dim $C \leqq 1$. The proof of the lemma is complete.

The original proof of Lemma 1.8 needed the assumption of algebraic closure of the base field. The use of the "Fitting" decomposition as in the present proof to dispense with this superfluous assumption was kindly suggested by Professor G. B. Seligman. However, a proof of this lemma can be alternatively modelled on [**6**, Theorem 1] in view of the fact that the base field can be assumed without loss of generality to be algebraically closed. For, $L_{A_K}{}^* = (L_A{}^*)_K$, where K is an extension of the base field F (see, in this connection, the remarks following the proof of Lemma 1.2).

Now we complete the proof of Theorem 1.7. We suppose that L^* is nilpotent and that A is not one-dimensional. By Lemma 1.8, $A = B \oplus C$, for ideals B, C such that every derivation of A is nilpotent on B and $CA = AC = 0$; dim $C \leqq 1$. Since dim $A > 1$, B is a non-zero ideal. If C is also a non-zero ideal, an appeal to Lemma 1.6 leads to the contradiction that $D(A)$ is not nilpotent. Hence $C = 0$, so that A is C-nilpotent (by Remark 1). The theorem is now proved.

Remark 5. When A is a one-dimensional nilpotent algebra over a field F with the base element x, $x^2 = 0$, and $L^* = \{\alpha I\}_{\alpha \in F}$, where I is the identity transformation. L^* is nilpotent as a Lie algebra. However, A is not C-nilpotent in this case, I being a non-nilpotent derivation of A. (In fact, any derivation of A is a scalar multiple of I.)

The following immediate corollary to Theorem 1.7 is a result of Leger and Tôgô [**6**, Theorem 1]. To obtain the corollary one need only observe the following:

(i) A C-nilpotent Lie algebra is nilpotent,
(ii) $L^* = D(L)$ for a Lie algebra L, and
(iii) The nilpotency of $D(L)$ implies the nilpotency of L.

COROLLARY 1.9. *A Lie algebra L is* C-*nilpotent if and only if $D(L)$ is a nilpotent Lie algebra and L is not one-dimensional.*

2. In this section we introduce a new concept of radical and the related notion of semisimplicity. Besides giving certain basic results relating to these concepts, we also give some examples.

Let B be a characteristic ideal of an algebra A. Let $D(A)$ be the Lie algebra of all derivations of A. Define $B^{[1]} = B$, $B^{[2]} = \{\sum x_i D_i | \, x_i \in B, D_i \in D(A)\}$, and inductively $B^{[k+1]} = \{B^{[k]} D(A)\}$. Since B is a characteristic ideal of A, we have $B \supseteq B^{[2]}$. $B^{[k]}$ is a characteristic subspace of A (by induction) and $B \supseteq B^{[2]} \supseteq B^{[3]} \supseteq \ldots$. We define B to be C-nilpotent if $B^{[k]} = 0$ for some k. We note that for a Lie algebra A, $B^{[k]}$ are characteristic ideals of A, and that a

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C-nilpotent characteristic ideal is a nilpotent ideal. A characteristic ideal B of A is a C-nilpotent ideal if and only if every derivation of A restricted to B is nilpotent. We then have the following result.

PROPOSITION 2.1. *If A is a direct sum of characteristic ideals A_i, then A is* C-*nilpotent if and only if the A_i are* C-*nilpotent ideals of A.*

Let B and C be two characteristic ideals of A. Then it can be easily seen that $(B + C)$ is a characteristic ideal of A satisfying the inclusion

$$(B + C)^{[k]} \subseteq B^{[k]} + C^{[k]}$$

for any integer k. This observation immediately yields the following result.

LEMMA 2.2. *The sum of two* C-*nilpotent characteristic ideals of an algebra A is again a* C-*nilpotent characteristic ideal.*

Lemma 2.2 enables us to define the new concept of radical mentioned in the Introduction.

Definition 2.3. The maximal C-nilpotent characteristic ideal R of an algebra A (which exists, by Lemma 2.2) is called the C_1-*radical* of A.

Remark 6. Recalling (see [**9**]) that the C-radical (C-nil radical) of a Lie algebra L is its maximal solvable (nilpotent) characteristic ideal, we note immediately that $R \subseteq$ C-nil radical of $L \subseteq$ nil radical N of $L \subseteq$ radical R' of L. Further, $R \subseteq$ C-radical of L. Of course, when the base field is of characteristic zero, the C-radical (C-nil radical) coincides with the radical (nil radical). The question as to when the nil radical N of a Lie algebra L coincides with R has been settled by Tôgô [**10**, Corollary 2 to Theorem 2] in a different form. It can be restated as follows: For a Lie algebra L over a field of characteristic zero, the C_1-radical R of L = nil radical N of L if and only if N is itself characteristically nilpotent as an algebra. In this case, $R = N =$ C-nil radical of L = radical of L = C-radical of L; further, $L = N \oplus S$ for a semisimple ideal S of L (see [**3**]).

Definition 2.4. An element x in an algebra A is said to be D-*nilpotent* if there exists an integer n (depending on x) such that $xD_1D_2 \ldots D_n = 0$ for any n derivations D_i.

PROPOSITION 2.5 (cf. [**8**, Theorem 3.7]). *The C_1-radical R of an associative algebra A is precisely the set of* D-*nilpotent elements x in A such that xy is* D-*nilpotent for all y in A (i.e., x is* D-*properly nilpotent in A).*

Proof. Evidently, the C_1-radical R of A is contained in the set N' of D-properly nilpotent elements of A. To prove that $N' \subseteq R$, it suffices to show that N' is a C-nilpotent characteristic ideal of A. N' is a subspace of A. It is easily seen that for x in N', y in A, $xy \in N'$; further, for a derivation D of A, xD is evidently D-nilpotent; $(xD)y = (xy)D - x(yD)$ is D-nilpotent for all y in A.

Hence, xD is D-properly nilpotent. Consequently, N' is a characteristic right ideal, and hence is a characteristic ideal of A ($y \rightarrow xy - yx$ is a derivation in A). By finite dimensionality of A, N' is a C-nilpotent characteristic ideal of A and $N' \subseteq R$. The proposition is proved.

Remark 7. For a Lie algebra L, the C_1-radical R,

$$R = N' \equiv \{x \in L \mid xD_1D_2 \ldots D_n = 0 \text{ for any } n \text{ derivations } D_i \text{ of } L, n \text{ being an integer depending on } x\}.$$

This assertion follows directly from the fact that N' is a characteristic subspace (hence an ideal) of L.

Definition 2.6. An algebra A is said to be *C_1-semisimple* if its C_1-radical is the zero ideal.

For a Lie algebra, the following implications are evident:

$$\begin{array}{ccccc} \text{simplicity} & \Rightarrow & \text{semisimplicity} & \Leftrightarrow & \text{nil semisimplicity} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{C-simplicity} & \Rightarrow & \text{C-semisimplicity} & \Leftrightarrow & \text{C-nil semisimplicity} \\ & & & & \Downarrow \\ & & & & C_1\text{-semisimplicity.} \end{array}$$

Remark 8. The concept of C_1-semisimplicity does not coincide with the classical semisimplicity even for a Lie algebra over a field of characteristic zero, unlike Seligman's concepts, as will be shown by our example later.

The following analogue of [**9**, Theorem 1] can be easily proved.

PROPOSITION 2.7. *If R is the C_1-radical of an algebra A, then A/R is C_1-semisimple.*

The following result is immediate from Proposition 2.5.

PROPOSITION 2.8. *A characteristic ideal of a C_1-semisimple associative algebra is itself C_1-semisimple as an algebra.*

Remark 9. The analogue of Proposition 2.8 holds for Lie algebras as well, in view of Remark 7. This observation has a direct proof, using the fact that a characteristic ideal C of a characteristic ideal B in a Lie algebra L, is again a characteristic ideal of L.

Remark 10. If A is a flexible algebra over a field of characteristic $\neq 2$, i.e., an algebra satisfying the identity $(xy)x = x(yx)$, then $x \rightarrow xy - yx$ is a derivation of the algebra A^+ associated with A (see [**8**, p. 146]), so that any characteristic ideal of A^+ is a characteristic ideal of A. Further, since any derivation of A is a derivation of A^+, it follows that the C_1-radical of A^+ is contained in the C_1-radical of A. Thus, if A is a C_1-semisimple flexible algebra, then so is A^+.

We note the following interesting facts about the new concept of C_1-radical. A zero algebra happens to be trivially C_1-semisimple. If $xy = 0$ for every x, y in an algebra A, every linear mapping of A into itself is a derivation; the only characteristic ideals are 0 and A. Evidently, A cannot be C-nilpotent and the C_1-radical coincides with **0**. The same example shows that a nilpotent algebra need not be C-nilpotent. Further, a characteristic ideal of a C-nilpotent algebra need not be C-nilpotent as an algebra: Let L be a C-nilpotent Lie algebra (e.g., see [**3**]). L being nilpotent, if $L^k = 0$, $L^{k-1} \neq 0$, then L^{k-1} is a characteristic abelian ideal of L which is not C-nilpotent as an algebra. A non-trivial example of a C_1-semisimple Lie algebra which is not C-semisimple is given by the Lie algebra with basis x_1, x_2, x_3 over a field F and multiplication defined by (see [**11**])

$$[x_1, x_2] = x_2; \qquad [x_1, x_3] = x_3; \qquad [x_2, x_3] = 0.$$

As an example of a C_1-semisimple associative algebra we can cite the algebra A with basis e, x and multiplication defined by $ex = 0 = x^2$; $e^2 = e$; $xe = x$.

For an example of a C-nilpotent non-associative (non-Lie) algebra, the reader is referred to [**12**].

An example of a C_1-semisimple associative algebra is the algebra $F[1, a] \equiv A$ over a field F of characteristic $p \neq 0$, with $a^p = 0$. A is C-simple, i.e. it does not have any characteristic ideals other than A and 0; $AA \neq 0$. However, A is not a C-nilpotent ideal, since the mapping $a \to a$ is a non-nilpotent derivation of A (see [**4**, p. 75]).

3. In this section we consider the concepts corresponding to those dealt with in earlier sections, for a Lie triple system, obtaining (Proposition 3.5) a sort of analogue for Mal′cev algebras of a theorem of Leger and Tôgô [**6**, Theorem 1].

Let T be a Lie triple system over a field F with composition $[x, y, z]$ (see [**7**] for the details regarding Lie triple systems). Let $D(T)$ be the Lie algebra of all derivations of T. Then, as for an algebra, T can be defined to be C-nilpotent if the series of subspaces

$$T^{[1]} = T,\ T^{[2]} = \{TD(T)\}, \ldots, T^{[k+1]} = \{T^{[k]}D(T)\}, \ldots$$

terminates with zero after a finite stage. The notions of C-nilpotent characteristic ideal, C_1-radical (which is well-defined here too) are clear. If R is the C_1-radical of T, T/R is C_1-semisimple (i.e., the C_1-radical of T/R is the zero ideal). Propositions 1.3 and 2.1 remain true for Lie triple systems.

Remark 11. If B were a characteristic ideal of T such that

$$B^{(1)} = B,\ B^{(2)} = [T, B, B], \ldots, B^{(n+1)} = [T, B^{(n)}, B^{(n)}], \ldots$$

is the derived series of B, then it is easily seen that $B^{(r)} \subseteq B^{[r]}$ for any integer r. Thus, when B is C-nilpotent, B is solvable. Hence, the C_1-radical of T is contained in the C-radical of T (maximal solvable characteristic ideal of T)

which is contained in the radical of T. In particular, a C-semisimple Lie triple system is C_1-semisimple.

Let us now consider a Mal′cev algebra A over a field of characteristic $\neq 2$ (this assumption on the characteristic is assumed without mention throughout the remainder of this section), i.e. an anticommutative algebra satisfying the identity $(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$ (see [**7**]); let T_A be the Lie triple system (see [**7**]) associated with A, with the composition

$$[x, y, z] = 2(xy)z - (yz)x - (zx)y.$$

Any derivation of A is a derivation of T_A, and any characteristic ideal of T_A is a characteristic ideal of A [**7**, Lemma 2]. Therefore

$$C_1\text{-radical of } T_A \subseteq C_1\text{-radical of } A.$$

Thus we have the following result.

PROPOSITION 3.1. *If A is a C_1-semisimple Mal′cev algebra, then T_A is a C_1-semisimple Lie triple system.*

We now introduce a stronger notion of C-nilpotency of a Mal′cev algebra based on its Lie triple system.

Definition 3.2. A Mal′cev algebra A is said to be *strongly C-nilpotent* (SC-nilpotent), if the associated Lie triple system T_A is C-nilpotent. The C_1-radical of T_A is called the SC_1-*radical* of A.

Remark 12. Evidently, the SC_1-radical of A is also a characteristic ideal of A contained in the C_1-radical of A, as was observed earlier. The notions of SC_1-semisimplicity of A and C_1-semisimplicity of T_A are equivalent. In view of the fact that the left multiplication L_x of A is a derivation of T_A [**7**, *Satz* 1], it follows that an SC-nilpotent Mal′cev algebra is a nilpotent algebra (i.e. there exists an integer n such that all products of n elements of A is zero, irrespective of associations). Thus we also see that SC_1-radical of A is contained in the C-radical of A (maximal solvable characteristic ideal). In particular, a C-semisimple Mal′cev algebra is SC_1-semisimple. It is not known as to whether a C-semisimple Mal′cev algebra is C_1-semisimple as an algebra or not.

Since a characteristic subspace of T_A is a characteristic ideal of T_A, the following characterization of the SC_1-radical is evident, as in Remark 7.

PROPOSITION 3.3. *The SC_1-radical of a Mal′cev algebra A is precisely the set $\{x \in T_A \mid xD_1D_2 \ldots D_n = 0$ for any n derivations D_i of T_A, n being an integer depending on $x\}$.*

The above characterization shows that a characteristic ideal R of T_A for an SC_1-semisimple Mal′cev algebra A is itself SC_1-semisimple as an algebra.

We shall call a Lie triple system T nilpotent if there exists an (odd) integer n such that all triple products in T involving n elements irrespective of the association are zero. Further, an ideal B of T is said to be an annihilator ideal if $[B, T, T] = 0$; obviously,

$$[B, T, T] = 0 \Leftrightarrow [T, B, T] = 0 = [T, T, B].$$

LEMMA 3.4 (cf. [**6**, Lemma 2] and Lemma 1.6). *Let T be a nilpotent Lie triple system such that T is the direct sum of two non-zero ideals, one of which is an annihilator ideal. Then $D(T)$ is not nilpotent.*

Now suppose that A is an SC-nilpotent Mal′cev algebra; then $D(T_A)$ is a nilpotent Lie algebra. Conversely, suppose that A is a Mal′cev algebra over a field F such that $D(T_A)$ is a nilpotent Lie algebra. Then we can assume F to be algebraically closed without loss of generality (see the remarks preceding Proposition 1.3 and note that the arguments hold for Lie triple systems). Now we can adapt the arguments of Leger and Tôgô [**6**, Theorem 1], noting that a characteristic subspace of T_A is a characteristic ideal of T_A, to show that $T_A = B \oplus C$, for characteristic ideals B, C of T_A with $D(T_A)$ nilpotent on B and with $[C, T_A, T_A] = 0$. As in the proof of Lemma 1.8, we can show that $\dim C \leqq 1$. The system T_A can be easily seen to be a nilpotent system. Thus we can appeal to Lemma 3.4 to deduce the following companion to Theorem 1.7, which is at the same time an analogue of the result of Leger and Tôgô [**6**, Theorem 1].

PROPOSITION 3.5. *A Mal′cev algebra A over a field of characteristic $\neq 2$ is* SC-*nilpotent if and only if $D(T_A)$ is nilpotent and A is not one-dimensional.*

Remark 13. A proof of Proposition 3.5 as in the case of Theorem 1.7 (Lemma 1.8) using the "Fitting" decomposition does not seem to be possible here.

Suppose now again that A is an SC-nilpotent Mal′cev algebra. Then there exists an integer n such that all products of n elements of $D(T_A)$ is zero. We can use this fact to show (by an easy computation) that the enveloping Lie algebra $L = T_A \oplus D(T_A)$ of T_A (with the multiplication in L being defined as in [**7**, § 4, p. 555]) is a nilpotent Lie algebra.

Conversely, if the enveloping Lie algebra L of T_A for a Mal′cev algebra A is nilpotent, then $D(T_A)$ is a nilpotent Lie algebra, being a subalgebra of L. Also, A cannot be one-dimensional. Suppose, to the contrary, that A is one-dimensional, i.e. $A = \{\alpha x\}_{\alpha \in F}$, $x^2 = 0$; $D'\colon x \to x$ is a derivation of T_A. By definition of multiplication in L (see [**7**, § 4]),

$$[\ldots[[x, D'], D'] \ldots, D' \ (r \text{ times})] = x$$

for any r, and L cannot be nilpotent, a contradiction. Consequently, we have from Proposition 3.5 that A is an SC-nilpotent Mal′cev algebra. We have thus proved the following result.

PROPOSITION 3.6. *A Mal′cev algebra A over a field of characteristic $\neq 2$ is* SC-*nilpotent if and only if the enveloping Lie algebra L $(\equiv T_A \oplus D(T_A))$ of T_A is a nilpotent Lie algebra.*

4. In this section we formulate a generalization of the notion of characteristic nilpotency and study its properties. (For an example of a characteristic nilpotent algebra which is not Lie, we again refer to [**12**].)

Definition 4.1. Let $\mathscr{D}$ be any collection of derivations of an algebra A. Define inductively $A^{[1]'} = \{A\mathscr{D}\}, \ldots, A^{[j+1]'} = \{A^{[j]'}\mathscr{D}\}, \ldots$. A is said to be D-*nilpotent* if $A^{[r]'} = 0$ for some integer r. The maximal D-nilpotent D-ideal of A arising out of such a definition can be called the D-*radical* of A.

Remark 14. For a Lie algebra L, if we take $\mathscr{D}$ to be a Lie algebra of derivations containing the inner derivations, then the C_1-radical is contained in the D-radical which is contained in the radical.

The notion of D-nilpotency can be used to advantage in proving the following theorem, which simultaneously generalizes a recent result of Tôgô [**12**, Theorem] and a result of Chao [**2**, Theorem 2]: For an ideal N of a non-associative algebra A, let us define N^j to be the subspace of N generated by all products of j elements in N (irrespective of association). N is nilpotent if $N^r = 0$ for some r. We further remark that for any D-ideal B of an algebra A, $\mathscr{D}$ induces a collection $\overline{\mathscr{D}}$ of derivations $\bar{D}$ on A/B defined by

$$(x + B)\bar{D} = xD + B \quad \text{for} \quad x \in A \quad \text{and} \quad D \in \mathscr{D}.$$

THEOREM 4.2. *Let A be a non-associative algebra over a field F and let $\mathscr{D}$ be a collection of derivations of A. Let N be a nilpotent* D-*ideal of A such that the N^j are ideals of A for all j. Then A is* D-*nilpotent if and only if A/N^n is* $\overline{\text{D}}$-*nilpotent for some integer $n \geqq 2$, $\overline{\mathscr{D}}$ being the set of induced derivations in A/N^n.*

Proof. If A is D-nilpotent, evidently A/N^t is $\overline{\text{D}}$-nilpotent (for any t). Conversely, suppose that A/N^n is $\overline{\text{D}}$-nilpotent for an $n \geqq 2$. Since N^n is a D-ideal of A, there exists an integer m such that $A\mathscr{D}^m \subset N^n$. We claim that $A\mathscr{D}^{3^k m} \subset N^{n+k}$ for all non-negative integers k. This being true for $k = 0$, by induction on k, we have

$$\begin{aligned} A\mathscr{D}^{3^{k+1}m} &= A\mathscr{D}^{3^k m} \cdot \mathscr{D}^{2\cdot 3^k \cdot m} \\ &\subseteq N^{n+k} \cdot \mathscr{D}^{2\cdot 3^k m} \\ &\subseteq \Bigg(\sum_{\substack{p+q=n+k;\\ p,q\geqq 1}} N^p N^q \Bigg) \mathscr{D}^{2\cdot 3^k m} \\ &\subseteq \sum_i \sum_{\substack{p+q=n+k;\\ p,q\geqq 1}} (N^p \mathscr{D}^i)(N^q \mathscr{D}^{2\cdot 3^k m - i}) \\ &\subseteq N^{n+p+k} \quad \text{or} \quad N^{n+q+k} \\ &\subseteq N^{n+k+1}. \end{aligned}$$

By the nilpotency of N, there exists an integer k for which $N^{n+k+1} = 0$ and A will be D-nilpotent.

Remark 15. We note that the above result can be proved also by applying verbally the proofs of [**12**, Lemma and Theorem].

COROLLARY 4.3 (see [**12**, Theorem]). *Let A be an algebra over F and let N be a nilpotent characteristic ideal of A such that the N^j are ideals of A for all positive integers j. Then A is characteristically nilpotent if and only if A/N^n is characteristically nilpotent for some integer $n \geqq 2$.*

COROLLARY 4.4 (cf. [**2**, Theorem 2]). *Let N be a nilpotent ideal of a Lie algebra L. Then L is nilpotent if and only if L/N^n is nilpotent for some integer $n \geqq 2$.*

In order to prove Corollary 4.4, it suffices to observe that:

(i) the nilpotency of a Lie algebra L is equivalent to its D-nilpotency for $\mathscr{D} = \{\text{ad } x\}_{x \in L}$,

(ii) the nilpotency of L/N^n is equivalent to its $\bar{\text{D}}$-nilpotency for the above $\mathscr{D}$,

(iii) the N^j are ideals of L and $\{N^j\}_j$ is the lower central series of N.

Remark 16. Since the N^j are automatically ideals of A for an alternative or Lie algebra A, this assumption is superfluous in Theorem 4.2 in these cases (see also [**12**, Corollary]).

The restriction stipulating that N^j be ideals can be waived provided we use the following stronger notion of nilpotency. Let B be a subspace of an algebra A. Define $B^{\langle k \rangle}$ to be the set of all finite sums of finite products of elements of A involving at least k elements of B (irrespective of their association). Then $B^{\langle k \rangle}$ are ideals of A. If B is an ideal of A, then $B = B^{\langle 1 \rangle} \supseteq B^{\langle 2 \rangle} \supseteq \ldots$, and B is called *strongly nilpotent* if $B^{\langle k \rangle} = 0$ for some k. Evidently, a strongly nilpotent algebra is nilpotent and vice versa. However, for an ideal of an algebra this concept differs in general from that of nilpotency.

We have the following result.

PROPOSITION 4.5. *Let A be a finite-dimensional non-associative algebra over a field F, let $\mathscr{D}$ be a collection of derivations of A, and let N be a strongly nilpotent* D-*ideal of A. Then A is* D-*nilpotent if and only if $A/N^{\langle n \rangle}$ is* $\bar{\text{D}}$-*nilpotent for some integer $n \geqq 2$ ($\bar{\mathscr{D}}$ being the set of derivations induced in $A/N^{\langle n \rangle}$ by $\mathscr{D}$).*

This proposition can be proved on similar lines as Theorem 4.2, with slight changes.

Remark 17. For an alternative or a Lie algebra, $N^{\langle n \rangle}$ is the same as N^n, and hence the above proposition simply reduces to Theorem 4.2 itself for these cases (see also Remark 16).

We finally consider some questions regarding the D-nilpotency series, namely the descending chain of subspaces

$$A^{[1]'} = \{A\mathscr{D}\}, \ldots, A^{[k+1]'} = \{A^{[k]'}\mathscr{D}\}, \ldots$$

of a D-nilpotent algebra A.

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THEOREM 4.6. *Let L be a non-abelian Lie algebra whose centre is one-dimensional. Then L cannot be any $N^{[i]'}$, $i \geqq 1$, of a* D-*nilpotent Lie algebra N for a collection $\mathscr{D}$ of derivations of N including all the inner derivations of N, where*

$$N = N^{[0]'} \supset N^{[1]'} \supset \ldots \supset N^{[t]'} \supset N^{[t+1]'} = \{0\}$$

is the D-*nilpotency series of N.*

Proof. Let, if possible, L be some $N^{[i]'}$, $1 \leqq i \leqq t$, for some N. Since $\mathscr{D}$ includes the inner derivations, $[N^{[t]'}, N^{[t]'}] \subseteq N^{[t+1]'} = 0$; $N^{[t]'}$ is abelian so that $L \neq N^{[t]'}$; i.e. $i \leqq t - 1$. Then, for any element $\sum x_i D_i$ of $N^{[1]'}$ $(x_i \in N, D_i \in \mathscr{D})$, $[N^{[t-1]'}, \sum x_i D_i] \subset N^{[t-1]'} \sum [\text{ad } x_i, D_i] \subset N^{[t+1]'} = 0$ (since ad $x_i \in \mathscr{D}$ and $[\text{ad } x, D] = \text{ad } xD$); i.e. $N^{[t-1]'} \subset$ centre of $N^{[1]'}$, hence of L. But dim $N^{[t-1]'}$ is at least 2, a contradiction to the hypothesis that the centre of L is one-dimensional.

COROLLARY 4.7 (cf. [**1**, Theorem 1]). *Let L be a non-abelian Lie algebra whose centre is one-dimensional. Then L cannot be any N_i, $i \geqq 1$, of a nilpotent Lie algebra N, where $N = N_0 \supset N_1 \supset \ldots \supset N_t \supset 0$ is the lower central series of N.*

For the proof of the corollary, it suffices to recall the observations following Corollary 4.4.

Remark 18. We note that Theorem 4.6 can also be stated in particular for characteristically nilpotent algebras.

We also note that the proof given for the above theorem essentially works for the case of an associative algebra to yield the following result.

THEOREM 4.8. *Let A be a non-commutative associative algebra whose centre is one-dimensional. Then A cannot be any $N^{[i]'}$, $i \geqq 1$, for a* D-*nilpotent associative algebra N (where $\mathscr{D}$ includes all the inner derivations D_y: $x \to xy - yx$ in N).*

Corollary 4.7 (or [**1**, Theorem 1]) can also be deduced from the following more general result.

PROPOSITION 4.9. *Let A be a non-zero $(AA \neq 0)$ non-associative algebra whose annihilator ideal I is one-dimensional. Then A cannot be any N^i, $i \geqq 2$, of a nilpotent algebra N, where $N = N^1 \supset N^2 \supset \ldots \supset N^t \supset 0$ is the nilpotency series of N.*

Proof. Suppose, to the contrary, that A is N^i $(i \geqq 2)$ for some N. Since $N^r N^s \subset N^{r+s}$ for any integers r, s, $A \neq N^t$; we have

$$N^{t-1}N^i \text{ (and } N^i N^{t-1}) \subset N^{i+t-1} = 0 \text{ (since } i + t - 1 \geqq t + 1),$$

so that N^{t-1} is contained in the annihilator ideal I of $N^i = A$. But the dimension of N^{t-1} is at least 2, a contradiction to the hypothesis on I.

Remark 19. The above theorem generalizes the result of Chao [**1**, Theorem 1] (see also Corollary 4.7) and simultaneously simplifies his proof.

In view of the generalized form (Theorem 4.6) of [**1**, Theorem 1], now available, [**1**, Theorem 2] can be generalized as follows.

PROPOSITION 4.10. *Let L be a non-abelian Lie algebra with* $\dim L/[L, L] = 2$. *Such an L cannot be any $N^{[i]'}$, $i \geqq 1$, of a* D*-nilpotent Lie algebra N, for a set $\mathscr{D}$ of derivations of N including the inner derivations, where*

$$N = N^{[0]'} \supset N^{[1]'} \supset \ldots$$

is the D*-nilpotency series of N.*

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