GENERALIZED CONVEX BODIES OF REVOLUTION

WM. J. FIREY

To Professor H. S. M. Coxeter on his sixtieth birthday

Introduction. The figures studied in this paper are special convex bodies in Euclidean three-dimensional space which we shall call generalized convex bodies of revolution (GCBR). Such a set is obtained by the following procedure. Let K_1 be a convex body of revolution and let x, y, z denote Cartesian coordinates in a system for which the z-axis is the axis of K_1. We map each point (x, y, z) into a point $(\bar{x}, \bar{y}, \bar{z})$ by a transformation of the following sort:

$$\text{(1)} \qquad \bar{x} = rx/f(x, y), \quad \bar{y} = ry/f(x, y), \quad \bar{z} = z, \quad r = \sqrt{(x^2 + y^2)},$$

if $r > 0$. The points $(0, 0, z)$ remain fixed. The function f is required to satisfy the characteristic conditions for the distance function of a plane convex body k, that is:

(i) $f(x, y) \geqslant 0$ with equality if and only if $x = y = 0$;

(ii) $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda > 0$;

(iii) $f(x + x_1, y + y_1) \leqslant f(x, y) + f(x_1, y_1)$ for any pairs (x, y) and (x_1, y_1).

Call the image of K_1 under this transformation K. A GCBR is any set K obtained by such a construction. It will be shown that K is a convex body.

The principal results obtained concern the behaviour of the volume V, surface area S, and total mean curvature or, to within a factor of 2π, mean width M under the process of Blaschke addition of certain pairs of GCBR. To give a rough idea of this composition process, imagine two convex bodies C_0, C_1 whose boundaries are of sufficient smoothness and regularity that they have reciprocal Gauss curvatures F_0 and F_1 defined as continuous functions over the unit spherical surface Ω of outer normal directions $\bar{u}$. It is a consequence of a theorem of Minkowski that F, defined over Ω by

$$F(\bar{u}) = F_0(\bar{u}) + F_1(\bar{u}),$$

is the reciprocal Gauss curvature function of a convex body which we call the Blaschke sum of C_0 and C_1. This sum is unique to within a translation.

Our conclusions regarding V, S, and M under Blaschke addition take the form of concavity and convexity theorems somewhat analogous to the Brunn–Minkowski theorem. With the use of these theorems, we deduce certain inequalities involving V, S, and M for GCBR. These are generalizations of inequalities, known for convex bodies of revolution, due to Hadwiger **(3)**.

Received April 17, 1967. The work was supported in part by a grant from the U.S. National Science Foundation, NSF-GP5665.

1. Generalized convex bodies of revolution. In this section we show that, in the notation of the Introduction, the set K constructed from K_1 is a convex body. We also discuss the approximation of general GCBR by special sorts of GCBR.

Consider a plane parallel to $z = 0$ which intersects K_1. If (x, y, z) is in this intersection, then

$$x^2 + y^2 \leqslant \rho^2(z)$$

where $\rho(z)$ is the radius of the intersection. With the aid of (1) and properties (i), (ii), we see that the image point $(\bar{x}, \bar{y}, \bar{z})$ satisfies

$$f(\bar{x}, \bar{y}) \leqslant \rho(\bar{z}). \tag{2}$$

Now k is the set of points in an $\bar{x}, \bar{y}$-plane for which

$$f(\bar{x}, \bar{y}) \leqslant 1.$$

Hence the intersection of our plane with K is a plane convex body $k(\bar{z})$. In fact, if we imagine k drawn in the plane $z = 0$, we have to within a translation

$$k(z) = \rho(z)k.$$

We call the z-axis the axis of K and any one of the sets $k(z)$ a directrix for K. The largest directrix is called the equatorial directrix. We always assume k to be non-degenerate, i.e. k is neither a segment nor a point.

Consider the half-plane $\mathscr{H}(\theta)$, bounded by the z-axis, which makes an angle of measure θ, $0 \leqslant \theta < 2\pi$, with the half-plane

$$y = 0, \qquad x \geqslant 0.$$

The intersection of $\mathscr{H}(\theta)$ with K_1 is made up of points $(r \cos\theta, r \sin\theta, z)$ which are characterized by

$$0 \leqslant r \leqslant \rho(z),$$

and the convexity of this intersection is reflected in the property:

(iv) $\qquad \rho(1 - \vartheta)z_0 + \vartheta z_1) \geqslant (1 - \vartheta)\rho(z_0) + \vartheta\rho(z_1), \qquad 0 \leqslant \vartheta \leqslant 1,$

which holds for any z_0, z_1 such that the planes $z = z_0$, $z = z_1$ intersect K_1. Under the transformation (1), $\mathscr{H}(\theta)$ is transformed into itself and if

$$(\bar{r} \cos\theta, \bar{r} \sin\theta, \bar{z})$$

is the image of $(r \cos\theta, r \sin\theta, z)$, then

$$0 \leqslant \bar{r} \leqslant \rho(\bar{z})/f(\cos\theta, \sin\theta),$$

with equality holding if and only if $(\bar{r} \cos\theta, \bar{r} \sin\theta, z)$ is a boundary point of K. For fixed θ, we call the set of such boundary points the meridian $\Gamma(\theta)$ of K; $\Gamma(0)$ is called the prime meridian.

It is clear that the functions f and ρ are determined by K and, given any pair f, ρ which satisfy conditions (i) through (iv), we can construct a unique K with

$$\bar{r} = \rho(\bar{z})/f(1, 0)$$

as the equation (in cylindrical coordinates) of its prime meridian and

$$f(\bar{x}, \bar{y}) \leqslant \max \rho(\bar{z})$$

as the description (in rectangular coordinates) of its equatorial directrix. More generally, $(\bar{x}, \bar{y}, \bar{z})$ is in K if and only if (2) holds.

To demonstrate the convexity of K we consider the segment of points

$$x = (1 - \vartheta)x_0 + \vartheta x_1, \quad y = (1 - \vartheta)y_0 + \vartheta y_1, \quad z = (1 - \vartheta)x_0 + \vartheta z_1 \qquad (0 \leqslant \vartheta \leqslant 1),$$

whose end points are in K and so satisfy (2). We have from (ii), (iii), and (iv)

$$f(x, y) \leqslant (1 - \vartheta)f(x_0, y_0) + \vartheta f(x_1, y_1) \leqslant (1 - \vartheta)\rho(z_0) + \vartheta\rho(z_1) \leqslant \rho(z)$$

which shows that (x, y, z) is in K.

Since we shall be concerned exclusively with rigid motion invariant properties of GCBR, we shall always take the z-axis to be the axis of any GCBR or the common axis of any collection of such figures which are coaxial. Further, we shall often place the equatorial directrices in the plane $z = 0$. Thus when we speak of two coaxial GCBR with similar directrices, we mean that a suitable magnification of the plane $z = 0$, with centre of magnification at the origin, carries one equatorial directrix into the other when the two GCBR are located in this special fashion.

Some examples of GCBR are these: convex bodies of revolution; right truncated cylinders and cones (whose vertices project orthogonally into their bases); the convex closure of the union of a plane convex body and a segment perpendicular to and meeting that plane body. We call figures of this last sort spindles. Of the five regular solids, only the tetrahedron, cube, and octahedron are GCBR.

If one meridian of a GCBR K is polygonal, then so are all its meridians and we say K is polygonal. Note that if K' is that part of K which lies between two planes, normal to the axis of K, which pass through successive meridian vertices, then K' is a truncated cone whose vertex lies on the axis of K. This follows from the similarity, with respect to axial points, between the directrices of K which lie in the truncating planes. K itself is the union of the finite set of such K'. Consider two coaxial, polygonal GCBR. We say these two figures are analogous if they have similar directrices, their prime meridians have the same number of vertices, and the pairs of edges joining corresponding vertices are parallel.

The Blaschke deviation between two convex bodies K_0, K_1 is defined by

$$\delta(K_0, K_1) = \max_{\bar{u}\in\Omega} |H_0(\bar{u}) - H_1(\bar{u})|$$

where H_i signifies the support function of K_i. This deviation is a metric in the space of all convex bodies and we shall use the metric-induced topology in what follows. The fundamental measures V, S, M are continuous in this topology. For details see **(1)**.

The following special approximation theorem will be of service.

THEOREM 1. *Given two coaxial* GCBR K_0 *and* K_1 *with directrices similar to a plane convex body* k*: for each* $\epsilon > 0$, *we can determine two analogous polygonal* GCBR $K_0(\epsilon)$ *and* $K_1(\epsilon)$, *with directrices similar to* k, *such that*

$$\delta(K_i, K_i(\epsilon)) \leqslant \epsilon \qquad (i = 0, 1). \tag{3}$$

Among the rays from the origin in the equatorial plane there is one whose intersection with k has maximal length; for convenience we choose this as the direction of the positive x-axis. This means that the passage from a prime meridian of a GCBR with directrix k to any other meridian involves a reduction of distances in the normal direction to the axis by a factor, fixed for each meridian.

Let A_i signify the convex closure of the prime meridian of K_i. In the plane of the prime meridian we form the outer parallel $A_i + (\epsilon/2)U$, where U is a unit circular body in the x, z-plane, and discard that part of the set which lies in the half-plane $x < 0$. Call this set B_i. We next select a finite set of supporting half-planes to B_i, including the half-plane $y = 0$, $x \geqslant 0$, whose intersection determines a convex polygon C_i such that

$$B_i \subseteq C_i, \qquad C_i \subseteq B_i + (\epsilon/2)U. \tag{4}$$

The existence of C_i is assured by a standard polygonal approximation theorem; cf. **(1)**. Let Θ_i be the set of outer unit normal directions to the half-planes whose intersection is C_i. We form $\Theta = \Theta_0 \cup \Theta_1$ and let D_i denote the intersection of the supporting half-planes to B_i whose outer normal directions are in Θ. Clearly

$$B_i \subseteq D_i \subseteq C_i$$

and so, in the x, z-plane,

$$\delta(D_i, B_i) \leqslant \epsilon/2.$$

This, together with (4), gives

$$\delta(A_i, D_i) \leqslant \epsilon. \tag{5}$$

Excepting the end points of that edge of B_i which is on the z-axis, B_i has no vertices, since the outer parallel $A_i + (\epsilon/2)U$ has none. Hence each of the half-planes whose intersection forms D_i contains an edge of D_i, and D_i has as many edges as there are directions in Θ. From this it follows that D_0 and D_1 have the same number of vertices and that edges joining corresponding pairs of vertices are parallel.

https://doi.org/10.4153/CJM-1967-089-x Published online by Cambridge University Press

Let P_i be the boundary of D_i with the edge lying on the z-axis removed. With P_i as a prime meridian and with an equatorial directrix similar to k, we form the polygonal GCBR $K_i(\epsilon)$. From the preceding comments, we see that $K_0(\epsilon)$, $K_1(\epsilon)$ are coaxial, analogous GCBR. We must now prove (3). This amounts to showing that the distance between support planes Π and $\Pi(\epsilon)$ to K_i and $K_i(\epsilon)$, which have the same outer normal direction, cannot exceed ϵ. Let $\mathscr{H}$ be a half-plane, bounded by the z-axis, which intersects Π in a boundary point of K_i. The similarity of the directrices of K_i and $K_i(\epsilon)$ and the parallelism of Π and $\Pi(\epsilon)$ show that $\mathscr{H}$ intersects $\Pi(\epsilon)$ in a boundary point of $K_i(\epsilon)$. Moreover, the lines

$$L = \mathscr{H} \cap \Pi, \qquad L(\epsilon) = \mathscr{H} \cap \Pi(\epsilon)$$

are parallel support lines to the meridians of K_i and $K_i(\epsilon)$ in $\mathscr{H}$.

If $\mathscr{H}$ is the half-plane of the prime meridians, i.e. $y = 0$, $x \geqslant 0$, then by (5) the distance between L and $L(\epsilon)$ does not exceed ϵ. As we noted at the beginning of the proof, if $\mathscr{H}$ is any other meridional half-plane, the corresponding meridians of K_i and $K_i(\epsilon)$ are obtained from the prime meridians by reducing all distances in directions normal to the z-axis by a fixed factor. Therefore, in this case too, the distance from L to $L(\epsilon)$ cannot exceed ϵ. Hence the distance from Π to $\Pi(\epsilon)$ is no more than ϵ. This completes the proof of the theorem.

Observe that the approximations $K_0(\epsilon)$, $K_1(\epsilon)$ can be chosen so that no meridional edge is parallel to the z-axis; we simply exclude the direction of the positive x-axis from the set Θ above. This will be of later use.

2. Blaschke sums. We next define Blaschke addition of convex bodies with more precision and then consider Blaschke sums of coaxial GCBR with similar directrices.

The area function $S(K, \omega)$ of a non-degenerate convex body is the set function, totally additive over the Borel sets ω of the surface Ω of the unit sphere, which is defined as follows. Let $\Pi(\bar{u})$ be the support plane of K with outer normal in the direction of the vector from the centre of E to $\bar{u}$ on Ω. Then $S(K, \omega)$ is the area of

$$\bigcup_{\bar{u}\in\omega} (\Pi(\bar{u}) \cap K).$$

The area function satisfies:

(a) $S(K, \omega) \geqslant 0$ and is positive if ω is an open hemisphere,

(b) $\int_\Omega (\bar{u}, \bar{v}) S(K, d\omega) = 0$ for all $\bar{u}$ on Ω.

Here $(\bar{u}, \bar{v})$ is the inner product and the integration with respect to $\bar{v}$ is of the Radon–Stieltjes type.

In **(2)**, Fenchel and Jessen proved that any totally additive set function over the Borel sets of Ω which meets conditions (a), (b) is the area function of a convex body which is unique to within a translation. In a less general and less satisfactory form, the theorem is much older and we shall refer to it as Minkowski's theorem.

The linear combination

$$\alpha_0 S(K_0, \omega) + \alpha_1 S(K_1, \omega) \qquad (\alpha_i \geqslant 0, i = 0, 1)$$

of area functions of convex bodies K_i satisfies (a) and (b) and so, by Minkowski's theorem, determines a convex body which we denote by

$$\alpha_0 \times K_0 \# \alpha_1 \times K_1.$$

We call this set the weighted Blaschke sum of K_0, K_1. If $0 \leqslant \vartheta \leqslant 1$ and

$$\alpha_0 = (1 - \vartheta), \alpha_1 = \vartheta,$$

then we write K_ϑ for the Blaschke sum. We note that $\lambda^2 \times K = \lambda K$ for $\lambda > 0$.

It is important for our purposes to remark that K_ϑ is continuous (in the sense of the Blaschke deviation) in the weights and in the summands by virtue of Theorem VIII of **(2)**.

Let K_0 and K_1 be coaxial GCBR with similar directrices. We aim for a description of their weighted Blaschke sum K_ϑ. As usual, we take the common axis of K_0 and K_1 to be the z-axis and let the intersection of the boundary of K_i with the half-plane $y = 0, x \geqslant 0$ be the prime meridian Γ_i of K_i.

We consider some special cases. Let C be a cone with vertex at the origin and with directrix k in a plane normal to the z-axis. We suppose that the z-axis pierces k in an interior point and this point is on the positive half of the z-axis. Let K_i be the set of points (x, y, z) in C for which

$$0 \leqslant z \leqslant \zeta_i, \qquad i = 0, 1.$$

In this case K_ϑ is that part of C for which

$$0 \leqslant z \leqslant \sqrt{[(1 - \vartheta)(\zeta_0)^2 + \vartheta(\zeta_1)^2]} = \zeta_\vartheta. \tag{6}$$

This is because the base areas of K_i and K_ϑ must be proportional to ζ_i^2 and ζ_ϑ^2 and, with the choice (6), the lateral areas swept out on K_i and K_ϑ by rays from the origin passing through an arc of the boundary of k will also be proportional to ζ_i^2 and ζ_ϑ^2. Note that, if ξ_ϑ signifies the x-coordinate of the intersection of the prime meridian with the plane $z = \zeta_\vartheta$, then

$$\xi_\vartheta = \sqrt{[(1 - \vartheta)(\xi_0)^2 + \vartheta(\xi_1)^2]} \tag{7}$$

in virtue of (6) and the fact that

$$\xi_\vartheta : \zeta_\vartheta = \xi_0 : \zeta_0 = \xi_1 : \zeta_1 = -\cot \beta$$

where β is the angle between the positive half of the z-axis and the outer normal to the prime meridian.

Next let K_i'' be the set of points of C for which

$$0 \leqslant \zeta_i' \leqslant z \leqslant \zeta_i,$$

https://doi.org/10.4153/CJM-1967-089-x Published online by Cambridge University Press

and let K_i' be the set of points of C for which

$$0 \leqslant z \leqslant \zeta_i'.$$

Then

$$K_i = K_i' \cup K_i''.$$

From our discussion of K_ϑ we see that the Blaschke sum

$$K_\vartheta' = (1-\vartheta) \times K_0' \# \vartheta \times K_1'$$

is that part of C for which

$$0 \leqslant z \leqslant \sqrt{[(1-\vartheta)(\zeta_0')^2 + \vartheta(\zeta_1')^2]} = \zeta_\vartheta'.$$

From this and the description of K_ϑ, it follows that

$$K_\vartheta'' = (1-\vartheta) \times K_0'' \# \vartheta \times K_1'' \tag{8}$$

is that part of C for which

$$\zeta_\vartheta' \leqslant z \leqslant \zeta_\vartheta.$$

Moreover, if ξ_ϑ' is the x-coordinate of the intersection of the prime meridian of K_ϑ'' with $z = \zeta_\vartheta'$, then

$$\xi_\vartheta' = \sqrt{[(1-\vartheta)(\xi_0')^2 + \vartheta(\xi_1')^2]}. \tag{9}$$

We observe that to specify K_ϑ'' it is enough to specify k, ξ_ϑ', ξ_ϑ, and β. We shall always keep the same directrix and so we denote K_ϑ'' by $K(\xi_\vartheta', \xi_\vartheta, \beta)$. The distance $\zeta_\vartheta - \zeta_\vartheta'$ between the truncating planes of $K(\xi_\vartheta', \xi_\vartheta, \beta)$ can be found from

$$\xi_\vartheta - \xi_\vartheta' : \zeta_\vartheta - \zeta_\vartheta' = -\cot\beta. \tag{10}$$

All this has been stated in terms of sets $K(\xi_\vartheta', \xi_\vartheta, \beta)$ for which

$$\xi_\vartheta' < \xi_\vartheta \quad \text{and} \quad \pi/2 < \beta < \pi.$$

Clearly formulas (7), (9), and (10) also hold if

$$\xi_\vartheta < \xi_\vartheta' \quad \text{and} \quad 0 < \beta < \pi/2.$$

It should be remarked that the description of $K(\xi_\vartheta', \xi_\vartheta, \beta)$ furnished by (7), (9), (10) makes no use of the fact that the z-coordinate of the vertex of C is zero: in short $K(\xi_\vartheta', \xi_\vartheta, \beta)$ is described only to within a translation in the direction of the z-axis.

Next let K_0 and K_1 be analogous GCBR with directrices similar to k. We denote the vertices of their prime meridians Γ_i by their x, z-coordinates:

$$(\xi_i^0, \zeta_i^0), \quad (\xi_i^1, \zeta_i^1), \quad \ldots, \quad (\xi_i^n, \zeta_i^n), \quad \xi_i^0 = \xi_i^n = 0, \tag{11}$$

where

$$\xi_0^k - \xi_0^{k+1} : \zeta_0^k - \zeta_0^{k+1} = \xi_1^k - \xi_1^{k+1} : \zeta_1^k - \zeta_1^{k+1} = -\cot\beta_k \tag{12}$$
$$(k = 0, 1, \ldots, n-1)$$

because K_0 and K_1 are analogous. Here β_k signifies the measure of the angle between the outer normal to the edge joining (ξ_i^k, ζ_i^k) to $(\xi_i^{k+1}, \zeta^{k+1})$ and the positive z-axis. The indexing is so chosen that

$$\zeta_i^k < \zeta_i^{k+1}, \qquad \beta_k > \beta_{k+1};$$

of course the last inequalities are a consequence of the first, coupled with the concavity of Γ_i with respect to the z-axis. We assume that none of the numbers β_k is $\pi/2$.

The part of K_i which lies in the set

$$\zeta_i^k \leqslant z \leqslant \zeta_i^{k+1}$$

is the GCBR which, in our earlier notation, is denoted by $K(\xi_i^k, \xi_i^{k+1}, \beta_k)$ and so

$$K_i = \bigcup_{k=1}^{n-1} K(\xi_i^k, \xi_i^{k+1}, \beta_k).$$

As in (10) we define, for $0 \leqslant \vartheta \leqslant 1$,

$$\xi_\vartheta^k = \sqrt{[(1-\vartheta)(\xi_0^k)^2 + \vartheta(\xi_1^k)^2]}, \tag{13}$$

and, in virtue of (12), we determine ζ_ϑ^k from

$$\xi_\vartheta^{k+1} - \xi_\vartheta^k : \zeta_\vartheta^{k+1} - \zeta_\vartheta^k = -\cot \beta_k, \tag{14}$$

and a choice of ζ_ϑ^0. In this way, by (13), (14), and our choice of ζ_ϑ^0, we determine points

$$(\xi_\vartheta^0, \zeta_\vartheta^0), \quad (\xi_\vartheta^1, \zeta_\vartheta^1), \quad \ldots, \quad (\xi_\vartheta^n, \zeta_\vartheta^n), \quad \xi_\vartheta^0 = \xi_\vartheta^n = 0$$

in the prime meridian plane. These are the vertices of a polygon Γ_ϑ which is concave with respect to the z-axis. This is a consequence of the inequalities

$$\zeta_\vartheta^k < \zeta_\vartheta^{k+1}, \qquad \beta_k > \beta_{k+1};$$

the second set of inequalities comes from the concavity of Γ_0 and Γ_1, and the first set is an easy consequence of (13) and (14).

With Γ_ϑ as a prime meridian and with a directrix similar to k, we form the GCBR

$$K_\vartheta = \bigcup_{k=1}^{n-1} K(\xi_\vartheta^k, \xi_\vartheta^{k+1}, \beta_k).$$

From the polygonal character of K_ϑ and the equation

$$K(\xi_\vartheta^k, \xi_\vartheta^{k+1}, \beta_k) = (1-\vartheta) \times K(\xi_0^k, \xi_0^k, \beta_k) \,\#\, \vartheta \times K(\xi_1^k, \xi_1^{k+1}, \beta_k),$$

which is (8) with altered notation, we conclude that K_ϑ is the weighted Blaschke sum of K_0 and K_1.

The general case is treated with the aid of Theorem 1. Thus if K_0 and K_1 are coaxial GCBR with directrices similar to k, then we may simultaneously approximate them by analogous polygonal bodies $\bar{K}_0$, $\bar{K}_1$ whose directrices are similar to k and are such that their prime meridians have no edges parallel to their common axis. The approximation can be made so that

$$\delta(K_i, \bar{K}_i) < \epsilon$$

for preassigned $\epsilon > 0$. Since

$$K_\vartheta = (1-\vartheta) \times K_0 \# \vartheta \times K_1$$

is continuous in K_0 and K_1, the weighted Blaschke sum $\bar{K}_\vartheta$ of $\bar{K}_0$ and $\bar{K}_1$ tends to K_ϑ as ϵ tends to zero.

To determine the equations corresponding to (13) and (14) in the general case, we let $\xi_i(\beta)$, $\zeta_i(\beta)$ be the x, z-coordinates of that point on the prime meridian Γ_i at which there is a support line in the x, z-plane with an outer normal which makes an angle β with the positive z-axis. Then, in lieu of (13), we conclude from our approximation argument that

$$\xi_\vartheta(\beta) = \sqrt{[(1-\vartheta)(\xi_0(\beta))^2 + \vartheta(\xi_1(\beta))^2]}. \tag{15}$$

To deal with (14) we rewrite it for the polygonal case:

$$\zeta_\vartheta^{k+1} = \zeta_\vartheta^{0} - \sum_{j=0}^{k} \tan \beta_j (\xi_\vartheta^{j+1} - \xi_\vartheta^{j}),$$

which gives in the limit the improper Stieltjes integral

$$\zeta_\vartheta(\beta) = \zeta_\vartheta(\pi) - \int_\pi^\beta \tan \bar{\beta}\, d\xi_\vartheta(\bar{\beta}). \tag{16}$$

The concavity of Γ_ϑ ensures that ξ_ϑ is of bounded variation. Equations (15) and (16) describe the prime meridian of K_ϑ, which, since its directrix is similar to k, suffices to determine K_ϑ to within a translation, or else exactly for preassigned $\xi_\vartheta(\pi)$.

One consequence of our description is that the Blaschke sum of coaxial GCBR with similar directrices is a GCBR.

For our next theorem it is useful to note that (13) and (14) remain valid even if we allow some of the vertices in (11) to coalesce; this follows from equations (15) and (16). Thus we may use (13) and (14) to determine K_ϑ in all those cases in which K_0 and K_1 are coaxial polygonal GCBR with similar directrices none of whose meridian edges are parallel to their common axis.

We close this section with a decomposition theorem and an approximation result.

THEOREM 2. *A polygonal* GCBR *K which has no meridian edges parallel to its axis can be represented as a finite Blaschke sum of spindles, all coaxial with K and having directrices similar to that of K.*

Let $n+1$ denote the number of vertices of the prime meridian Γ of K, including the end points of Γ. We have $n \geqslant 2$. The proof will be inductive on n; since K is a spindle if $n = 2$, that case is settled.

Suppose the theorem true for $n \leqslant N$, where $N \geqslant 2$, and assume that Γ has $N + 2$ vertices. In accordance with our earlier notation, let the x, z-coordinates of these points be

$$(\xi^0, \zeta^0), \quad (\xi^1, \zeta^1), \quad \ldots, \quad (\xi^{N+1}, \zeta^{N+1}), \quad \xi^0 = \xi^{N+1} = 0,$$

and let β_k $(k = 0, 1, \ldots, N)$ signify the angle between the edge joining (ξ^k, ζ^k) to (ξ^{k+1}, ζ^{k+1}) and the positive z-axis. By assumption, no β_k equals $\pi/2$. Let us assume that

$$\xi^1 \geqslant \xi^N; \tag{17}$$

if the reversed inequality holds, our argument will proceed along obviously similar lines.

Let K_0 be the spindle

$$K(0, \xi^N, \beta_0) \cup K(\xi^N, 0, \beta_N)$$

whose directrix is similar to that of K. We view the prime meridian of K_0 as having vertices

$$(\xi_0{}^0, \zeta_0{}^0), \quad (\xi_0{}^1, \zeta_0{}^1) = (\xi_0{}^2, \zeta_0{}^2) = \ldots = (\xi_0{}^N, \zeta_0{}^N), \quad (\xi_0{}^{N+1}, \zeta_0{}^{N+1}),$$

where

$$\xi_0{}^N = \xi^N, \qquad \xi_0{}^0 = \xi_0{}^{N+1} = 0.$$

That is to say we view $(\xi_0{}^N, \zeta_0{}^N)$ as the result of the coalescence of N vertices.

Next we determine points in the prime meridian plane whose x, z-coordinates are

$$(\xi_1{}^0. \zeta_1{}^0), \quad \ldots, \quad (\xi_1{}^N, \zeta_1{}^N) = (\xi_1{}^{N+1}, \zeta_1{}^{N+1}), \quad \xi_1{}^0 = \xi_1{}^N = \xi_1{}^{N+1} = 0, \tag{18}$$

where

$$\xi_1{}^k = \sqrt{[(\xi^k)^2 - (\xi_0{}^k)^2]} \qquad (k = 0, 1, \ldots, N + 1). \tag{19}$$

These numbers are well defined since (17) and the concavity of Γ imply that $\xi^k \geqslant \xi^N$ for $k = 1, 2, \ldots, N$. Also

$$\xi_1{}^0 = \xi_1{}^{N+1} = 0.$$

The numbers $\zeta_1{}^k$ are determined by preassigning $\zeta_1{}^0 = \zeta^0$ and using the rule

$$\xi_1{}^k - \xi_1{}^{k+1} : \zeta_1{}^k - \zeta_1{}^{k+1} = -\cot \beta_k \qquad (k = 0, 1, \ldots, N). \tag{20}$$

The concavity of Γ and equations (20) allow us to conclude that (18) is a set of vertices of a polygon Γ_1, concave with respect to the z-axis, in the prime meridian plane. Take K_1 to be the GCBR with Γ_1 as its prime meridian and with directrix similar to that of K.

We claim that

$$K = K_0 \sharp K_1.$$

Indeed, by (20),

$$\xi^k = \sqrt{[(\xi_0{}^k)^2 + (\xi_1{}^k)^2]}$$

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and, with the use of the angles β_k, we recover the numbers ζ_k in the usual way. We note that the prime meridian Γ_1 has $N + 1$ or N distinct vertices depending on whether inequality (17) is strict or not. In either case, we may apply our induction hypothesis to K_1 and in this way the proof of the theorem is completed.

3. The mean width of Blaschke sums. In this section we shall first study the behaviour of the total mean curvature M of the weighted Blaschke sum K_ϑ in the special case in which K_0 and K_1 are coaxial convex bodies of revolution. We assume these figures all to be in standard position so that the directrices of K_ϑ, for $0 \leqslant \vartheta \leqslant 1$, are circles centred on the z-axis in planes normal to that axis.

First suppose K_0 and K_1 are analogous. We denote the x, z-coordinates of the vertices of the prime meridian Γ_i as in (11) and we assume that equations (12) hold. We also suppose that no β_k is $\pi/2$. Hence K_ϑ is polygonal with prime meridian vertices given by (13) and (14).

Hadwiger in **(3)** gives a convenient representation for M in the case of polygonal convex bodies of revolution:

$$M(K_\vartheta) = \pi \sum_{k=1}^{n-1} \xi_\vartheta^k[f(\beta_k) - f(\beta_{k-1})],$$

where

$$f(\beta) = \tan \beta - \beta.$$

We note that

$$f(0) = 0, \qquad f(\pi) = -\pi$$

and

$$df(\beta)/d\beta = \sec^2\beta - 1 > 0 \qquad \text{for } 0 < \beta < \pi, \beta \neq \pi/2.$$

Thus, if $m (0 \leqslant m \leqslant n - 1)$ is such that, in the decreasing sequence $\{\beta_k\}$ of angular measures, we have

$$\beta_{m-1} > \pi/2, \qquad \beta_m < \pi/2,$$

then

$$f(\beta_k) - f(\beta_{k-1}) < 0 \qquad \text{for } k \neq m,$$

$$f(\beta_m) - f(\beta_{m-1}) > 0.$$

Consequently, if we define the positive numbers p_ϑ^k by

$$p_\vartheta^k = \pi\xi_\vartheta^k|f(\beta_k) - f(\beta_{k-1})| \qquad (k = 1, 2, \ldots, n - 1),$$

we have

$$p_\vartheta^m - M(K_\vartheta) = \textstyle\sum_{k\neq m} p_\vartheta^k. \tag{21}$$

From the definition of p_ϑ^k and equations (13) and (14) we have

$$p_\vartheta^k = \sqrt{[(1 - \vartheta)(p_0^k)^2 + \vartheta(p_1^k)^2]} \tag{22}$$

and, by Minkowski's inequality,

$$\sum_{k\neq m} p_\vartheta^k \geqslant \sqrt{[(1-\vartheta)(\sum_{k\neq m} p_0^k)^2 + \vartheta(\sum_{k\neq m} p_1^k)^2]}.$$

Consequently, using (21), we get

$$[p_\vartheta^m - M(K_\vartheta)]^2 \geqslant (1-\vartheta)[p_0^m - M(K_0)]^2 + \vartheta[p_1^m - M(K_1)]^2.$$

In view of (22), this may be written

$$(23)\quad M^2(K_\vartheta) - (1-\vartheta)M^2(K_0) - \vartheta M^2(K_1)$$
$$\geqslant 2p_\vartheta^m\{M(K_\vartheta) - [(1-\vartheta)(p_0^m/p_\vartheta^m)M(K_0) + \vartheta(p_1^m/p_\vartheta^m)M(K_1)]\}.$$

To the expression in square brackets on the right we apply Cauchy's inequality and obtain, with the aid of (22),

$$(24)\quad M^2(K_\vartheta) - (1-\vartheta)M^2(K_0) - \vartheta M^2(K_1) \geqslant 2p_\vartheta^m$$
$$\{M(K_\vartheta) - \sqrt{[(1-\vartheta)M^2(K_0) + \vartheta M^2(K_1)]}\}.$$

We shall prove in a moment that the convex bodies of revolution K_0 and K_1 can be simultaneously altered in such a way as to make p_ϑ^m arbitrarily large without altering the numbers $M(K_\vartheta)$, $M(K_0)$, and $M(K_1)$ by more than an arbitrarily small preassigned positive quantity. From this it follows that

$$(25)\qquad M^2(K_\vartheta) \leqslant (1-\vartheta)M^2(K_0) + \vartheta M^2(K_1).$$

Let ϵ be a positive number less than any of the positive differences

$$\xi_0^m - \xi_0^{m+1}, \qquad \xi_0^m - \xi_0^{m-1}, \qquad \xi_1^m - \xi_1^{m+1}, \qquad \xi_1^m - \xi_1^{m-1}.$$

For $i = 0, 1$ we consider, in the prime meridian plane, the line L_i:

$$x = \xi_i^m - \epsilon$$

parallel to the z-axis. The prime meridians Γ_i of K_i cut off equal segments on Γ_i. With these as bases we construct isosceles triangles T_i of altitude $\lambda\epsilon$, $0 < \lambda < 1$, lying in the half-plane

$$(26)\qquad x \geqslant \xi_i^m - \epsilon.$$

Clearly T_0, T_1 are translates of each other. Let β'_{m-1}, β'_m denote the measures of the angles between the outer normals to the legs of these triangles and the positive z-axis. We restrict λ to be small enough so that

$$\beta_{m-1} > \beta'_{m-1} > \pi/2 > \beta_m' > \beta_m.$$

We replace that part of Γ_i which lies in (26) by the legs of T_i; this yields concave polygons Γ_i' which can serve as prime meridians of new convex bodies of revolution K_i'. Since Γ_0', Γ_1' have the same number of vertices and since sides joining corresponding pairs of vertices are parallel, K_0' and K_1' are analogous. Of course K_0', K_1' depend on ϵ and λ.

Regardless of the choice of λ, as ϵ tends to zero the bodies K_0', K_1' tend to K_0, K_1 and the Blaschke sums

$$K_\vartheta' = (1 - \vartheta) \times K_0' \# \vartheta \times K_1'$$

tend to K_ϑ in virtue of the continuity of K_ϑ' in its summands which we mentioned earlier. Finally we recall that $M(K)$ is continuous in K. Consequently, we can choose ϵ small enough so that in (24) the left side and the expression in curly brackets on the right side, when evaluated at K_0', K_1', K_ϑ', differ from their values for K_0, K_1, K_ϑ by less than a preassigned positive number whatever the choice of λ.

Consider the quantity p_ϑ', defined for K_ϑ' as p_ϑ^m is defined for K_ϑ, viz.

$$p_\vartheta' = \pi \xi_\vartheta'^m (f(\beta_m') - f(\beta'_{m-1})) = \pi \xi_\vartheta'^m (2 \tan \beta_m' + 2\beta_m' - \pi)$$

in view of the isosceles character of T_0 and T_1. Clearly the equatorial radius $\xi_\vartheta'^m$ is bounded away from zero as λ tends to zero. On the other hand, the remaining factor in p_ϑ' grows large without bound as λ tends to zero since this causes β_m' to tend to $\pi/2$ from below. Hence we can alter K_0, K_1, K_ϑ in the fashion originally asserted, and the proof of (25) is complete.

Let us return to (23). From (25) and the positivity of p_ϑ^m, we see that the expression in curly brackets is non-positive. Since p_0^m, p_1^m, and p_ϑ^m are proportional to the equatorial radii

$$\xi_0^m = a(K_0), \qquad \xi_1^m = a(K_1), \qquad \xi_\vartheta^m = a(K_\vartheta)$$

of the analogous convex bodies of revolution K_0, K_1, K_ϑ, we get

$$a(K_\vartheta)M(K_\vartheta) \leqslant (1 - \vartheta)a(K_0)M(K_0) + \vartheta a(K_1)M(K_1). \tag{27}$$

In particular, when $a(K_0) = a(K_1)$, then $a(K_\vartheta)$ has the same value and (27) becomes

$$M(K_\vartheta) \leqslant (1 - \vartheta)M(K_0) + \vartheta M(K_1). \tag{28}$$

This is an improvement on (25), for the special case considered, because the arithmetic mean is less than the root mean square.

Theorem 1 and the remark immediately following it show that any convex bodies of revolution can be arbitrarily well approximated by convex bodies of revolution of the special type which, up to this point, we have allowed for K_0 and K_1. Consequently, from the continuity of K_ϑ in the summands K_0 and K_1 and the continuity of $a(K)$, $M(K)$ in K, we deduce our next theorem.

THEOREM 3. *If K_0 and K_1 are coaxial convex bodies of revolution and K_ϑ is their weighted Blaschke sum, then inequalities* (25) *and* (27) *hold where $M(K)$ and $a(K)$ signify the total mean curvature and equatorial radius of K. When K_0 and K_1 have equal equatorial radii, then* (28) *is true.*

In preparation for the extension of (25) and (28) to a more general case let us consider a GCBR K whose axis is, as usual, the z-axis and whose equatorial directrix k is in the plane $z = 0$. We introduce geographic coordinates (θ, ϕ) on the unit spherical surface Ω centred at $(0, 0, 0)$ so that $(0, 0, 1)$ is the north pole and the points of longitude $\theta = 0$ lie in the half-plane $y = 0$, $x \geqslant 0$, that is, the prime meridian half-plane. Let $H(\theta, \phi)$ signify the distance from the origin to the support plane $\Pi(\theta, \phi)$ of K whose outer unit normal has geographic coordinates (θ, ϕ). The total mean curvature of K is—cf. **(1)**

$$M(K) = \int_0^{2\pi} \int_0^{\pi} H(\theta, \phi) \sin \phi \, d\phi d\theta. \tag{29}$$

In the special case in which K is a convex body of revolution, H is independent of θ and we have, for any choice of θ,

$$M(K) = 2\pi \int_0^{\pi} H(\theta, \phi) \sin \phi \, d\phi. \tag{30}$$

For fixed θ, the planes $\Pi(\theta, \phi)$, $0 \leqslant \phi \leqslant \pi$, envelope a cylindrical surface $C(\theta)$ which is tangential to K. The generators of $C(\theta)$ are perpendicular to the half-plane $\mathscr{H}(\theta)$, bounded by the z-axis, which forms an angle of measure θ with the prime meridian half-plane. Suppose $\Pi(\theta, 0)$, $\Pi(\theta, \pi/2)$, and $\Pi(\theta, \pi)$ touch K in single points. Simple similarity considerations, applied in the planes $z =$ const. which contain directrices similar to k, show that $C(\theta)$ touches K along a meridian Γ'. Let ψ be the angle between $\mathscr{H}(\theta)$ and the half-plane containing Γ' and denote the intersection of $C(\theta)$ with $\mathscr{H}(\theta)$ by $Q(\theta)$.

The curve $Q(\theta)$ satisfies the requirements for a curve to be a meridian of a convex body of revolution $K(\theta)$ with axis along the z-axis. If we set

$$\mu(\theta) = M(K(\theta)),$$

then by (30)

$$\mu(\theta)/2\pi = \int_0^{\pi} H(\theta, \phi) \sin \phi \, d\phi \tag{31}$$

and so, from (29), we have

$$M(K) = \int_0^{2\pi} \mu(\theta) \, d\theta/2\pi. \tag{32}$$

Notice that if ξ is the distance from a point in Γ' to the z-axis, then the corresponding point on $Q(\theta)$ is at a distance $\xi \cos \psi$ from the z-axis. In particular, for such points in the equatorial plane of K, we find that

$$\xi \cos \psi = H(\theta, \pi/2) = h(\theta) \tag{33}$$

is the equatorial radius of $K(\theta)$, where $h(\theta)$ is the distance from the origin to that support line of k, in the equatorial plane, whose outer normal makes an angle θ with the positive x-axis.

The restriction that $\Pi(\theta, 0)$, $\Pi(\theta, \pi/2)$, and $\Pi(\theta, \pi)$ touch K in single points is inessential; if this is not so, it is still possible to find at least one meridian Γ' and associated angle ψ for which the foregoing discussion is valid.

Let K_i $(i = 0, 1)$ be coaxial GCBR with identical equatorial directrices k. Their weighted Blaschke sum K_ϑ has equatorial directrix λk. However, $\lambda = 1$. This is because the similarity ratio of the directrices of K_0 and K_1 is

$$1 = \xi_0(\pi/2) : \xi_1(\pi/2)$$

in the notation of (15). From that same equation it follows that

$$\xi_\vartheta(\pi/2) : \xi_1(\pi/2) = \lambda = 1.$$

Thus if $H_\vartheta(\theta, \phi)$ is the support distance for K_ϑ, we have

$$H_\vartheta(\theta, \pi/2) = h(\theta) \qquad \text{for } 0 \leqslant \vartheta \leqslant 1. \tag{34}$$

Let $Q_\vartheta(\theta)$, $K_\vartheta(\theta)$ be the figures associated with K_ϑ in the same way in which $Q(\theta)$, $K(\theta)$ were associated with K above. Thus, from (31), we obtain

$$M(K_\vartheta(\theta)) = \mu_\vartheta(\theta) = 2\pi \int_0^\pi H_\vartheta(\theta, \phi)\, d\phi.$$

We shall prove that

$$K_\vartheta(\theta) = (1 - \vartheta) \times K_0(\theta) \# \vartheta \times K_1(\theta). \tag{35}$$

When this has been done, we deduce for the total mean curvature $\mu_\vartheta(\theta)$ of $K_\vartheta(\theta)$:

$$\mu_\vartheta(\theta) \leqslant (1 - \vartheta)\mu_0(\theta) + \vartheta\mu_1(\theta) \tag{36}$$

from (28) and the fact that $K_0(\theta)$ and $K_1(\theta)$ have equal equatorial radii by (34). In turn, because of (32), we have from integration of (36) with respect to θ

$$M(K_\vartheta) \leqslant (1 - \vartheta)M(K_0) + \vartheta M(K_1).$$

That is to say (28) holds when K_0 and K_1 are GCBR with identical equatorial directrices.

It remains to prove (35).

In keeping with earlier notation, let Γ_ϑ' be the meridian along which the tangential cylinder with generators perpendicular to $\mathscr{H}(\theta)$ touches K_ϑ; thus $Q_\vartheta(\theta)$ is the result of projecting Γ_ϑ' orthogonally onto $\mathscr{H}(\theta)$. As before, ψ denotes the measure of the dihedral angle formed by $\mathscr{H}(\theta)$ and the half-plane of Γ_ϑ'. In turn Γ_ϑ' is obtained from the prime meridian Γ_ϑ of K_ϑ by multiplying all the distances from Γ_ϑ to the z-axis by a fixed factor $\tau > 0$, and then rotating the resulting figure about the z-axis into the half-plane of Γ_ϑ'. Hence $Q_\vartheta(\theta)$ can be constructed from Γ_ϑ by multiplying all the distances from Γ_ϑ to the z-axis by $\tau \cos \psi$ so as to obtain a curve G_ϑ and then rotating G_ϑ about the z-axis into $\mathscr{H}(\theta)$. The quantities τ and ψ depend on θ and the directrix of K_ϑ. Since this directrix is k for all θ, the factor $\tau \cos \psi$ is independent of ϑ. Of course, G_ϑ is the prime meridian of the convex body of revolution $K_\vartheta(\theta)$.

In the x, z-plane Γ_ϑ is the set of points $(\xi_\vartheta(\beta), \zeta_\vartheta(\beta))$, where β is the angle between the positive z-axis and the outer normal in the x, z-plane to Γ_ϑ at that point. From the preceding discussion, the plane $z = \zeta_\vartheta(\beta)$ cuts G_ϑ in a point whose distance $\bar{\xi}_\vartheta(\beta)$ to the z-axis satisfies

(37) $$\bar{\xi}_\vartheta(\beta) = \xi_\vartheta(\beta)\tau\cos\psi.$$

By (15), followed by (37), we have

(38) $$\begin{aligned}\bar{\xi}_\vartheta(\beta) &= \sqrt{[(1-\vartheta)(\xi_0(\beta))^2\tau^2\cos^2\psi + \vartheta(\xi_1(\beta))^2\tau^2\cos^2\psi]}\\ &= \sqrt{[(1-\vartheta)(\bar{\xi}_0(\beta))^2 + \vartheta(\bar{\xi}_1(\beta))^2]}.\end{aligned}$$

Thus the prime meridian of $K_\vartheta(\theta)$ is the set of points $(\bar{\xi}_\vartheta(\beta), \zeta_\vartheta(\beta))$ for $0 \leqslant \beta \leqslant \pi$.

In general β does not have the significance of the measure of the angle between the positive z-axis and the outer normal in the x, z-plane to G_ϑ. However, simple similarity considerations show that at $(\bar{\xi}_\vartheta(\beta), \zeta_\vartheta(\beta))$ this angle has measure ϕ given by

(39) $$\tan\phi = (\tan\beta)/\tau\cos\psi, \qquad \text{if } \beta \neq \pi/2.$$
$$\phi = \pi/2, \qquad \text{if } \beta = \pi/2.$$

Thus if we let g be defined by $g(\phi) = \beta$, then

(40) $$g(\pi) = \pi.$$

Moreover, the functions x_ϑ, z_ϑ, defined by

(41) $$x_\vartheta(\phi) = \bar{\xi}_\vartheta(g(\phi)), \qquad z_\vartheta(\phi) = \zeta_\vartheta(g(\phi)),$$

give a parametric representation of G_ϑ such that the outer normal to G_ϑ at $(x_\vartheta(\phi), z_\vartheta(\phi))$ forms an angle of measure ϕ with the positive z-axis.

From (38) we obtain

(42) $$x_\vartheta(\phi) = \sqrt{[(1-\vartheta)(x_0(\phi))^2 + \vartheta(x_1(\phi))^2]}.$$

Since $\zeta_\vartheta(\beta)$ is described by (16), $z_\vartheta(\phi)$ is given by

$$z_\vartheta(\phi) = \zeta_\vartheta(g(\pi)) - \int_\pi^{g(\phi)} \tan\bar{\beta}\, d\bar{\xi}_\vartheta(\bar{\beta}).$$

By the use of (37), (39), and (40) we deduce that

(43) $$\begin{aligned}z_\vartheta(\phi) &= z_\vartheta(\pi) - \int_\pi^{\phi} \tan\bar{\phi}\, d\bar{\xi}_\vartheta(g(\bar{\phi}))\\ &= z_\vartheta(\pi) - \int_\pi^{\phi} \tan\bar{\phi}\, dx_\vartheta(\bar{\phi})\end{aligned}$$

in virtue of the first equation in (41). The comparison of (42), (43) with (15) and (16) and the fact that $K_0(\theta)$, $K_1(\theta)$ have circular directrices shows that (35) is true.

We can now prove that (25) holds for the Blaschke sum K_ϑ of any pair K_0, K_1 of coaxial GCBR with similar directrices. In this case the equatorial directrices of K_0, K_1 are $\lambda_0 k$, $\lambda_1 k$ for some plane convex body k and some positive numbers λ_0, λ_1. The convex bodies

$$K_i' = (1/\lambda_i)K_i = (1/\lambda_i)^2 \times K_i \qquad (i = 0, 1),$$

have identical equatorial directrices. Set

$$\vartheta' = \vartheta\lambda_1^2/[(1-\vartheta)\lambda_0^2 + \vartheta\lambda_1^2] \qquad (0 \leqslant \vartheta \leqslant 1)$$

and

$$K'_{\vartheta'} = (1-\vartheta') \times K_0' \# \vartheta' \times K_1'.$$

Then, by (28),

$$M(K'_{\vartheta'}) \leqslant (1-\vartheta')M(K_0') + \vartheta' M(K_1').$$

Since

$$M(\lambda^2 \times K) = M(\lambda K) = \lambda M(K)$$

and

$$K'_{\vartheta'} = (1/[(1-\vartheta)\lambda_0^2 + \vartheta\lambda_1^2]) \times [(1-\vartheta) \times K_0 \# \vartheta \times K_1],$$

we deduce that

$$(44) \quad M(K_\vartheta) \leqslant [(1-\vartheta)\lambda_0 M(K_0) + \vartheta\lambda_1 M(K_1)]/\sqrt{[(1-\vartheta)\lambda_0^2 + \vartheta\lambda_1^2]}.$$

Cauchy's inequality, applied to the numerator on the right side of this last inequality, proves (25).

Suppose α to be a set function, defined over all non-degenerate plane convex bodies k, which is rigid motion invariant, positive, and homogeneous of degree one, i.e.

$$(45) \qquad \alpha(\lambda k) = \lambda\alpha(k) \qquad (\lambda > 0).$$

For example, we may take $\alpha(k)$ to be the perimeter of k. In turn define the set function a over all GCBR by

$$(46) \qquad a(K) = \alpha(k),$$

where k is the equatorial directrix of K. Clearly a is homogeneous of degree one. In the notation of the preceding paragraphs

$$\lambda_i = a(K_i)/a(K).$$

It follows from equation (15) that

$$\lambda_\vartheta = \sqrt{[(1-\vartheta)\lambda_0^2 + \vartheta\lambda_1^2]},$$

where $\lambda_\vartheta k$ is the directrix of K_ϑ. This and (44) yield

$$(47) \qquad a(K_\vartheta)M(K_\vartheta) \leqslant (1-\vartheta)a(K_0)M(K_0) + \vartheta a(K_1)M(K_1).$$

We summarize our results.

THEOREM 4. *If K_0, K_1 are coaxial* GCBR *with similar directrices and K_ϑ is their weighted Blaschke sum, then inequalities* (25) *and* (47) *hold where $M(K)$ is the total mean curvature of K and $a(K)$ is a set function of the type described by*

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(45) *and* (46). *When* K_0 *and* K_1 *have identical equatorial directrices, then* (28) *is true.*

4. Blaschke sums of cylinders. If K_0, K_1 are not coaxial GCBR with similar directrices, then inequality (25) need not be true.

Let k_0, k_1 be two convex bodies in the x, y-plane. We form the cylinders K_0 and K_1 with k_0 and k_1 as directrices and with generators parallel to the z-axis. We suppose both these cylinders to be truncated by the planes $z = 0$, $z = \zeta$ where $\zeta > 0$.

We first describe the construction of the Blaschke sum K_ϑ. Denote by $\sigma(k)$ the area of the plane convex body k. Form the weighted vector sum

$$k_\vartheta = \mu_\vartheta[(1 - \vartheta)k_0 + \vartheta k_1] \tag{48}$$

where

$$\mu_\vartheta = \sqrt{\left[\frac{(1 - \vartheta)\sigma(k_0) + \vartheta\sigma(k_1)}{\sigma((1 - \vartheta)k_0 + \vartheta k_1)}\right]}. \tag{49}$$

Also set

$$\zeta_\vartheta = \zeta/\mu_\vartheta \tag{50}$$

and define K to be the cylinder with directrix k_ϑ, having generators parallel to the z-axis and which is truncated by the planes $z = 0$, $z = \zeta_\vartheta$.

Let ω be a Borel set on the spherical surface Ω; we write ω' for the intersection of ω with the plane $z = 0$. Denote by $ds_\vartheta(\phi)$ $(0 \leqslant \vartheta \leqslant 1)$ the arc element at the boundary point of k_ϑ at which the outer normal makes an angle ϕ

$$(0 \leqslant \phi < 2\pi)$$

with the positive x-axis. Then

$$S(K, \omega) = \zeta_\vartheta \int_{\omega'} ds_\vartheta(\phi) + \nu(\omega)\sigma(k_\vartheta)$$

where $\nu(\omega)$ is 0, 1, or 2 according to the number of points in the intersection of ω with the z-axis.

From well-known properties of vector addition and from (48), (49), and (50), we have:

$$\zeta_\vartheta\, ds_\vartheta(\phi) = \zeta[(1 - \vartheta)ds_0(\phi) + \vartheta ds_1(\phi)],$$

$$\sigma(k_\vartheta) = \mu_\vartheta^2\sigma((1 - \vartheta)k_0 + \vartheta k_1) = (1 - \vartheta)\sigma(k_0) + \vartheta\sigma(k_1).$$

From these last three equations we deduce that

$$S(K, \omega) = (1 - \vartheta)S(K_0, \omega) + \vartheta S(K_1, \omega)$$

and so $K = K_\vartheta$.

For figures of this type we have

$$M(K_\vartheta) = \pi[s(k_\vartheta)/2 + \zeta_\vartheta],$$

where $s(k)$ signifies the perimeter of k. Equations (48), (49), (50) yield

$$M(K_\vartheta) = \pi[\mu_\vartheta((1-\vartheta)s(k_0) + \vartheta s(k_1))/2 + \zeta/\mu_\vartheta]. \tag{51}$$

Set

$$D(\vartheta) = [M^2(K_\vartheta) - (1-\vartheta)M^2(K_0) - \vartheta M^2(K_1)]/\pi^2;$$

our goal is to show that we have $D(\vartheta) > 0$ when $0 < \vartheta < 1$ for some choices of k_0, k_1, ζ—that is to say for some choices of K_0 and K_1. For this purpose, we write $\sigma(k_0, k_1)$ for the mixed area of k_0 and k_1 and define $A(\vartheta)$ and $B(\vartheta)$ by

$$A(\vartheta) = 2\vartheta(1-\vartheta)[\sigma(k_0, k_1) - (\sigma(k_0) + \sigma(k_1))/2]/[(1-\vartheta)\sigma(k_0) + \vartheta_0(k_1)],$$

$$B(\vartheta) = [(1-\vartheta)s(k_0) + \vartheta s(k_1)]^2/(1 + A(\vartheta)) - [(1-\vartheta)s^2(k_0) + \vartheta s^2(k_1)].$$

Using (49) and (51), we find by direct computation that

$$D(\vartheta) = A(\vartheta)\zeta^2 + B(\vartheta).$$

The quantities $A(\vartheta)$, $B(\vartheta)$ depend on ϑ, k_0, k_1 only. Moreover, given three positive numbers $\sigma_0, \sigma_1, \sigma_{01}$ subject to

$$\sigma_{01}{}^2 \geqslant \sigma_0\sigma_1,$$

there are plane convex bodies k_0, k_1 such that

$$\sigma_0 = \sigma(k_0), \qquad \sigma_1 = \sigma(k_1), \qquad \sigma_{01} = \sigma(k_0, k_1).$$

From this we see that we can make $A(\vartheta) > 0$ by any choice of k_0, k_1 for which

$$\sigma(k_0, k_1) > [\sigma(k_0) + \sigma(k_1)]/2. \tag{52}$$

With such a choice of k_0, k_1 and with a suitably large choice of ζ, we have $D(\vartheta) > 0$ as asserted.

In connection with choosing k_0, k_1 so that (52) holds, we attach the following remark. If k_0, k_1 have the same width in some direction, then it is known that

$$\sigma((1-\vartheta)k_0 + \vartheta k_1) \geqslant (1-\vartheta)\sigma(k_0) + \vartheta\sigma(k_1); \tag{53}$$

cf. (**1**, p. 94). Steiner's formula shows that there is equality if and only if the inequality sign in (52) is replaced by equality. The cases of equality in (53) occur when one of the bodies k_0, k_1 is obtained from the other by the vector addition of a segment lying in a direction perpendicular to their common direction of equal width. So, for example, if k_0 fails to have a centre of symmetry, we may take k_1 to be the reflection of k_0 in a point and (52) will be satisfied. From this it is seen that, by taking k_0 sufficiently near to being circular, we can construct convex bodies K_0, K_1 which are as close as we please to convex bodies of revolution and for which (25) is false.

5. The volume of Blaschke sums. The study of the behaviour of the volume V and the surface area S of weighted Blaschke sums is much simpler than is the case for M.

According to the definition of Blaschke addition, the area function S_ϑ of the Blaschke sum K_ϑ of any two convex bodies K_0, K_1 satisfies

$$S_\vartheta(\omega) = (1-\vartheta)S_0(\omega) + \vartheta S_1(\omega) \tag{54}$$

for any Borel set ω on Ω. In particular, with the choice $\omega = \Omega$, we have

$$S(K_\vartheta) = (1-\vartheta)S(K_0) + \vartheta S(K_1). \tag{55}$$

H. Kneser and W. Süss **(4)** and, even earlier, Minkowski **(5)** showed that

$$V^{2/3}(K_\vartheta) \geqslant (1-\vartheta)V^{2/3}(K_0) + \vartheta V^{2/3}(K_1), \tag{56}$$

for any pair of convex bodies K_0, K_1. Their method of proof can be adapted to give a modified result when K_0, K_1 are coaxial GCBR with similar directrices.

First suppose the equatorial directrices of the coaxial GCBR K_0, K_1 are identical. Their Blaschke sum has the same equatorial directrix. In this case, a modified form of the Brunn–Minkowski theorem reads—cf. **(1**, p. 94**)**—

$$V((1-t)K_i + tK_\vartheta) \geqslant (1-t)V(K_i) + tV(K_\vartheta) \qquad (0 \leqslant t \leqslant 1, i = 0, 1).$$

It is a direct consequence of this concavity theorem that

$$3V(K_i, K_i, K_\vartheta) \geqslant 2V(K_i) + V(K_\vartheta) \tag{57}$$

where the expression on the left is a mixed volume. Consider the volume $V(K_\vartheta)$ given by

$$V(K_\vartheta) = \int_\Omega H_\vartheta\, S_\vartheta(d\omega)/3$$

where H_ϑ is the support function of K_ϑ. By (54) and the integral representations of the mixed volumes $V(K_i, K_i, K_\vartheta)$ we deduce:

$$V(K_\vartheta) = (1-\vartheta)V(K_0, K_0, K_\vartheta) + \vartheta V(K_1, K_1, K_\vartheta).$$

In virtue of inequalities (57) we obtain

$$V(K_\vartheta) \geqslant (1-\vartheta)V(K_0) + \vartheta V(K_1). \tag{58}$$

If K_0, K_1 are coaxial GCBR with equatorial directrices $\lambda_0 k$, $\lambda_1 k$, then K_ϑ has equatorial directrix $\lambda_\vartheta k$ where λ_ϑ is given by

$$\lambda_\vartheta = \sqrt{[(1-\vartheta)\lambda_0^2 + \vartheta\lambda_1^2]}.$$

Hence, by applying (58) to the sets $K_\vartheta' = (1/\lambda_\vartheta)K_\vartheta$, we have

$$V(K'_{\vartheta'}) \geqslant (1-\vartheta')V(K_0') + \vartheta' V(K_1')$$

where

$$\vartheta' = \vartheta\lambda_1^2/\lambda_\vartheta^2.$$

It follows from $V(\lambda K) = \lambda^3 V(K)$ that

$$V(K_\vartheta)/\lambda_\vartheta \geqslant (1-\vartheta)V(K_0)/\lambda_0 + \vartheta V(K_1)/\lambda_1.$$

In this inequality we may replace λ_ϑ by $a(K_\vartheta)$ where a is defined by (45) and (46); the argument is the same as that which led to (47).

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For later reference we gather together those results pertinent to the Blaschke addition of GCBR.

THEOREM 5. *If* K_0, K_1 *are coaxial* GCBR *with similar directrices and* K_ϑ *is their weighted Blaschke sum, then* (55) *and* (56) *are true as well as*

$$V(K_\vartheta)/a(K_\vartheta) \geqslant (1-\vartheta)V(K_0)/a(K_0) + \vartheta V(K_1)/a(K_1), \tag{59}$$

where $a(K)$ *is a set function of the type described by* (45) *and* (46). *When* K_0 *and* K_1 *have identical equatorial directrices then* (58) *holds.*

6. Generalized inequalities of Hadwiger. This section is devoted to some consequences of the theorems developed so far. These will take the form of inequalities involving $V(K)$, $S(K)$, and $M(K)$ where K is a GCBR. The general method for obtaining these inequalities is this. We construct a finite-valued set function F, defined over all those GCBR with directrices similar to some fixed k, and such that F has the following properties:

(I) F is continuous over non-degenerate GCBR in the sense that, if $\{K_j\}$ is a sequence of such figures which converges to the non-degenerate GCBR, then $\{F(K_j)\}$ converges to $F(K)$;

(II) $F(\lambda K) = F(K)$ for $\lambda > 0$;

(III) $F(K) \geqslant \min\{F(K_0), F(K_1)\}$ whenever K is the Blaschke sum of K_0 and K_1;

(IV) F is bounded below in its values over the set of spindles with directrix k.

Let m be the greatest lower bound of F over spindles with directrix k. By (II) and (III) F satisfies

$$F(K) \geqslant m \tag{60}$$

over the set of all polygonal GCBR with directrices similar to k because these latter are finite sums of spindles by virtue of Theorem 2. Finally, condition (I) and Theorem 1 show that (60) holds for any GCBR with directrix similar to k. Since F will be formed from the set functions V, S, M and functions of the type a, described in (45) and (46), which depend only on k, (60) will be an inequality of the sort we seek.

Further refinement of (60) is then possible depending on the choice of the functions of the type a. Roughly, we replace the occurrences of those functions of the type α which arise in the definition of the functions a in one of two ways. If we make a specific choice of k, we get an inequality for a class of GCBR; for example, we shall take k to be a circle, and get inequalities of type (60) which hold for convex bodies of revolution. The ones we obtain will be those found by Hadwiger **(3)** using other methods. Alternatively, one may replace the α by appropriate extreme values so as to give inequalities valid for all GCBR.

For F we choose

$$F(K) = [2c_1 S(K) - c_2\, a(K)M(K) + 3c_3\, V(K)/b(K)]/c^2(K), \tag{61}$$

where a, b, c are positive functions of the type described in (45) and (46) and c_1, c_2, c_3 are constants as yet not chosen. We let α, β, γ be set functions of plane convex bodies such that, if K has equatorial directrix k, then

$$(62) \qquad \alpha(k) = a(K), \qquad \beta(k) = b(K), \qquad \gamma(k) = c(K).$$

Thus the positive functions α, β, γ satisfy (45), are rigid motion invariant, and vanish only for degenerate k. Clearly F satisfies conditions (I) and (II).

Next, if we restrict c_2 and c_3 to be non-negative, the numerator of F is concave in virtue of Theorems 4 and 5. As to the denominator, if K_0, K_1 have equatorial directrices $\lambda_0 k$, $\lambda_1 k$, then the equatorial directrix of the Blaschke sum K_ϑ has equatorial directrix $\lambda_\vartheta k$ where

$$\lambda_\vartheta^2 = (1-\vartheta)\lambda_0^2 + \vartheta\lambda_1^2.$$

If we multiply this equation by $\gamma^2(k)$ and use (45), we obtain from (62)

$$c^2(K_\vartheta) = (1-\vartheta)c^2(K_0) + \vartheta c^2(K_1).$$

Hence, denoting the numerator of F by N, we have

$$F(K_\vartheta) \geqslant \frac{(1-\vartheta)N(K_0) + \vartheta N(K_1)}{(1-\vartheta)c^2(K_0) + \vartheta c^2(K_1)} \geqslant \min\left\{\frac{N(K_0)}{c^2(K_0)}, \frac{N(K_1)}{c^2(K_1)}\right\}$$

which shows that F satisfies (III).

It remains to examine (IV). Let C be a spindle with equatorial directrix k in the plane $z = 0$ and denote the area and perimeter of k by $v(k)$ and $u(k)$. Further, let l be the length of the segment on the z-axis whose convex closure with k gives C. We denote the end points of this segment by $(0, 0, -l_0)$ and $(0, 0, l_1)$ so that

$$(62) \qquad l_0 \geqslant 0, \qquad l_1 \geqslant 0, \qquad l_0 + l_1 = l.$$

Clearly

$$(63) \qquad V(c) = lv(k)/3.$$

In the plane $z = 0$, let Q be the boundary point of k such that the segment from the origin to Q makes an angle of measure θ with the positive x-axis. Suppose $q(\theta)$ to be the length of this segment and $ds(\theta)$ to be the arc element of the boundary of k at Q. For the surface area of C we have

$$S(C) = \int_0^{2\pi} \{\sqrt{(l_0^2 + q^2(\theta))} + \sqrt{(l_1^2 + q^2(\theta))}\} ds(\theta)/2.$$

From (62) and the inequalities

$$0 \leqslant \sqrt{(l_i^2 + q^2(\theta))} - l_i \leqslant q(\theta) \qquad (i = 0, 1),$$

we get

$$(64) \qquad lu(k)/2 \leqslant S(C) \leqslant lu(k)/2 + 2v(k).$$

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Finally we turn to an estimate of $M(C)$. The cylinder C' with directrix k and generators parallel to the z-axis which is truncated by planes perpendicular to this axis through the vertices of C contains C. The monotonicity of M as a set function gives

$$M(C) \leqslant M(C') = \pi l + \pi u(k)/2. \tag{65}$$

To estimate $F(C)$ from below we use (63), (64), and (65) to obtain

$$\gamma^2(k)F(C) \geqslant l[c_1 u(k) - c_2 \pi\alpha(k) + c_3 v(k)/\beta(k)] \\ - [4|c_1|v(k) + c_2 \alpha(k)u(k)/2].$$

This shows that, for fixed k, $F(C)$ will be bounded below if the coefficient of l is non-negative.

Since F satisfies conditions (I) through (IV) for the choices of c_1, c_2, c_3 indicated, we have the following theorem.

THEOREM 6. *Suppose c_1, c_2, c_3 are constants which satisfy*

$$c_2 \geqslant 0, \qquad c_3 \geqslant 0, \qquad c_1 u(k) - c_2 \pi\alpha(k) + c_3 v(k)/\beta(k) \geqslant 0, \tag{66}$$

where u and v are the perimeter and area of the plane convex set k and α, β are positive rigid-motion invariants of k which are positively homogeneous of degree one. The number m, defined by

$$m = \operatorname*{g.l.b.}_{\{C\}} [2c_1 S(C) - c_2 \alpha(k)M(C) + 3c_3 V(C)/\beta(k)]/\gamma^2(k),$$

where γ is a function of the same type as α, β and $\{C\}$ is the set of all spindles with directrices similar to k, exists and we have

$$2c_1 S(K) - c_2 \alpha(k)M(K) + 3c_3 V(K)/\beta(k) \geqslant m\gamma^2(k) \tag{67}$$

for all GCBR with directrices similar to k.

To give some specimen cases of this theorem, let k be a circle of radius R and choose

$$\alpha(k) = \beta(k) = R, \qquad \gamma(k) = R\sqrt{\pi}.$$

Then (66) reads

$$2c_1 - c_2 + c_3 \geqslant 0. \tag{68}$$

We fix our attention on convex bodies of revolution and consider three cases:

(1) $c_1 = 1/2, \quad c_2 = 1, \quad c_3 = 0;$

(2) $c_1 = 0, \quad c_2 = 1, \quad c_3 = 1;$

(3) $c_1 = -1/2, \quad c_2 = 0, \quad c_3 = 1,$

all of which satisfy (68).

The functions F in these three cases will be written F_1, F_2, F_3 and the corresponding lower bounds m will be denoted by m_1, m_2, m_3. Here $\{C\}$ is the

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class of all spindles of rotation or double cones. For such figures we find, in terms of earlier notation,

$$V(C) = \pi l R^2/3, \qquad S(C) = \pi R\{\sqrt{(l_0^2 + R^2)} + \sqrt{(l_1^2 + R^2)}\},$$

$$M(C) = \pi R\{l_0/R + \operatorname{arccot} l_0/R + l_1/R + \operatorname{arccot} l_1/R\}.$$

The last equation comes from the general formula for M given at the beginning of the third section.

Set

$$t_i = l_i/R, \qquad G_1(t) = \sqrt{(1 + t^2)} - t - \operatorname{arccot} t.$$

Then

$$F_1(C) = G_1(t_0) + G_1(t_1)$$

and so

$$\underset{\{C\}}{\text{g.l.b.}}\ F_1(C) = 2 \min_{t \geqslant 0} G_1(t) = 2 - \pi = m_1.$$

Thus, in case (1), (67) reads

$$S(K) - RM(K) + \pi(\pi - 2)R^2 \geqslant 0$$

for all convex bodies of revolution K with equatorial radius R. This is inequality (5a) of **(3)**.

In the second case, set

$$G_2(t) = -\operatorname{arccot} t.$$

Then

$$F_2(C) = G_2(t_0) + G_2(t_1)$$

and so

$$\text{g.l.b.}\ F_2(C) = 2 \min_{t \geqslant 0} G_2(t) = -\pi = m_2.$$

Thus, in case (2), (67) can be written

$$\pi^2 R^3 - R^2 M(K) + 3V(K) \geqslant 0$$

for the same K as in case (1). This is (6a) of **(3)**.

Finally, set

$$G_3(t) = t - \sqrt{(1 + t^2)}.$$

Then

$$F_3(C) = G_3(t_2) + G_3(t_1)$$

and

$$\underset{\{C\}}{\text{g.l.b.}}\ F_3(C) = 2 \min_{t \geqslant 0} G_3(t) = -2 = m_3.$$

Here (67) takes the form, for the same K as before,

$$2\pi R^3 - RS(K) + 3V(K) \geqslant 0$$

which is (7a) of **(3)**.

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Oregon State University,
Corvallis, Oregon, U.S.A., and
University of Otago,
Dunedin, N.Z.