CONSTRUCTION OF PRIMITIVES OF GENERALIZED DERIVATIVES WITH APPLICATIONS TO TRIGONOMETRIC SERIES

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1. Introduction. This paper is an extension of the ideas discussed in **(3**, §§ 14-16**)**; the extension consisting of the use of the third and fourth symmetric Riemann derivative instead of the Schwarz or second symmetric Riemann derivative.

The J_2-integral, due to James **(1)**, is defined in **(3)** as follows. Let $f(x)$ be measurable on $[a, b]$ and finite at each point; if there exists a continuous function $F(x)$ such that $D^2F = f$ everywhere on (a, b),

$$D^2F = \lim_{h\to 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2}$$

then

$$\int_{a,b}^{x} f(t)d_2t = F(x) - \frac{x-b}{a-b}F(a) - \frac{x-a}{b-a}F(b) = H_2(F: a, b, x). \tag{1.1}$$

The definition is unique since if $F(x)$ and $G(x)$ are continuous and $D^2F = D^2G$ everywhere then

$$H_2(F: a, b, x) = H_2(G: a, b, x).$$

This integral has application to convergent trigonometric series, **(3)**.

Using the third and fourth symmetric Riemann derivatives J_3- and J_4-integrals are defined and applied to $(C, 1)$ and $(C, 2)$ summable trigonometric series.

2. Definitions. With the notation of Kassimatis, **(4)**, we write for any function $F(x)$ defined at the points x_1, x_2, x_3, x_4,

$$H_3(F: x_1, x_2, x_3, x_4) = F(x_4) - F(x_3)\frac{(x_4-x_1)(x_4-x_2)}{(x_3-x_1)(x_3-x_2)} - F(x_2)\frac{(x_4-x_3)(x_4-x_1)}{(x_2-x_3)(x_2-x_1)} - F(x_1)\frac{(x_4-x_2)(x_4-x_3)}{(x_1-x_2)(x_1-x_3)}, \tag{2.1}$$

$$V_3(F: x_1, x_2, x_3, x_4) = \frac{H_3(F: x_1, x_2, x_3, x_4)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}. \tag{2.2}$$

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V_3 is then the third divided difference of $F(x)$. In particular if $h > k > 0$ we write

$$(2.3)\quad w_3(F\colon x; h, k) = w_3(x; h, k) = 3!V_3(F\colon x + h, x + k, x - k, x - h)$$
$$= \frac{3}{h^2 - k^2}\left\{\frac{F(x+h) - F(x-h)}{h} - \frac{F(x+k) - F(x-k)}{k}\right\},$$

$$(2.4)\quad w_3(F\colon x; 3h, h)$$
$$= \frac{\Delta^3(F\colon 2h)}{(2h)^3} = \frac{F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h)}{(2h)^3}.$$

From (2.3) and (2.4) we define

$$(2.5)\qquad \Delta'''F(x) = \overline{\lim_{h,k\to 0}}\, w_3(x; h, k), \qquad \delta'''F(x) = \underline{\lim_{h,k\to 0}}\, w_3(x\colon h, k),$$

$$(2.6)\qquad \overline{D}^3F(x) = \overline{\lim_{h\to 0}}\, w_3(x\colon 3h, h), \qquad \underline{D}^3F(x) = \underline{\lim_{h\to 0}}\, w_3(x\colon 3h, h),$$

and if $\underline{D}^3F(x) = \overline{D}^3F(x)$ we say that $F(x)$ has a third symmetric Riemann derivative at x and write it $D^3F(x)$.

Clearly

$$(2.7)\qquad \delta'''F \leqslant \underline{D}^3\, F \leqslant \overline{D}^3\, F \leqslant \Delta'''F.$$

The following lemma, which generalizes Theorem 19, **(3)**, is needed later.

LEMMA 2.1. *If F'' exists in an interval containing x and if $\Delta_1(\delta_1)$ is the greater (smaller) of the first derivates of F'' then*

$$(2.8)\qquad \delta_1 \leqslant \delta''' \leqslant \Delta''' \leqslant \Delta_1.$$

All points will be assumed to be interior to the interval mentioned in the statement of the lemma. It is sufficient to prove $\delta_1 \leqslant \delta$ as a similar argument will complete (2.8). Further we may obviously assume $\delta_1 > -\infty$. The proof is in two parts.

(a) Assume $\delta_1 < \infty$. From the definition of δ_1, if $\epsilon > 0$ is given, there exists $\mu > 0$ such that if $0 < \eta, \xi < \mu$ then

$$F''(x+\eta) - F''(x) > \eta(\delta_1 - \epsilon),$$
$$F''(x-\xi) - F''(x) < \xi(\delta_1 - \epsilon).$$

Consider the function $X(u)$ defined by

$$X(u) = F(x+u) - F(x) - uF'(x) - \frac{u^2}{2!}F''(x) - \frac{u^3}{3!}(\delta_1 - \epsilon).$$

The following properties of $X(u)$ are immediate,

$$X'(u) = F'(x+u) - F'(x) - uF''(x) - \frac{u}{2!}(\delta_1 - \epsilon),$$

$$X''(u) = F''(x+u) - F''(x) - u(\delta_1 - \epsilon),$$
$$X(0) = X'(0) = X''(0) = 0,$$

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$$X''(u) > 0 \text{ if } 0 < u < \mu, \quad X''(u) < 0 \text{ if } -\mu < u < 0, \tag{2.9}$$

$$w_3(X:0; h, k) = w_3(F:x; h, k) - (\delta_1 - \epsilon). \tag{2.10}$$

It follows, from (2.10), that it is sufficient to show that, for all h, k small enough, $w_3(X:0;\, h, k) \geqslant 0$. To do this define, for $0 < u < \mu$,

$$Y(u) = \frac{X(u) - X(-u)}{u}.$$

Then by (2.3)

$$w_3(X:0; h, k) = \frac{3}{h^2 - k^2}\,(Y(h) - Y(k)).$$

If, therefore, $Y(u)$ is monotonic increasing for all u small enough, then $w_3(X:0;\, h, k) \geqslant 0$. Now

$$\begin{aligned} Y'(u) &= -\frac{1}{u^2}\,[\{X(u) - uX'(u)\} - \{X(-u) - (-u)X'(-u)\}] \\ &= -\frac{1}{u^2}\,[Z(u) - Z(-u)] \end{aligned}$$

where $Z(u) = X(u) - uX'(u)$. $Z(u)$ is clearly defined wherever $X(u)$ and $X'(u)$ are defined and

$$Z'(u) = -\,uX''(u) \leqslant 0$$

by (2.9). Hence $Z(-u) \leqslant Z(u)$ and hence $Y'(u) \geqslant 0$ wherever $Y'(u)$ is defined.

Thus we have shown that $Y(u)$ is monotonic increasing and the result follows.

(b) Assume $\delta_1 = \infty$. Then in the above argument replace $\delta_1 - \epsilon$ by an arbitrary positive number A to arrive at $\delta''' = \delta_1 = \infty$.

The following lemma due to Saks, **(5)**, will be required later.

LEMMA 2.2. *If $F'(x)$ exists everywhere in $[a, b]$ and $\underline{D}^3F > 0$ in (a, b) then $F'(x)$ is continuous, convex, and*

$$\Delta^3 F(x: 2h) \geqslant 0 \tag{2.11}$$

for every x, $h > 0$ $(a \leqslant x - 3h < x + 3h \leqslant b)$.

3. We now wish to define a class of funtions for which $D^3F = D^3G$ everywhere in an interval implies $H_3(F: x_1, x_2, x_3, x_4) = H_3(g: x_1, x_2, x_3, x_4)$ for all sets of four points in that interval. As has been pointed out by Kassimatis, **(4)**, continuity of F and G is not enough.

LEMMA 3.1. *If $F'(x)$ exists and is continuous in $[a, b]$ then*

$$\min_{a<x<b} \underline{D}^3F(x) \leqslant 3!\; V_3(F: x_1, x_2, x_3, x_4) \leqslant \max_{a<x<b} \overline{D}^3F(x) \tag{3.1}$$

for all x_1, x_2, x_3, x_4 in $[a, b]$.

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The argument is that of Verblunsky, **(6)**. Define

$$f(x) = F(x) - (ax^3 + bx^2 + cx + d) \tag{3.2}$$

where a, b, c, d are determined by the conditions

$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$$

for some x_1, x_2, x_3, x_4 in (a, b). Simple calculations then show that

$$a = V_3(F; x_1, x_2, x_3, x_4).$$

Since $f(x)$ has four zeros, $f'(x)$ has three zeros and a maximum at some point ξ, say. Then we must have

$$w_3(f: \xi; 3h_i, h_i) \leqslant 0 \tag{3.3}$$

for a sequence of h_i, $h_i \to 0$. For if not, then for all h small enough

$$\Delta^3 \frac{(f: \xi: 2h)}{(2h)^3} > 0$$

that is,

$$\frac{f(\xi + 3h) - f(\xi - 3h)}{6h} > \frac{f(\xi + h) - f(\xi - h)}{2h}$$
$$> \ldots > f'(\xi),$$

which contradicts the fact that $f'(x)$ has a maximum at ξ. As we have (3.3) it follows that

$$\underline{D}^3 f(\xi) \leqslant 0$$

that is,

$$\underline{D}^3 F(\xi) \leqslant 3!\, a = 3!\; V_3(F: x_1, x_2, x_3, x_4).$$

This proves the left-hand inequality of (3.1). The right-hand inequality comes from applying the above result to $-F$. Finally the result holds for x_1, x_2, x_3, x_4, in $[a, b]$ by continuity of $F(x)$.

An immediate corollary of Lemma 3.1 is

LEMMA 3.2. *If the relation* (3.1) *holds for* $F(x) - G(x)$, *in particular if* $(F - G)'$ *is continuous, then* $D^3\,(F - G) = 0$ *implies*

$$H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4) \tag{3.4}$$

for all x_1, x_2, x_3, x_4 *in* $[a, b]$.

Let

$$F_1(x) = H_3(F: x_1, x_2, x_3, x)$$
$$G_1(x) = H_3(G: x_1, x_2, x_3, x).$$

Then $D^3(F_1 - G_1) = 0$ and hence, by (3.1),

$$V_3(F_1 - G_1: y_1, y_2, y_3, y_4) = 0$$

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for all y_1, y_2, y_3, y_4. $F_1(x) - G_1(x)$ is therefore a polynomial of degree at most 2 but it is zero at x_1, x_2, x_3 and hence is identically zero.

The following lemma, due to Kassimatis, **(4)**, obtains (3.4) under weaker conditions than the continuity of $(F - G)'$ but it is quite possible that (3.1) holds under less restrictive conditions which would then generalize Lemma 3.2.

LEMMA 3.3. *If $F(x)$ and $G(x)$ are defined in $[a, b]$ and* (i) *$F - G$ is continuous in $[a, b]$,* (ii) *$(F - G)'$ exists in (a, b) then $D^3(F - G) = 0$ implies* (3.4).

4. The J_3-integral. Let $f(x)$ be defined and measurable on $[a, b]$ and finite at each point. If there exists a function $F(x)$, continuous on $[a, b]$ and differentiable on (a, b) such that $D^3F = f$ then we define the J_3-integral of f to be

$$\int_{x_1,x_2,x_3}^{x} f(t)d_3t = H_3(F: x_1, x_2, x_3, x) \tag{4.1}$$

where x_1, x_2, x_3, x are any four points of $[a, b]$ and $a \leqslant x_1 < x_2 < x_3 \leqslant b$. Lemma 3.3 ensures that this definition is unique.

If $f(x)$ is complex valued and $f(x) = u(x) + iv(x)$ then we define the J_3-integral of $f(x)$, if it is defined for both $u(x)$ and $v(x)$, by

$$\int_{x_1,x_2,x_3}^{x} f(t)d_3t = \int_{x_1,x_2,x_3}^{x} u(t)d_3 + i\int_{x_1,x_2,x_3}^{x} v(t)d_3(t).$$

The following elementary properties of this integral are immediate.

(a) If $f(x)$ and $g(x)$ are J_3-integrable on $[a, b]$ so is $\alpha f(x) + \beta g(x)$ for any numbers α, β and

$$\int_{x_1,x_2,x_3}^{x} \{\alpha f(t) + \beta g(t)\}d_3t = \alpha\int_{x_1,x_2,x_3}^{x} f(t)d_3t + \beta\int_{x_1,x_2,x_3}^{x} g(t)d_3t.$$

(b) If $f(x)$ is J_3-integrable on $[a, b]$ it is also J_3-integrable on any subinterval $[\alpha, \beta]$ and if

$$F(x) = \int_{a,\gamma,b}^{x} f(t)d_3t \quad \text{then if } \alpha \leqslant \delta \leqslant \beta \quad \text{and} \quad \alpha \leqslant x \leqslant \beta$$

$$\int_{\alpha,\delta,\beta}^{x} f(t)d_3t = H_3(F: \alpha, \delta, \beta, x).$$

5. Application to trigonometric series.

THEOREM 1. *Let $f(t)$ be the $(C, 1)$ sum of the series*

$$\sum_{-\infty}^{\infty} c_n e^{int}, \; c_0 = 0,$$

and let

$$F(t) = \sum_{-\infty}^{\infty} \frac{c_n e^{int}}{(in)^3}, \tag{5.1}$$

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then

$$\int_{x_1,x_2,x_3}^{x} f(t)d_3t = H_3(F\colon x_1, x_2, x_3, x). \tag{5.2}$$

To obtain this result we need the following lemma **(6**, II, p. 69**)**.

LEMMA 5.1. "*If*

$$\sum_{-\infty}^{\infty} c_n e^{inx}, \quad c_0 = 0,$$

is summable (C, α), $\alpha \geqslant -1$ *at* x_0 *to* s *then it is summable* R_r *at* x_0 *to* s *provided* $r > 1 + \alpha$. *By this we mean that*

$$\lim_{h\to 0} \sum_{-\infty}^{\infty} c_n e^{inx}\left(\frac{\sin n\, h}{inh}\right)^r = s."$$

The result as stated in **(8)** requires $\alpha > -1$ but it is in fact true when $\alpha = -1$ when it is the result of **(8**, I, p. 322**)**.

Simple calculations give

$$\frac{F(x+h) - F(x-h)}{2h} = \sum_{-\infty}^{\infty} \frac{c_n}{in^2} e^{inx}\left(\frac{\sin nh}{inh}\right)$$

$$\frac{F(x+2h) - 2F(x) + F(x-2h)}{(2h)^2} = \sum_{-\infty}^{\infty} \frac{c_n}{i^3 n} e^{inx}\left(\frac{\sin nh}{inh}\right)^2$$

$$\frac{F(x+3h) - 3F(x+h) + 3F(x-h) - 3F(x+3h)}{(2h)^3} = \sum_{-\infty}^{\infty} \frac{c_n}{i} e^{inx}\left(\frac{\sin nh}{inh}\right)^3.$$

By hypothesis the series $\sum c_n e^{int}$ is summable $(C, 1)$ and hence the series

$$\sum \frac{c_n}{n} e^{inx} \quad \text{and} \quad \sum \frac{c_n}{n^2} e^{inx}$$

are summable $(C, 0)$ and $(C, -1)$ respectively.

Hence by Lemma 5.1 $D^3F = f$ everywhere and D^2F and $D\ F$ exist and this implies the existence of $F'(x)$, (4). This then proves that f is J_3-integrable and gives (5.2).

THEOREM 2. *If*

$$\sum_{-\infty}^{\infty} c_n e^{int}$$

is summable $(C, 1)$ *to* $f(t)$ *then*

$$c_n = -\frac{3}{8\pi^3}\int_{-4\pi,-2\pi,2\pi}^{0} f(t)e^{-int}\, d_3t. \tag{5.3}$$

We first calculate c_0. In Theorem 1 we assumed for simplicity that $c_0 = 0$ but this clearly involves no loss in generality.

Hence we know that

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$$\int_{-4\pi,-2\pi,2\pi}^{0} f(t)d_3t = H_3(F: -4\pi, -2\pi, 2\pi, 0)$$
$$= F(0) + \frac{1}{3}F(-4\pi) - F(-2\pi) - \frac{1}{3}F(2\pi),$$

where

$$F(x) = \frac{c_0x^3}{3'} + \sum_{-\infty}^{\infty}{}' \frac{c_ne^{inx}}{(in)^3}.$$

Since the last term on the right-hand side is periodic its contribution to the integral is zero. Therefore,

$$\int_{-4\pi,-2\pi,2\pi}^{0} f(t)d_3t = \frac{1}{3}\frac{c_0}{6}(-4\pi)^3 - \frac{c_0}{6}(-2\pi)^3 - \frac{1}{3}\frac{c_0}{6}(2\pi)^3.$$
$$= -\frac{8\pi^3}{3}c_0.$$

To calculate c_n, $n > 0$, requires $f(x)e^{inx}$ to be expressed as the $(C, 1)$ sum of a trigonometric series with constant term c_n. This has been done by James, **(2)**, and then a similar calculation to the one above completes the proof of (5.3).

6. Construction of the J_3-integral. The J_3-integral can be constructed by methods used in Jeffery, **(3)**, to construct the J_2-integral.

THEOREM 3. *Let $f(x) \in L(a, b)$ and let $f(x)$ be the finite third symmetric Riemann derivative of a function continuous in $[a, b]$ and differentiable in (a, b). Further let*

$$\Phi(x) = \int_a^x \int_a^u \int_a^v f(t)\, dt\, dv\, du, \tag{6.1}$$

then

$$\int_{x_1,x_2,x_3}^{x} f(t)d_3t = H_3(\Phi: x_1, x_3, x_3, x). \tag{6.2}$$

Since the construction follows the lines of **(3)** it is only sketched here to point out certain differences.

We first determine a sequence of continuous functions $U_n(x)$ such that $\underline{D}^3U_n(x) > f(x)$ and which converges uniformly to $\Phi(x)$.

As in **(1)** define $A_n(x)$ such that, with the notation of Lemma 2.1, $\delta_1A_n(x) >$ (x) and $A_n(x)$ converges uniformly to $\int_a^x f(t)\, dt$ as $n \to \infty$.

Then the required $U_n(x)$ is

$$U_n(x) = \int_a^x du \int_a^u A_n(t)\, dt,$$

which clearly converges uniformly to $\Phi(x)$ and, from the continuity of $A_n(x)$,

$$U_n'(x) = \int_0^x A_n(t)\, dt, \qquad U_n''(x) = A_n(x).$$

By Lemma 2.1

$$\delta''' U_n(x) \geqslant \delta_1 U_n = \delta_1 \mathrm{A}_n > f(x) = D^3 F(x)$$

where $F(x)$ is some function continuous on $[a, b]$ and differentiable on (a, b). Hence

$$\underline{D}^3(U_n(x) - F(x)) > 0$$

and so, by Lemma 2.2,

$$\Delta^3(U_n - f)(x: 2h) > 0$$

for all x, $h > 0$, $a < x - 3h < x + 3h < b$

Hence letting $n \in \infty$

$$\Delta^3(\Phi - F)(x\ 2h) \geqslant 0$$

which implies

$$\underline{D}^3(\Phi - F) \geqslant 0.$$

In a similar manner it can be shown that

$$\bar{D}^3(\Phi - F) \leqslant 0$$

which together with the previous inequality implies

$$D^3(\Phi - F) = 0.$$

From Lemma 3.2 this gives

$$H_3(F: x_1, x_2, x_3, x_4) = H_3(\Phi: x_1, x_2, x_3, x_4),$$

completing the proof of the theorem.

A function is said to be Lebesgue integrable at a point x_0 if it is Lebesgue integrable in every sufficiently small neighbourhood of x_0.

As in **(3)** the above result can be extended to functions $f(x)$ which have a finite number of points at which they are not Lebesgue integrable. This can be done provided only that if β is such a point, then

$$\lambda(x) = \int_\alpha^x \int_\alpha^u f(t)\, dt\, du$$

is Denjoy integrable in some interval (α, γ) containing β.

7. The fourth symmetric derivative. We now indicate the definitions and results in the case of the fourth symmetric Riemann derivative. As in § 2 we define

$$(7.1)\quad H_4(F: x_1, x_2, x_3, x_4, x_5) = F(x_5) - F(x_4)\frac{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} - F(x_3)\frac{(x_5 - x_4)(x_5 - x_1)(x_5 - x_2)}{(x_3 - x_4)(x_3 - x_1)(x_3 - x_2)} - F(x_2)\frac{(x_5 - x_3)(x_5 - x_4)(x_5 - x_1)}{(x_2 - x_3)(x_2 - x_4)(x_2 - x_1)} - F(x_1)\frac{(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)},$$

$$(7.2)\quad V_4(F: x_1, x_2, x_3, x_4, x_5) = \frac{H_4(F: x_1, x_2, x_3, x_4, x_5)}{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}.$$

It may be noted in passing that the function $f(x)$ in (3.2) is equal to $H_4(F: x_1, x_2, x_3, x_4, x)$.

In particular if $h > k > 0$ we write

$$(7.3)\quad w_4(F: x; h, k) = 4!\ V_4(F: x + h, x + k, x - k, x - h)$$
$$= \frac{12}{h^2 - k^2}\left\{\frac{F(x + h) - 2F(x) + F(x - h)}{h^2} - \frac{F(x + k) - 2F(x) + F(x - k)}{k^2}\right\},$$

$$(7.4)\quad \frac{\Delta^4(F: x; h)}{h^4} = w_4(F: x; 2h, h)$$
$$= \frac{F(x + 2h) - 4F(x + h) + 6F(x) - 4F(x - h) + F(x - 2h)}{h^4}.$$

Using (7.3) and (7.4) we define

$$(7.5)\quad \Delta^{(iv)}F = \overline{\lim_{h,k\to 0}}\, w_4(h, k),\ \delta^{(iv)}F = \underline{\lim}_{h,k\to 0}\, w_4(h, k)$$

$$(7.6)\quad \overline{D}^4F = \overline{\lim_{h\to 0}}\, w_4(2h, h),\ \underline{D}^4F = \underline{\lim}_{h\to 0}\, w_4(2h, h),$$

and if $\overline{D}^4F = \underline{D}^4F$ we say that F has a fourth symmetric Riemann derivative and write it D^4F. Clearly

$$(7.7)\quad \delta^{(iv)}F \leqslant \underline{D}^4F \leqslant \overline{D}^4F < \Delta^{(iv)}F,$$

and we have the following lemmas.

LEMMA 7.1. *If $F'''(x)$ exists in an interval containing x and if $\Delta_1(\delta_1)$ is the greater (smaller) of the first derivates of F then*

$$(7.8)\quad \delta_1 \leqslant \delta^{(iv)} \leqslant \Delta^{(iv)} \leqslant \Delta_1.$$

LEMMA 7.2. *If $F''(x)$ exists everywhere in $[a, b]$ and $\underline{D}^4F > 0$ in (a, b) then $F''(x)$ is continuous, convex, and*

$$(7.9)\quad \Delta^4F(x: h) \geqslant 0$$

for every x, $h > 0$ $(a \leqslant x - 4h < x + 4h \leqslant b)$.

The proof of Lemma 7.1 is very similar to that of Lemma 2.1. Making all the obvious changes define

$$X(u) = F(x + u) - F(x) - uF'(x) - \frac{u^2}{2!}F''(x) - \frac{u^3}{3!}F'''(x) - \frac{u^4}{4!}(\delta_1 - \epsilon).$$

As in the previous proof it is sufficient to show that $w_4(X: 0; h, k) \geqslant 0$ for all h, k small enough.

Defining

$$Y(u) = \frac{X(u) - 2X(0) + X(-u)}{u^2} = \frac{X(u) + X(-u)}{u^2}$$

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it is sufficient to prove $Y(u)$ to be monotonic for u small enough, $u > 0$. Then if we define

$$Z(u) = 2X(u) - uX'(u)$$

it is sufficient to show that $Z(u)$ has a local maximum at $u = 0$. This follows since $Z'(u)$ is monotonic decreasing, $Z''(u)$ being $-uX'''(u)$ which by a result similar to (2.9) is always negative.

The proof of Lemma 7.2 is exactly similar to that of Lemma 2.2 owing to the reasons given by Verblunsky in **(7)**.

LEMMA 7.3. *If $F''(x)$ exists and is continuous in $[a, b]$ then*

$$\min_{a<x<b} \underline{D}^4F(x) < 4'\ V_4(F: x_1, x_2, x_3, x_4, x_5) \leqslant \max_{a<x<b} \overline{D}^4F(x) \tag{7.10}$$

for all x_1, x_2, x_3, x_4, x_5 in $[a, b]$.

LEMMA 7.4. *If* (7.10) *holds for $F(x) - G(x)$, in particular if $(F - G)''$ is continuous, then $D^4(F - G) = 0$ implies*

$$H_4(F: x_1, x_2, x_3, x_4, x_5) = H_4(G: x_1, x_2, x_3, x_4, x_5). \tag{7.11}$$

LEMMA 7.5. *If $F(x)$ and $G(x)$ are defined in $[a, b]$ and* (i) *$(F - G)$ is continuous in $[a, b]$,* (ii) *$(F - G)''$ exists in (a, b) then $D^4(F - G) = 0$ implies* (7.11).

As the proofs of the corresponding lemmas 3.1, 3.2, and 2.3 depend on **(6)** the proof of these are exactly the same but are based on **(7)**.

Now let $f(x)$ be defined at each point and measurable. If there exists a function $F(x)$, continuous on $[a, b]$ and with a second derivative on (a, b) such that $D^4F = f$, we define the J_4-integral of f to be

$$\int_{x_1,x_2,x_3,x_4}^{x} f(t)d_4t = H_4(F: x_1, x_2, x_3, x_4, x) \tag{7.12}$$

where x_1, x_2, x_3, x_4, x are any five points of $[a, b]$ and $a \leqslant x_1 < x_2 < x_3 < x_4 \leqslant b$. The discussion of § 4 applies with obvious changes to this definition. Further, the following theorems can be proved.

THEOREM 4. *If*

$$\sum_{-\infty}^{\infty} c_n e^{inx}, c_0 = 0,$$

is $(C, 2)$ summable everywhere to $f(t)$ and $c_n = o(n)$ and if

$$F(t) = \sum \frac{c_n e^{int}}{(in)^4}, \tag{7.13}$$

then

$$\int_{x_1,x_2,x_3,x_4}^{x} f(t)d_4t = H_4(F; x_1, x_2, x_3, x_4, x) \tag{7.14}$$

where $x_1 < x_2 < x_3 < x_4, x$ are any five numbers.

THEOREM 5. *If*

$$\sum_{-\infty}^{\infty} c_n e^{inx},$$

has $c_n = o(n)$ *and is* $(C, 2)$ *summable to* $f(t)$ *then*

$$c_n = \frac{3}{8\pi^4} \int_{-4\pi,-2\pi,2\pi,4\pi}^{0} f(t) e^{-int} d_4 t.$$

THEOREM 6. *Let* $f(x)$ *be the finite fourth Riemann symmetric derivative of a function continuous on* $[a, b]$ *and with a second derivative on* (a, b). *Let* $f(x)$ *be Lebesgue integrable except at a finite number of points* $\beta_1, \ldots, \beta_n$. *Further suppose that*

$$\lambda(x) = \int_{a_i}^{x} \int_{a_i}^{y} \int_{a_i}^{u} f(t)\, dt\, du\, dy \quad i = 1, 2, \ldots, n$$

is Denjoy integrable in some interval (α_i, γ_i) *containing* β_i. *Then if we define*

$$\Phi(x) = \int_{a}^{x} \int_{a}^{y} \int_{a}^{u} \int_{a}^{v} f(t)\, dt\, dv\, du\, dy$$

then

$$\int_{x_1,x_2,x_3,x_4}^{x} f(t) d_4 t = H_4(\Phi : x_1, x_2, x_3, x_4, x).$$

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