whence taking $a_1 + a_2 + \dots + a_n = nA_n$, we immediately obtain the inequality $A_n \geqslant G_n$. Finally, since by the Lemma, in the above inequality the equality holds if, and only if, $a_1 = a_2 = \dots = a_n = G_n$, it follows that $A_n = G_n$ is true if, and only if, $a_1 = a_2 = \dots = a_n$.

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## 107.33 On the antiderivatives of a monotone function and its inverse

As far as we know, computing antiderivatives of the inverse trigonometric functions and logarithmic function all are initial examples of integration by parts in calculus. In this Note we are motivated by the question that is it possible to compute the above mentioned antiderivatives without using integration by parts? The common property of these functions is their monotonicity. Based on this fact, we demonstrate a geometric argument to relate the antiderivatives of a monotone function $f$ and its inverse $f^{-1}$. Although our geometric implication essentially carries ideas from integration by parts, it avoids many computational details, more precisely when we have an antiderivative for $f$.

*Theorem* 1: Let $f : [a, b] \to \mathbb{R}$ be positive, continuous and monotone, and let $F' = f$. Then, we have

$$\int_a^b f^{-1}(t)\,dt = bf^{-1}(b) - F\left(f^{-1}(b)\right) - af^{-1}(a) + F\left(f^{-1}(a)\right). \qquad (1)$$

*Proof*: We give the proof only for the case $f$ is increasing. Thus, $f^{-1}$ is also positive and increasing, and we may consider the following diagram, picturing the grey areas given by $\int_a^b f^{-1}(t)dt$ and $\int_{f^{-1}(a)}^{f^{-1}(b)} f(y)dy = F\left(f^{-1}(b)\right) - F\left(f^{-1}(a)\right)$.

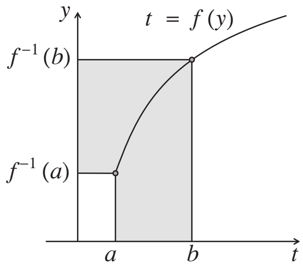


FIGURE 1: The key to relate antiderivatives of a monotone function $f$ and its inverse $f^{-1}$

On the other hand, the sum of these areas, the area of the grey L-shape, is equal to $bf^{-1}(b) - af^{-1}(a)$. Thus, we obtain (1). The same proof works for the case $f$ is decreasing.

Now we may explore the antiderivatives of a monotone function and its inverse. The fundamental theorem of calculus asserts that the function $x \to \int_a^x f^{-1}(t)\,dt$ is an antiderivative of $f^{-1}(x)$. Also, considering (1) with $b = x$, we obtain

$$\int_a^x f^{-1}(t)\,dt = xf^{-1}(x) - F\left(f^{-1}(x)\right) - af^{-1}(a) + F\left(f^{-1}(a)\right).$$

Thus, if $f$ is positive, continuous and monotone, and $F' = f$, then we have the following antiderivative formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F\left(f^{-1}(x)\right) + C. \qquad (2)$$

Differentiating the right-hand side of (2) we observe that the relation (2) actually holds for any continuous and monotone function $f$.

Let us show usefulness of the identity (2) by some examples. In the following, we always let $F(x) = \int f(x)\,dx$.

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*Example* 1: Let $f(x) = \sin x$. Since $F(x) = -\cos x$, from (2) we conclude that

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$$

$$= x \sin^{-1} x + \sqrt{1 - x^2} + C.$$

*Example* 2: For $f(x) = \tan x$ we have $F(x) = \ln \sec x$. Since

$$\sec\left(\tan^{-1} x\right) = \sqrt{x^2 + 1},$$

by using (2) we obtain

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \ln \sqrt{x^2 + 1} + C.$$

*Example* 3: Let $f(x) = e^x$. We have $F(x) = e^x$. Thus, from (2) we get

$$\int \ln x \, dx = x \ln x - e^{\ln x} + C = x \ln x - x + C.$$

*Example* 4: The Lambert $W$ function is defined as the inverse of the function $f(x) = xe^x$. Thus, it satisfies $x = W(x) e^{W(x)}$. We get $e^{W(x)} = x / W(x)$. Also, since $F(x) = (x - 1) e^x$, from (2) we deduce that

$$\int W(x) \, dx = xW(x) - \left(W(x) - 1\right) e^{W(x)} + C$$

$$= xW(x) - x + \frac{x}{W(x)} + C.$$

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## 107.34 On a staircase function

Let $x \in \left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right)$ and consider the series defined as follows: the first term is equal to $x$ and the cosine of the sum $S_n(x)$ of the first $n$ terms equals the $(n + 1)$th term. We regard $|S_n|$ as the length of an arc on the unit circle. In this Note, by using an elementary geometric argument, we show that $S_n$ is a monotone sequence that converges to $\frac{1}{2}\pi$. Due to periodicity of the cosine function, we also have convergence to $2k\pi + \frac{1}{2}\pi$ for any