ON RIESZ OPERATORS

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1. Introduction

In (4) Vala proves a generalization of Schauder's theorem (3) on the compactness of the adjoint of a compact linear operator. The particular case of Vala's result that we shall be concerned with is as follows. Let t_1 and t_2 be non-zero bounded linear operators on the Banach spaces Y and X respectively, and denote by ${}_1T_2$ the operator on $B(X, Y)$ defined by

$${}_1T_2(s) = t_1 s t_2 \quad (s \in B(X, Y)).$$

Vala shows that ${}_1T_2$ is compact if and only if both t_1 and t_2 are compact. In this paper we prove similar results for Riesz operators. We first show that ${}_1T_2$ is a Riesz operator whenever t_1 and t_2 are both Riesz operators. The proof of this result simply consists of two applications of Ruston's characterization of Riesz operators [(2), Theorem 3.1]. A converse result is established, and we also determine the spectrum and the spectral projections of ${}_1T_2$ in terms of those of t_1 and t_2.

The problem can also be transferred to a general Banach algebra setting where for elements a and c in a Banach algebra A, we consider the operator ${}_aT_c$ on A where ${}_aT_c(b) = abc$ $(b \in A)$. The situation when ${}_aT_a$ is compact for each element a in A is considered in a previous paper (1) by the author. The situation when ${}_aT_c$ is a Riesz operator for some elements a and c in A is considered in the author's Ph.D. thesis for Edinburgh University, of which the present paper is a part.

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2. Notation and preliminaries

X and Y will denote Banach spaces over the complex field C, and we denote by X^* and Y^* the conjugate space of X and Y respectively. We denote by $B(X)$ and $B(Y)$ the algebra of bounded linear operators on X and Y respectively. As no confusion should arise, we use e to denote the identity operator on both X and Y. We denote by $B(X, Y)$ the space of bounded linear transformations of X into Y. The spectrum of an element t in $B(X)$ or $B(Y)$ is denoted by $\sigma(t)$ and the spectral radius by $r(t)$. We write $\sigma_0(t)$ for the non-zero spectrum $\sigma(t)\setminus\{0\}$. The range and the null space of an operator t will be denoted by $R(t)$ and $N(t)$ respectively. The restriction of t to an invariant subspace M will be written $t \mid M$. For f in X^* and y in Y, we denote by $y \otimes f$ the operator in $B(X, Y)$ defined by

$$(y \otimes f)(x) = f(x)y \quad (x \in X).$$

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Let t be in $B(X)$. In (5) West defines a Riesz point of $\sigma(t)$ to be a point λ in $\sigma(t)$ for which there exist closed subspaces $N(\lambda;\ t)$ and $F(\lambda;\ t)$ such that $X = N(\lambda;\ t)\oplus F(\lambda;\ t)$, $\dim N(\lambda;\ t)<\infty$, $N(\lambda;\ t)$ and $F(\lambda;\ t)$ are invariant under t, and $t-\lambda e$ restricted to $N(\lambda;\ t)$ is nilpotent while $t-\lambda e$ restricted to $F(\lambda;\ t)$ is a homeomorphism. Take the projection of X into $N(\lambda;\ t)$ given by this decomposition of X. Then it is easily seen that λ is a non-zero Riesz point of $\sigma(t)$ if and only if $\lambda \in \sigma_0(t)$ and there is a projection q_λ of finite rank in $B(X)$ which commutes with t and for which $\lambda \notin \sigma(t(e-q_\lambda))$ and $(t-\lambda e)^\nu q_\lambda = 0$ for some positive integer ν.

Let λ be a Riesz point of $\sigma(t)$. Then clearly λ has finite index $\nu = \nu(\lambda;\ t)$ and

$$N(\lambda;\ t) = N((t-\lambda e)^\nu),$$

$$F(\lambda;\ t) = R((t-\lambda e)^\nu).$$

In (5), Theorem 2.1, West shows that, if $\lambda \neq 0$, then λ is an isolated point in $\sigma(t)$. Let $\gamma(\lambda)$ be a circle in C of centre λ such that λ is the only point of $\sigma(t)$ in or on $\gamma(\lambda)$. Then $p_\lambda = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma(\lambda)} (\mu e-t)^{-1}d\mu$ is the spectral projection associated with λ and t. It is easily seen that the range of p_λ is $N(\lambda;\ t)$ and the null space of p_λ is $R(\lambda;\ t)$. Then, if $\lambda \neq 0$, q_λ will be the spectral projection p_λ.

The operator t is said to be a Riesz operator on X if $t \in B(X)$ and if each point in $\sigma_0(t)$ is a Riesz point. In particular, a quasi-nilpotent operator is a Riesz operator. Let $R(X)$ denote the class of Riesz operators on X and let $K(X)$ denote the class of compact operators on X. The spectral theory of compact operators shows that $K(X)\subseteq R(X)$. The quotient algebra $B(X)/K(X)$ is a Banach algebra under the infimum norm. Let $t\to[t]$ be the canonical mapping of $B(X)$ onto $B(X)/K(X)$. In (2), Theorem 3.1, Ruston shows that $t \in R(X)$ if and only if $[t]$ is a quasi-nilpotent element in $B(X)/K(X)$, i.e.

$$\inf\left\{\inf_{c\in K(X)} \| t^n-c \|\right\}^{1/n} = 0.$$

Ruston also shows that this result remains true if we replace $K(X)$ by the closure of the ideal of finite rank operators in $B(X)$.

Let t be a Riesz operator on X. Since each non-zero point in $\sigma(t)$ is isolated, it follows that $\sigma(t)$ is at most countable with 0 as the only possible accumulation point. We shall need the following well-known result for spectral projections.

Proposition 1. *Let $t \in R(X)$, and let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\sigma_0(t)$ with associated spectral projections $p_1, \ldots, p_n$. Put $p = \sum_{i=1}^{n} p_i$. Then*

(i) $p_ip_j = 0$ if $i \neq j \quad (1 \leqq i, j \leqq n)$,

(ii) $\sigma_0(tp) = \{\lambda_i\}_{i=1}^n$,

(iii) $\sigma_0(t(e-p)) = \sigma_0(t)\diagdown\{\lambda_i\}_{i=1}^n$.

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3. Main results

Throughout this section t_1 will be a bounded linear operator on Y and t_2 will be a bounded linear operator on X. We denote by ${}_{t_1}T_{t_2}$ the bounded linear operator on $B(X, Y)$ defined by

$$ {}_{t_1}T_{t_2}(s) = t_1 s t_2 \quad (s \in B(X, Y)). $$

This will usually be abbreviated to ${}_1T_2$; the full notation is needed in the statement of Theorem 4.

Lemma 1. (i) $\| {}_1T_2 \| = \| t_1 \| \| t_2 \|$,

(ii) $r({}_1T_2) = r(t_1)r(t_2)$.

Proof. (i) Clearly $\| {}_1T_2 \| \leqq \| t_1 \| \| t_2 \|$. To prove equality we proceed as follows. Given $\varepsilon > 0$, take y in Y and f in X^* such that $\| y \| = \| f \| = 1$ and

$$ \| t_1 y \| \geqq \| t_1 \|(1-\varepsilon), \; \| t_2^* f \| \geqq \| t_2^* \|(1-\varepsilon) = \| t_2 \|(1-\varepsilon). $$

Then $\| y \otimes f \| = \| y \| \| f \| = 1$ and

$$ \| {}_1T_2(y \otimes f) \| = \| t_1 y \otimes t_2^* f \| = \| t_1 y \| \| t_2^* f \| \geqq (1-\varepsilon)^2 \| t_1 \| \| t_2 \|. $$

Hence $\| {}_1T_2 \| = \| t_1 \| \| t_2 \|$.

(ii) Let n be a positive integer. Since ${}_1T_2^n(s) = t_1^n s t_2^n$ for s in $B(X, Y)$, it follows from (i) that $\| {}_1T_2^n \| = \| t_1^n \| \| t_2^n \|$. Hence $r({}_1T_2) = r(t_1)r(t_2)$.

If t_1 and t_2 have rank one, let $t_1 = y \otimes g$ and $t_2 = x \otimes f$ where $x \in X$, $f \in X^*$, $y \in Y$, $g \in Y^*$. Then ${}_1T_2(s) = g(s(x))(y \otimes f)$ for s in $B(X, Y)$. Thus it is easily verified that ${}_1T_2$ is a non-zero operator of finite rank if both t_1 and t_2 are non-zero and have finite rank. The converse is also true but is not used in this paper. In Theorem 3 of (**4**) Vala shows that ${}_1T_2$ is a non-zero compact operator on $B(X, Y)$ if and only if t_1 and t_2 are non-zero compact operators. The following results show how the " Riesz properties " of t_1 and t_2 are carried over to ${}_1T_2$.

Theorem 1. *Let t_1 and t_2 be Riesz operators. Then ${}_1T_2$ is a Riesz operator on $B(X, Y)$.*

Proof. Let $\varepsilon > 0$ be given. The Ruston characterization of Riesz operators shows that there exist finite rank operators t_3 and t_4, t_3 in $B(Y)$, t_4 in $B(X)$, and a positive integer n such that

$$ \| t_1^n - t_3 \| \leqq \varepsilon^n, \quad \| t_2^n - t_4 \| \leqq \varepsilon^n. $$

Let s be in $B(X, Y)$. Then

$$ \begin{aligned} \| {}_1T_2^n(s) - {}_3T_4(s) \| &= \| (t_1^n - t_3) s t_2^n + t_3 s (t_2^n - t_4) \| \\ &\leqq \varepsilon^n (\| t_2^n \| + \| t_3 \|) \| s \| \\ &\leqq \varepsilon^n (\| t_1 \|^n + \| t_2 \|^n + \varepsilon^n) \| s \| \\ &\leqq 3\varepsilon^n M^n \| s \| \end{aligned} $$

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where $M = \max(\| t_1 \|, \| t_2 \|, \varepsilon)$. Hence

$$\| {}_1T_2^n - {}_3T_4 \| \leqq 3M^n\varepsilon^n.$$

Since ${}_3T_4$ has finite rank, it follows from Ruston's characterization that ${}_1T_2$ is a Riesz operator.

Putting $Y = C$ and $t_1 = e$, we see that Theorem 1 generalizes the result that the adjoint of a Riesz operator t is a Riesz operator—see Theorem 3.2 in (5). It is clear that $\sigma_0(t^*) = \sigma_0(t)$ and that, if $\lambda \in \sigma_0(t)$, the spectral projection associated with λ and t^* is the adjoint of the spectral projection associated with λ and t. We shall generalize these results later. We now give a converse result.

Theorem 2. (i) *${}_1T_2$ is a quasi-nilpotent operator if and only if either t_1 or t_2 is a quasi-nilpotent operator.*

(ii) *Let ${}_1T_2$ be a Riesz operator but not a quasi-nilpotent operator. Then t_1 and t_2 are Riesz operators.*

Proof. (i) This follows immediately from Lemma 1 (ii).

(ii) Let ${}_1T_2$ be a Riesz operator with a non-zero point λ in its spectrum. We first show that t_1 and t_2^* have non-zero eigenvalues. Let s be a non-zero element in $N({}_1T_2 - \lambda I)$ where I is the identity operator on $B(X, Y)$. For each positive integer n, $t_1^n s$ lies in $N({}_1T_2 - \lambda I)$ which is finite-dimensional. It follows that for some positive integer n the set $\{s, t_1 s, \ldots, t_1^n s\}$ is linearly dependent but the set

$$\{s, t_1 s, \ldots, t_1^{n-1} s\}$$

is linearly independent. Let p be a polynomial of degree n such that $p(t_1)s = 0$. Now $t_1^n s \neq 0$ since $t_1^n s t_2^n = \lambda^n s$, and hence p has a non-zero factor λ_1. Let $p(\xi) = (\xi - \lambda_1)q(\xi)$ for some polynomial q of degree $n-1$. Then $q(t_1)s \neq 0$ but $(t_1 - \lambda_1 e)q(t_1)s = 0$. Hence there is a non-zero point y_1 in Y such that $t_1 y_1 = \lambda_1 y_1$. Similarly, there exists a non-zero complex number λ_2 and a non-zero point f_2 in X^* such that $t_2^* f_2 = \lambda_2 f_2$.

Let U be the set $\{y \otimes f_2;\ y \in Y\}$. Then U is a closed subspace of $B(X, Y)$, and U is invariant under ${}_1T_2$ since

$${}_1T_2(y \otimes f_2) = t_1 y \otimes t_2^* f_2 = \lambda_2(t_1 y \otimes f_2) \quad (y \in Y).$$

The map $y \to y \otimes f_2$ is a homeomorphism between Y and U. In the resulting algebraic homeomorphism between $B(Y)$ and $B(U)$ the operator t_1 corresponds to $1/\lambda_2({}_1T_2 \mid U)$. The restriction of a Riesz operator to an invariant subspace is a Riesz operator—see Theorem 5.3 (i) in (6). Thus ${}_1T_2 \mid U$ is a Riesz operator on U, and hence t_1 is a Riesz operator on Y.

A similar argument shows that t_2^* is a Riesz operator. It follows from Theorem 3.2 in (5) that t_2 is a Riesz operator.

Remark. In the proof of Theorem 2 (ii) we have used the fact that a bounded linear operator is a Riesz operator if its adjoint is a Riesz operator. Theorem 2 generalizes this result—take $Y = C$ and $t_1 = e$.

We now determine the spectral structure of ${}_1T_2$ in terms of the spectral structure of t_1 and t_2.

Theorem 3. *Let t_1 and t_2 be Riesz operators on Y and X respectively and let λ be in C. Then $\lambda \in \sigma_0({}_1T_2)$ if and only if there exist complex numbers λ_1 in $\sigma_0(t_1)$ and λ_2 in $\sigma_0(t_2)$ such that $\lambda = \lambda_1\lambda_2$.*

Proof. If $\lambda \in \sigma_0({}_1T_2)$, let s be a non-zero element in $N({}_1T_2 - \lambda I)$. The proof of Theorem 2 (ii) shows that there is a non-zero complex number λ_1 and a polynomial q such that $(t_1 - \lambda_1 e)q(t_1)s = 0$ and $q(t_1)s \neq 0$. Since

$$q(t_1)s \in N({}_1T_2 - \lambda I),$$

it follows that

$$(\lambda_1 q(t_1)s)t_2 = t_1 q(t_1)st_2 = \lambda q(t_1)s.$$

Thus λ/λ_1 is in $\sigma_0(t_2)$.

Now suppose that $\lambda_1 \in \sigma_0(t_1)$, $\lambda_2 \in \sigma_0(t_2)$ and $\lambda = \lambda_1\lambda_2$. There exists a non-zero point y in Y and a non-zero point f in X^* such that $t_1 y = \lambda_1 y$ and $t_2^* f = \lambda_2 f$ since t_1 and t_2^* are Riesz operators. Since ${}_1T_2(y \otimes f) = \lambda_1\lambda_2(y \otimes f)$, it follows that $\lambda \in \sigma_0({}_1T_2)$.

We need the following elementary lemma from (6), Lemma 2.1 (i), in order to prove the next theorem.

Lemma 2. *Let s_i $(1 \leqq i \leqq n)$ be bounded linear operators on X such that $s_i s_j = 0$ if $i \neq j$. Then $\sigma\left(\sum_{i=1}^{n} s_i\right) = \bigcup_{i=1}^{n} \sigma(s_i)$.*

Let t_1 and t_2 be Riesz operators with non-zero spectral radii, and let λ be in $\sigma_0({}_1T_2)$. If μ is a non-zero complex number such that $\lambda/\mu \in \sigma_0(t_1)$, then $|\mu| \geqq |\lambda|/r(t_1)$. Since $\sigma(t_2)$ is compact and is at most countable with zero as the only possible accumulation point, it follows that the set

$$\{\mu;\ \mu \in \sigma_0(t_2),\ \lambda/\mu \in \sigma_0(t_1)\}$$

is finite. Let this set consist of the distinct points $\mu_1, \ldots, \mu_n$, and put $\lambda_i = \lambda/\mu_i$ $(1 \leqq i \leqq n)$. For each i, $1 \leqq i \leqq n$, let p_i and q_i denote the spectral projections associated with λ_i, t_1 and μ_i, t_2 respectively.

Theorem 4. $P_\lambda = \sum_{i=1}^{n} {}_{p_i}T_{q_i}$ *is the spectral projection associated with λ and ${}_1T_2$.*

Proof. It is sufficient to show that P_λ is a finite rank projection on $B(X, Y)$, commuting with ${}_1T_2$, such that $({}_1T_2 - \lambda I)^\nu P_\lambda = 0$ for some positive integer ν and such that $\lambda \notin \sigma_0({}_1T_2(I - P_\lambda))$.

For each i, $1 \leqq i \leqq n$, ${}_{p_i}T_{q_i}$ has finite rank since p_i and q_i have finite rank. By Proposition 1 (i), $p_i p_j = q_i q_j = 0$ if $i \neq j$, and thus P_λ is a projection. Clearly P_λ commutes with ${}_1T_2$.

For each i, $1 \leqq i \leqq n$,

$$ {}_1T_2 - \lambda I = {}_{(t_1-\lambda_i e)}T_{t_2} + \lambda_i\, {}_eT_{(t_2-\mu_i e)}. $$

Then, if m is a positive integer,

$$ ({}_1T_2 - \lambda I)^m(s) = \sum_{j=1}^{m} \binom{m}{j} \lambda_i^{m-j}(t_1-\lambda_i e)^j s(t_2-\mu_i e)^{m-j} t_2^j \quad (s \in B(X, Y)). $$

Taking $v_i = v(\lambda_i)+v(\mu_i)-1$, where $v(\lambda_i)$ and $v(\mu_i)$ are the indices of λ_i and μ_i respectively, it follows that $({}_1T_2-\lambda I)^{v_i}{}_{p_i}T_{q_i} = 0$. Thus for $v = \sup_{1 \leqq i \leqq n} v_i$ we have

$$ ({}_1T_2 - \lambda I)^v P_\lambda = 0. $$

Let $q = \sum_{i=1}^{n} q_i$. Then

$$ {}_1T_2(I-P_\lambda) = \sum_{i=1}^{n} {}_{t_1(e-p_i)}T_{t_2 q_i} + {}_{t_1}T_{t_2(e-q)}. $$

The product of a Riesz operator with a bounded linear operator which commutes with it is also a Riesz operator—see Theorem 3.1 in (**5**). Then $t_1(e-p_i)$, $t_2 q_i$ and $t_2(e-q)$ are Riesz operators. It then follows from Proposition 1 and Theorem 3 that, for each i, $1 \leqq i \leqq n$, λ does not lie in the spectrum of ${}_{t_1(e-p_i)}T_{t_2 q_i}$ and that λ does not lie in the spectrum of ${}_{t_1}T_{t_2(e-q)}$. Hence by Lemma 2, $\lambda \notin \sigma_0({}_1T_2(I-P_\lambda))$.

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