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ON THE PROBABILITY THAT THE *K*TH CUSTOMER FINDS AN *M/M/*1 QUEUE EMPTY

HARSHINDER SINGH* AND
RAMESHWAR D. GUPTA,** *University of New Brunswick*

Abstract

A result relating the probability that kth customer finds the system empty to the distribution of the number of customers served in a busy period, for an $M/M/1$ queue, has been obtained. This relationship is similar to the relationship between the probability that the queue is empty at time t and the distribution of the length of the busy period.

BUSY PERIOD; POISSON PROCESS

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1. Introduction

Consider an $M/M/1$ queue system with interarrival times X_1, $i = 1, 2, \cdots$, which are independent identically distributed (i.i.d.) random variables each with probability density function (p.d.f.) $\lambda e^{-\lambda x}$, $x \geqq 0$. The service times Y_i, $i = 1, 2, \cdots$ are i.i.d. random variables each with p.d.f. $\mu e^{-\mu y}$, $y \geqq 0$. Let $\rho = \lambda/\mu$ denote the traffic intensity of the system. The initial customer (say 0th customer) arrives at time 0 and starts getting service; his service time is Y_1. Let N be the serial number of the customer (after the 0th customer) who will be the first one to find the queue empty. Note that N also denotes the number of customers served (including the 0th customer) during the first busy period of the server. The probability distribution of N has been discussed in the literature (Riordan (1962), p. 65) and is given by

$$f_k = P[N = k] = \frac{(2k-2)!}{(k-1)!\,k!}\left(\frac{1}{1+\rho}\right)^k\left(\frac{\rho}{1+\rho}\right)^{k-1}, \quad k = 1, 2, \cdots. \tag{1.1}$$

The coefficients

$$a_{k-2} = \frac{(2k-2)!}{(k-1)!\,k!}$$

in (1.1) are called the Catalan numbers and satisfy the relationship

$$a_n = 2a_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-k-2}, \quad n \geqq 2 \tag{1.2}$$

(see Stanton and White (1986), p. 102, Exercise 1). Let p_k be the probability that the kth customer will find the queue empty on its arrival, $k = 1, 2, \cdots$. In Section 2, we derive the expression for p_k in terms of the distribution of N and give some properties of p_k.

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Postal address: Department of Statistics, Panjab University, Chandigarh-160014, India.

** Postal address: Division of Mathematics, Engineering and Computer Science, University of New Brunswick, Saint John, N.B., Canada, E2L 4L5.

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2. Derivation of probabilities p_k

The following theorem is the main result of the paper.

Theorem 2.1. $p_k = 1 - \rho\, P\,(N \leqq k),\ k = 1, 2, \cdots$

Proof. Since the process governing arrivals is Poisson, it follows that for $k \geqq 1$

$$p_k = \sum_{j=1}^{k} f_j p_{k-j}. \tag{2.1}$$

Thus

$$p_k - p_{k+1} = \sum_{j=1}^{k} f_j (p_{k-j} - p_{k+1-j}) - f_{k+1}.$$

Now the use of induction and (1.2) easily gives

$$p_k - p_{k+1} = \rho f_{k+1} \text{ for all } k \geqq 1$$

and the theorem follows by observing from (1.1) that $f_1 = 1 - \rho f_1$.

From (2.1), it also follows that the p_k's satisfy the properties (i) $\sum_{n=1}^{\infty} p_n = \infty$ if $\rho \leqq 1$ (ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} p_j = (1-\rho)^{-1}$, for $\rho < 1$. The proof is on similar lines to that in Parzen (1962) pp. 217, 219.

Remark. The result of Theorem 2.1 is similar to that of Corollary 4.2.3 in Abate and Whitt (1988), where they establish that $P_{00}(t) = \rho B(t)$, where $P_{00}(t)$ is the probability that the queue is empty at time t and $B(t)$ is the cumulative distribution function of the length of the busy period.

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