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# Homogeneous Einstein Finsler Metrics on $(4n+3)$-dimensional Spheres

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*Abstract.* In this paper, we study a class of homogeneous Finsler metrics of vanishing $S$-curvature on a $(4n+3)$-dimensional sphere. We find a second order ordinary differential equation that characterizes Einstein metrics with constant Ricci curvature 1 in this class. Using this equation we show that there are infinitely many homogeneous Einstein metrics on $S^{4n+3}$ of constant Ricci curvature 1 and vanishing $S$-curvature. They contain the canonical metric on $S^{4n+3}$ of constant sectional curvature 1 and the Einstein metric of non-constant sectional curvature given by Jensen in 1973.

## 1 Introduction

A Finsler metric $F$ on an $n$-dimensional manifold $M$ is said to be *Einstein* if its Ricci curvature satisfies

$$\mathrm{Ric} = (n-1)\kappa F^2,$$

where $\kappa = \kappa(x)$ is a constant. In particular, $F$ is said to be *Ricci-constant* if $\kappa$ is a constant. Ricci-constant (or Einstein and dimension $\geq 3$) Finsler metrics are the natural extension of Einstein Riemann metrics.

In [3], S. S. Chern has asked: *does every smooth manifold admit a Ricci-constant (or Einstein) Finsler metric* (also see [1, 6, 8])? Recently, the first author showed that if the Lie group $G$ is nilpotent and noncommutative, then $G$ does not admit any left invariant Ricci-constant Finsler metric [6]. Very recently, the first author studied a class of homogenous Finsler metrics of vanishing $S$-curvature on a 7-dimensional sphere $S^7$. He found a second order ODE that characterizes Einstein metrics with constant Ricci curvature 1 in this class [8]. Moreover, Huang showed that its two linear solutions correspond to the canonical metric on $S^7$ of constant sectional curvature 1 and the Einstein metric of non-constant sectional curvature given by Jensen in 1973 [9], respectively.

We know that the standard action of $Sp(2)$ on the sphere $S^7 \subset \mathbb{H}^2$ is transitive with isotropy subgroup $Sp(1)$. Thus, $S^7 = Sp(2)/Sp(1)$ is a reductive homogeneous space on which $Sp(2)$ acts transitively. Recall that a Finsler metric $F$ on $S^7 = Sp(2)/Sp(1)$ is said to be *homogeneous* if $F$ is invariant by $Sp(2)$. Similarly, we can define the notion of homogeneous Finsler metric on reductive homogeneous space $S^{4n+3} = Sp(n+1)/Sp(n)$ (see Section 2).

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This paper studies a class of homogeneous Finsler metrics on $S^{4n+3}$. We show that these homogeneous Finsler metrics are of vanishing $S$-curvature (see Proposition 3.6 below). The $S$-curvature is one of most important non-Riemannian quantities in Finsler geometry. It interacts with the flag curvature in a delicate way [11]. It is shown that the Bishop–Gromov volume comparison holds for Finsler manifold with vanishing $S$-curvature [13]. Shen proves that the $S$-curvature and the Ricci curvature determine the local behavior for the Busemann–Hausdorff measure of small metric balls around a point [14]. We know that the $S$-curvature vanishes for Berwald metrics including Riemannian metrics [13].

In the spirit of [8], we establish a second order ordinary differential equation that characterizes Einstein metrics with constant Ricci curvature 1 among these homogeneous metrics on $S^{4n+3}$. Precisely, we prove the following theorem.

***Theorem 1.1*** *The Finsler metric $F$, defined in (2.14), has constant Ricci curvature,* $\mathrm{Ric} = (4n+2)F^2$, *if and only if*

$$(1.1)\quad (8n+4)\phi = 4n+5+3\psi+(2n\psi^2-4n\psi-3\psi-2n-1)s+2s(1-s)(1-s\psi)\psi',$$

*where $\psi$ is given in (3.10).*

By investigating (1.1) and the regularity condition, we show that there are only two linear solutions of ODE (1.1). Furthermore, two linear solutions satisfy regularity condition and correspond to the canonical metric on $S^{4n+3}$ of constant sectional curvature 1 and to the Einstein metric of non-constant sectional curvature given by Jensen in 1973 [9,17], respectively.

Combining this with the theory of vector fields, we prove the following theorem.

***Theorem 1.2*** *On $S^{4n+3}$, there are infinitely many homogeneous Einstein Finsler metrics with constant Ricci curvature* 1 *and vanishing $S$-curvature. Furthermore, these metrics depend on two parameters.*

We will prove Theorem 1.2 in Section 4. For related results of Einstein Finsler metrics, we refer the reader to [2,5,12,15,16].

## 2 Preliminaries

Let $F$ be a Finsler metric on a manifold $M$ and let $(x^i)$ be a local chart on $M$. Then we have a natural local coordinate $(x^i, y^i)$ on $TM\backslash\{0\}$. Let $g_y = g_{ij}dx^i \otimes dx^j$, where $g_{ij} = \frac{1}{2}(F^2)_{y^i y^j}$. Furthermore, we can define the *Cartan tensor* by $C_y = C_{ijk}dx^i \otimes dx^j \otimes dx^k$, where $C_{ijk} = \frac{1}{2}(g_{ij})_{y^k}$.

The standard action of $Sp(n+1)$ on the sphere $S^{4n+3} \subset \mathbb{H}^n$ is transitive with isotropy subgroup $Sp(n)$ at the point $o = (0,\dots,0,1)$. Thus, $S^{4n+3} = Sp(n+1)/Sp(n)$ is a reductive homogeneous space. A Finsler metric $F$ on $S^{4n+3} = Sp(n+1)/Sp(n)$ is said to be *homogeneous* if $F$ is invariant under the action of $Sp(n+1)$ [8]. A $(4n+3)$-dimensional Finsler sphere $(S^{4n+3}, F)$ is said to be *homogeneous* if $F$ is homogeneous.

In the following, we will discuss homogeneous Finsler sphere $(S^{4n+3}, F)$. To assign a $Sp(n+1)$-invariant Finsler metric on $S^{4n+3} = Sp(n+1)/Sp(n)$, it suffices to assign a $Sp(n)$-invariant Minkowski norm on $T_oS^{4n+3}$, where $o = (0, \ldots, 0, 1)$, and then translate to other tangent space by the action of $Sp(n+1)$ [4]. Similarly, every $Sp(n+1)$-invariant object on $S^{4n+3}$ can be viewed as an $Sp(n)$-invariant on $T_oS^{4n+3}$.

Since $Sp(n+1)$ is compact, there exists an $\mathrm{Ad}(Sp(n))$-invariant subspace $\mathfrak{m}$ of $\mathfrak{sp}(n+1)$ that is complimentary to $\mathfrak{sp}(n)$, namely, we have the direct sum decomposition $\mathfrak{sp}(n+1) = \mathfrak{sp}(n) + \mathfrak{m}$. The $\mathrm{Ad}(Sp(n))$-invariance of $\mathfrak{m}$ is equivalent to $[\mathfrak{sp}(n), \mathfrak{m}] \subset \mathfrak{m}$, because $Sp(n)$ is connected.

For each $X \in \mathfrak{sp}(n+1)$, the action of the 1-parameter subgroup $\varphi_t = \exp(tX)$ on $S^{4n+3}$ induces a vector field $X^*$ on $S^{4n+3}$. The map sending $X$ to $X^*(o)$ is a linear isomorphism between $\mathfrak{m}$ and $T_oS^{4n+3}$. From now on, we will always identity $T_oS^{4n+3}$ with $\mathfrak{m}$ in this manner. Henceforth, the Minkowski norm $F$ on $T_oS^{4n+3}$ will be identified with a Minkowski norm on $\mathfrak{m}$.

Define $Q : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$Q(u, v) = -\operatorname{tr}(uv), \tag{2.1}$$

where $\mathfrak{g} := \mathfrak{sp}(\mathfrak{n}+1)$ is the Lie algebra of $Sp(n+1)$. Then $Q$ is a (positive definite) inner product. A simple calculation gives the formula

$$Q\big([u, v], w\big) + Q\big([v, [u, w]]\big) = 0. \tag{2.2}$$

It follows that $Q$ is $\mathrm{Ad}(G)$ invariant, where $G = Sp(n+1)$. Moreover, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}_0 + \mathfrak{m}_1$ and the subspaces $\mathfrak{h}, \mathfrak{m}_0, \mathfrak{m}_1$ are mutually orthogonal with respect to $Q$, where

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\} = \mathfrak{sp}(n), \tag{2.3}$$

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \,\middle|\, a + \overline{a} = 0, a \in \mathbb{H} \right\}, \tag{2.4}$$

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & \xi \\ -\xi^* & 0 \end{pmatrix} \,\middle|\, \xi \in \mathbb{H} \right\} \simeq \mathbb{H}^n. \tag{2.5}$$

***Lemma 2.1*** *For $y_0 \in \mathfrak{m}_0$ and $y_1 \in \mathfrak{m}_1$, we have*

$$Q\big([y_0, y_1], [y_0, y_1]\big) = Q(y_0, y_0) Q(y_1, y_1).$$

**Proof** This is proved by

$$y_0 = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad y_1 = \begin{pmatrix} 0 & \xi \\ -\xi^* & 0 \end{pmatrix}. \tag{2.6}$$

Taking this together with (2.1), we obtain

$$\begin{aligned} Q(y_0, y_0) &= -\operatorname{tr} \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} = -a^2, \\ Q(y_1, y_1) &= -\operatorname{tr} \begin{pmatrix} -\xi\xi^* & 0 \\ 0 & -\xi^*\xi \end{pmatrix} = \operatorname{tr}(\xi\xi^*) + \xi^*\xi = 2\xi^*\xi. \end{aligned} \tag{2.7}$$

On the other hand, from (2.6) one obtains $[y_0, y_1] = \begin{pmatrix} 0 & 0 \\ -a\xi^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & \xi a \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \xi a \\ a\xi^* & 0 \end{pmatrix}$. Combining this with (2.1), (2.6) and (2.7), we get

$$Q\big([y_0, y_1], [y_0, y_1]\big) = -\operatorname{tr}\begin{pmatrix} -\xi a^2 \xi^* & 0 \\ 0 & -a\xi^* \xi a \end{pmatrix} = Q(y_0, y_0)Q(y_1, y_1). \qquad \blacksquare$$

***Lemma 2.2*** *Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{m}_1$ with respect to the inner product $Q$. Then for $y_1 \in \mathfrak{m}_1$, we have*

$$\Sigma_i Q\big([y_1, e_i], [y_1, e_i]_{\mathfrak{m}}\big) = 3Q(y_1, y_1). \tag{2.8}$$

**Proof** Define $f : \mathfrak{m}_1 \to \mathbb{R}^+ \cup \{0\}$ by

$$f(X) := Q\big([y_1, X], [y_1, X]_{\mathfrak{m}}\big). \tag{2.9}$$

We claim that the sum $\Sigma_i f(e_i)$ does not depend on the choice of the orthonormal basis $\{e_i\}$. In fact,

$$f(e_i) = Q\big([y_1, e_i], [y_1, e_i]_{\mathfrak{m}}\big) = -Q\big(e_i, [y_1, [y_1, e_i]]_{\mathfrak{m}}\big), \tag{2.10}$$

where we have made use of (2.2) and (2.9). By using (2.5), (2.3), and (2.4), one obtains

$$[\mathfrak{m}_0, \mathfrak{m}_0] \subset \mathfrak{m}_0, \quad [\mathfrak{m}_0, \mathfrak{m}_1] \subset \mathfrak{m}_1, \quad [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_0 + \mathfrak{h} \tag{2.11}$$

and $[\mathfrak{h}, \mathfrak{m}_0] = 0, [\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1$. From (2.11) we have, for $y_1 \in \mathfrak{m}_1$, $[y_1, [y_1, v]]_{\mathfrak{m}} \in \mathfrak{m}_1$. It follows that $P(v) := -[y_1, [y_1, v]]_{\mathfrak{m}}, v \in \mathfrak{m}_1$ defined a linear operator $P : \mathfrak{m}_1 \to \mathfrak{m}_1$. Furthermore, $trP = \Sigma_i Q(e_i, -[y_1, [y_1, e_i]]_{\mathfrak{m}}) = \Sigma_i f(e_i)$, where we have used (2.10). As a result, $\Sigma_i f(e_i)$ is independent of the choice of the orthonormal basis $\{e_i\}$.

We describe an orthonormal basis of $\mathfrak{m}_1$ as follows. Let $\{Y_\alpha\}$ be the standard basis of $\mathbb{H}^n$ over $\mathbb{H}$. Then $\{Y_\alpha, \mathbf{i}Y_\alpha, \mathbf{j}Y_\alpha, \mathbf{k}Y_\alpha\}$ is a basis of $\mathbb{H}^n$ over $\mathbb{R}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are pure imaginary quaternions satisfying

$$\mathbf{ii} = -1, \quad \mathbf{jj} = -1, \quad \mathbf{kk} = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{ik} = \mathbf{j}.$$

For $\xi \in \mathbb{H}^n$, denote $\begin{pmatrix} 0 & \xi \\ -\xi^* & 0 \end{pmatrix}$ by $\widehat{\xi}$. Then $Q(\widehat{\xi}, \widehat{\eta}) = -\operatorname{tr}\begin{pmatrix} -\xi\eta^* & 0 \\ 0 & -\xi^*\eta \end{pmatrix} = \operatorname{tr}(\xi\eta^*) + \xi^*\eta = 2Re(\xi^*\eta)$, where $\eta \in \mathbb{H}^n$. It follows that

$$\Big\{\frac{1}{\sqrt{2}}\widehat{Y_\alpha}, \frac{1}{\sqrt{2}}\widehat{\mathbf{i}Y_\alpha}, \frac{1}{\sqrt{2}}\widehat{\mathbf{j}Y_\alpha}, \frac{1}{\sqrt{2}}\widehat{\mathbf{k}Y_\alpha}\Big\} \tag{2.12}$$

is an orthonormal basis of $\mathfrak{m}_1$. Let $y_1, w \in \mathfrak{m}_1$. We can assume that $y_1 = \widehat{\xi}$, $w = \widehat{\sigma}$, where $\xi, \sigma \in \mathbb{H}^n$. Direct calculations yield

$$[y_1, w] = \begin{pmatrix} \sigma\xi^* - \xi\sigma^* & 0 \\ 0 & \sigma^*\xi - \xi^*\sigma \end{pmatrix}, \quad [y_1, w]_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^*\xi - \xi^*\sigma \end{pmatrix},$$

where we have used (2.3). Together with (2.1) and (2.9) we have

$$\begin{aligned} f(w) = Q([y_1, w], [y_1, w]_{\mathfrak{m}}) &= -\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & (\sigma^*\xi - \xi^*\sigma)^2 \end{pmatrix} \\ &= -(\sigma^*\xi - \xi^*\sigma)^2. \end{aligned} \tag{2.13}$$

For $\tau \in \mathbb{H}$, we have the identity

$$-(\tau - \tau^*)^2 - (-\mathbf{i}\tau - \tau^*\mathbf{i})^2 - (-\mathbf{j}\tau - \tau^*\mathbf{j})^2 - (-\mathbf{k}\tau - \tau^*\mathbf{k})^2 = \tau^*\tau.$$

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Note that $\Sigma_\alpha Y_\alpha Y_\alpha^*$ is an unit matrix. Combining this with (2.7), (2.12), and (2.13), we obtain (2.8). ∎

Now we are going to describe our Finsler metrics. Note that $S^{4n+3} = Sp(n+1)/Sp(n)$ is single colored and $\mathfrak{m}(=\mathfrak{m}_0+\mathfrak{m}_1) \simeq T_o S^{4n+3}$, where $o=(0,\dots,0,1)\in S^{4n+3}$. We think of the Finsler metric on $S^{4n+3}$ as a Minkowski norm on $\mathfrak{m}$ [7,8].

Let $y \in \mathfrak{m}\setminus\{0\}$ and let $y_0$ and $y_1$ be the component of $y$ in $\mathfrak{m}_0$ and $\mathfrak{m}_1$ respectively. Define

$$F(y) := \sqrt{Q(y,y)\phi(s)}, \quad s := \frac{Q(y_0,y_0)}{Q(y,y)}. \tag{2.14}$$

From (2.2) we have

$$Q(\mathrm{Ad}(h)y, \mathrm{Ad}(h)y) = Q(y,y) \tag{2.15}$$

for $h \in Sp(n)$, $y \in \mathfrak{m}$. On the other hand, by [16],

$$\mathfrak{m}_0 = \left\{ X \in \mathfrak{m} \mid \mathrm{Ad}(h)X = X, \forall h \in Sp(n)\right\}. \tag{2.16}$$

It follows that $\mathrm{Ad}(h)\mathfrak{m}_1 = \mathfrak{m}_1$. Combining this with (2.16) yields $(\mathrm{Ad}(h)y)_0 = y_0$. From which, together with (2.14) and (2.15), we obtain $F(\mathrm{Ad}(h)y) = F(y)$ for $h \in Sp(n)$, $y \in \mathfrak{m}$. Thus the Finsler metric (2.14) is invariant under the action of $Sp(n+1)$ (see [7] and [4, Theorem 1.3]).

To ensure regularity, the $C^\infty$ function $\phi : [0,1] \to \mathbb{R}^+$ should satisfy

$$\begin{aligned} &\phi + (1-s)\phi' > 0, \quad \phi + (1-s)\phi' + 2s(1-s)\phi'' > 0,\\ &\phi - s\phi' > 0, \quad \phi - s\phi' + 2s(1-s)\phi'' > 0. \end{aligned} \tag{2.17}$$

Let $\phi = \frac{1}{2}(1+s)$. Then $F(y) = 1/\sqrt{2}\sqrt{2Q(y_0,y_0)+Q(y_1,y_1)}$. The metric $F$ is the standard metric on $S^{4n+3}$ of constant sectional curvature 1 [16]. Let

$$\phi = \frac{4n^2+14n+9}{2(2n+1)(2n+3)}\Big(1 - \frac{2n+1}{2n+3}s\Big).$$

Then

$$F(y) = \sqrt{\frac{4n^2+14n+9}{2(2n+1)(2n+3)}}\sqrt{\frac{2}{2n+3}Q(y_0,y_0) + Q(y_1,y_1)}.$$

The metric $F$ is the Einstein metric given by Jensen in 1973 [9,16,17].

## 3 The Einstein Equation

In this section, we are going to calculate the Ricci curvature of the Finsler metric given in (2.14) and give the proof of Theorem 1.1. We define a trivial flat connection $D$, which is just directional derivatives.

***Lemma 3.1*** *For any $v \in \mathfrak{m}$, we have $D_v s = ds(v) = \frac{2}{Q(y,y)}[Q(y_0,v_0) - sQ(y,v)]$, where $v_0$ (resp., $y_0$) is the component of $v$ (resp., $y$) in $\mathfrak{m}_0$.*

**Proof** By simple calculations, we have

$$D_v y = v, \quad D_v y_0 = v_0. \tag{3.1}$$

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It follows that

$$D_v[Q(y,y)] = Q(D_v y, y) + Q(y, D_v y) = 2Q(y,v). \tag{3.2}$$

Similarly, we have $D_v[Q(y_0,y_0)] = 2Q(y_0,v)$, where we have used the second equation of (3.1). Combining this with (2.14) and (3.2), we obtain

$$\begin{aligned} D_v s &= D_v\Big[\frac{Q(y_0,y_0)}{Q(y,y)}\Big] \\ &= \frac{Q(y,y)D_v[Q(y_0,y_0)] - Q(y_0,y_0)D_v[Q(y,y)]}{[Q(y,y)]^2} \\ &= \frac{2}{Q(y,y)}\big[Q(y_0,v_0) - sQ(y,v)\big]. \end{aligned}$$

■

***Lemma 3.2*** *For any $v \in \mathfrak{m}$, we have $g_y(y,v) = D_v(\frac{F^2}{2}) = \phi' Q(y_0,v_0) + (\phi - s\phi')Q(y,v)$.*

**Proof** In fact,

$$\begin{aligned} g_y(y,v) &= g_{ij}y^i v^j = v^j \frac{\partial}{\partial y^j}\Big(\frac{F^2}{2}\Big) \\ &= D_v\Big(\frac{F^2}{2}\Big) = \frac{1}{2}Q(y,y)D_v\phi + \frac{\phi}{2}D_v\big[Q(y,y)\big] \\ &= \frac{\phi'}{2}Q(y,y)D_v s + \phi Q(y,v) = \phi' Q(y_0,v_0) + (\phi - s\phi')Q(y,v), \end{aligned}$$

where we have made use of (3.2) and Lemma 3.1 ■

***Lemma 3.3*** *For any $v, w \in \mathfrak{m}$, we have*

$$\begin{aligned} g_y(v,w) &= \phi' Q(v_0,w_0) + (\phi - s\phi')Q(v,w) \\ &\quad + \frac{2\phi''}{Q(y,y)}\big[Q(y_0,v_0) - sQ(y,v)\big]\big[Q(y_0,w_0) - sQ(y,w)\big]. \end{aligned}$$

**Proof** Using Lemma 3.2, we obtain

$$\begin{aligned} g_y(w,v) &= g_{ij}w^i v^j = w^i \frac{\partial}{\partial y^i}\Big[v^j \frac{\partial}{\partial y^j}\Big(\frac{F^2}{2}\Big)\Big] \\ &= D_w D_v\Big(\frac{F^2}{2}\Big) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}), \end{aligned} \tag{3.3}$$

where

$$(\mathrm{I}) := (D_w\phi')Q(y_0,v_0) = \frac{2\phi''}{Q(y,y)}Q(y_0,v_0)\big[Q(y_0,w_0) - sQ(y,w)\big], \tag{3.4}$$

where we used the fact $D_w\phi' = \phi'' D_w s = \frac{2\phi''}{Q(y,y)}[Q(y_0,w_0) - sQ(y,w)]$. In (3.3),

$$\begin{aligned} (\mathrm{II}) &:= \phi' D_w\big[Q(y_0,v_0)\big] \\ &= \phi'\big[Q(D_w y_0, v_0) + Q(y_0, D_w v_0)\big] = \phi' Q(v_0,w_0), \end{aligned} \tag{3.5}$$

where we have used the fact $D_v y_0 = v_0, D_w y_0 = 0$. In (3.3),

$$\text{(III)} := \big[D_w(\phi - s\phi')\big] Q(y,v) = \frac{-2\phi'' s}{Q(y,y)} Q(y,v)\big[\, Q(y_0, w_0) - sQ(y,w)\big], \tag{3.6}$$

where we have used Lemma 3.1. In (3.3),

$$\text{(IV)} := (\phi - s\phi')\big[\, Q(D_w y, v) + Q(y, D_w v)\big] = (\phi - s\phi')Q(v,w), \tag{3.7}$$

where we have made use of the first equation of (3.1). Substituting (3.6), (3.7), (3.4), and (3.5) into (3.3) we obtain Lemma 3.3. ∎

For each $y \in \mathfrak{m}\backslash\{0\}$, there is a unique vector $\eta$ in $\mathfrak{m}$ satisfying $g_y(\eta, v) = g_y(y, [v,y]_{\mathfrak{m}})$, for all $v \in \mathfrak{m}$, where the subscript $\mathfrak{m}$ means a projection to the subspace $\mathfrak{m}$ [7, 8]. The vector $\eta$ is called the *spray vector* at $y$.

***Lemma 3.4*** *The spray vector $\eta$ at $y$ satisfies $\eta = \frac{-\phi'}{\phi - s\phi'}[y_0, y_1]$, where $y_0$ (resp., $y_1$) is the component of $y$ in $\mathfrak{m}_0$ (resp., $\mathfrak{m}_1$).*

**Proof** Set $\widetilde{\eta} = \frac{-\phi'}{\phi - s\phi'}[y_0, y_1]$. Note that the subspaces $\mathfrak{m}_0, \mathfrak{m}_1$ are mutually orthogonal with respect to $Q$. Combining this with Lemma 3.2 and (2.1), we get

$$g_y\big(y, [v,y]_{\mathfrak{m}}\big) = -\phi' Q\big(y_0, [y,v]\big) = -\phi' Q\big([y_0, y_1], v\big). \tag{3.8}$$

According to Lemma 3.3, we obtain

$$\begin{aligned} g_y([y_0,y_1],v) &= \phi' Q([y_0,y_1]_{\mathfrak{m}_0}, v_0) + (\phi - s\phi')Q([y_0,y_1],v) \\ &\quad + \frac{2\phi''}{Q(y,y)}\text{(I)}\big[\, Q(y_0,v_0) - sQ(y,v)\big], \end{aligned}$$

where

$$\begin{aligned} \text{(I)} :&= Q\big(y_0, [y_0,y_1]_{\mathfrak{m}_0}\big) - sQ\big(y, [y_0,y_1]\big) \\ &= -sQ\big(y_0,[y_0,y_1]\big) - sQ\big(y_1,[y_0,y_1]\big) = 0. \end{aligned}$$

Plugging (3.8) into (3.7) and using the second equation of (2.15) we have $g_y([y_0,y_1],v) = (\phi - s\phi')Q([y_0,y_1],v)$. It follows that $g_y(\widetilde{\eta}, v) = \frac{-\phi'}{\phi - s\phi'} g_y([y_0,y_1],v) = -\phi' Q([y_0,y_1],v)$. Combining this with (3.6) we have $g_y(\widetilde{\eta},v) = g_y(y,[v,y]_{\mathfrak{m}})$, for all $v \in \mathfrak{m}$. Now our conclusion can be obtained from the uniqueness of the spray vector at $y$. ∎

***Lemma 3.5*** *For any $v \in \mathfrak{m}$ we have*

$$D_v \eta = \psi' D_v s \cdot [y_0, y_1] + \psi[v_0, y_1] + \psi[y_0, v_1], \tag{3.9}$$

*where*

$$\psi := \psi(s) = \frac{\phi'}{\phi - s\phi'}. \tag{3.10}$$

*Consequently, $D_v \eta \in \mathfrak{m}_1$ for any $v \in \mathfrak{m}$.*

**Proof** By (3.1), we obtain $D_v y_1 = D_v(y - y_0) = D_v y - D_v y_0 = v - v_0 = v_1$. It follows that

$$D_v \eta = D_v \Big\{ \frac{-\phi'}{\phi - s\phi'}[y_0, y_1] \Big\} = \psi' D_v s \cdot [y_0, y_1] + \psi[v_0, y_1] + \psi[y_0, v_1],$$

where we have used (3.1) and (3.10). ∎

For each $y$ in $\mathfrak{m}\backslash\{0\}$, there is, by [7,8], a unique $(1,1)$ tensor $N$ on $\mathfrak{m}$ satisfying

$$2g_y\big(N(v), u\big) = g_y\big([u, v]_{\mathfrak{m}}, y\big) + g_y\big([u, y]_{\mathfrak{m}}, v\big) + g_y\big([v, y]_{\mathfrak{m}}, u\big) - 2C_y(u, v, y), \quad \forall u, v \in \mathfrak{m}.$$

This tensor $N$ is called the *connection operator at* $y$. By using the trivial flat connection $D$, the connection operator is given by

$$N = \frac{1}{2} D\eta - \frac{1}{2}\operatorname{ad}_{\mathfrak{m}}(y), \tag{3.11}$$

where $ad_{\mathfrak{m}}(y)$ denotes the linear map sending $v \in \mathfrak{m}$ to $[y, v]_{\mathfrak{m}}$. It follows that

$$N^2 = N \circ N = \frac{1}{4} D\eta \circ D\eta - \frac{1}{4}\operatorname{ad}_{\mathfrak{m}}(y) \circ D\eta - \frac{1}{4} D\eta \circ \operatorname{ad}_{\mathfrak{m}}(y) + \frac{1}{4}\operatorname{ad}_{\mathfrak{m}}(y) \circ \operatorname{ad}_{\mathfrak{m}}(y)$$

Taking the trace of this equation yields

$$\operatorname{tr}(N^2) = \frac{1}{4}\operatorname{tr}(D\eta \circ D\eta) - \frac{1}{2}\operatorname{tr}(D\eta \circ \operatorname{ad}_{\mathfrak{m}}(y)) + \frac{1}{4}\operatorname{tr}(\operatorname{ad}_{\mathfrak{m}}(y) \circ \operatorname{ad}_{\mathfrak{m}}(y)). \tag{3.12}$$

***Proposition 3.6*** *The Finsler metric defined by* (2.14) *has vanishing S-curvature.*

**Proof** The compactness of $G = Sp(n+1)$ implies that $G$ is unimodular [6]. Together with [10, Lemma 6.3] we obtain $\operatorname{tr}\operatorname{ad}(y) = 0$. Since $\operatorname{ad}(y)$ maps $\mathfrak{h}$ into $\mathfrak{m}$, we have

$$\operatorname{tr}\operatorname{ad}_{\mathfrak{m}}(y) = \operatorname{tr}\operatorname{ad}(y) = 0. \tag{3.13}$$

There is a simple relation between the connection operator and the S-curvature.

$$S(y) = \operatorname{tr}(N) + \operatorname{tr}\operatorname{ad}_{\mathfrak{m}}(y) = \operatorname{tr}(N), \tag{3.14}$$

where we have used (3.13). Combining this with (3.11) and (3.13), we obtain

$$S(y) = \operatorname{tr}\Big(\frac{1}{2} D\eta - \frac{1}{2}\operatorname{ad}_{\mathfrak{m}}(y)\Big) = \frac{1}{2} trD\eta. \tag{3.15}$$

Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{m}_1$ with respect to the inner product $Q$. Lemma 3.5 tells us that

$$\begin{aligned} \operatorname{tr} D\eta &= \operatorname{tr}_{\mathfrak{m}_1} D\eta = \Sigma_i Q(e_i, D_{e_i}\eta) \\ &= \Sigma_i Q\big(e_i, \psi' D_{e_i} s \cdot [y_0, y_1] + \psi[y_0, e_i]\big) \\ &= \psi' \Sigma_i D_{e_i} s \cdot Q\big(e_i, [y_0, y_1]\big) + \psi \Sigma_i Q\big(e_i, [y_0, e_i]\big), \end{aligned} \tag{3.16}$$

where we have made use of

$$(e_i)_0 = 0. \tag{3.17}$$

From (2.2), we obtain

$$Q\big(e_i, [y_0, e_i]\big) = Q\big(y_0, [e_i, e_i]\big) = 0. \tag{3.18}$$

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By (3.17) and Lemma 3.1 we have

$$D_{e_i}s = ds(v) = -\frac{2s}{Q(y,y)}Q(y,e_i). \tag{3.19}$$

Plugging (3.18) and (3.19) into (3.16) we have

$$\begin{aligned}\operatorname{tr} D\eta &= -\frac{2s\psi'}{Q(y,y)}\Sigma_i Q(y,e_i)Q\big(e_i,[y_0,y_1]\big)\\ &= -\frac{2s\psi'}{Q(y,y)}\Sigma_i Q\big(y_1,[y_0,y_1]\big) = -\frac{2s\psi'}{Q(y,y)}\Sigma_i Q\big(y_0,[y_1,y_1]\big) = 0,\end{aligned} \tag{3.20}$$

where we have used (2.2) and the second equation of (2.11). Plugging (3.20) into (3.15) yields Proposition 3.6. ∎

***Lemma 3.7*** *Let F be a Finsler metric on $S^{4n+3}$ defined in (2.12). Then its spray vector $\eta$ at $y$ satisfies*

$$\operatorname{tr}(D\eta\circ D\eta) = 4s\psi\big[s(1-s)\psi' - n\psi\big]Q(y,y), \tag{3.21}$$

*where $\psi$ is defined in (3.10).*

**Proof** For any $v\in\mathfrak{m}$, we have $D\eta\circ D\eta(v) = D\eta\,(D\eta(v)) = D_{D\eta(v)}\eta = D_{D_v\eta}\eta$ from which, together with (2.2) and (3.9), we obtain

$$\begin{aligned}\operatorname{tr}(D\eta\circ D\eta) &= \Sigma_i Q\big(e_i, D\eta\circ D\eta(e_i)\big)\\ &= \Sigma_i Q(e_i, D_{D_{e_i}\eta}\eta) = \psi'\Sigma_i(\mathrm{I})_i\, Q\big(e_i,[y_0,y_1]\big) + \psi\Sigma_i(\mathrm{II})_i,\end{aligned} \tag{3.22}$$

where

$$(\mathrm{II})_i := Q([e_i,y_0], D_{e_i}\eta) = Q\big([e_i,y_0], \psi' D_{e_i}s\cdot[y_0,y_1] + \psi[y_0,e_i]\big), \tag{3.23}$$

where $\{e_i\}$ is an orthonormal basis of $\mathfrak{m}_1$ with respect to the inner product $Q$. In (3.22),

$$\begin{aligned}(\mathrm{I})_i := D_{D_{e_i}\eta}s &= \frac{2}{Q(y,y)}\big[Q(y_0,(D_{e_i}\eta)_0) - sQ(y,D_{e_i}\eta)\big]\\ &= -\frac{2s}{Q(y,y)}Q(y,D_{e_i}\eta),\end{aligned} \tag{3.24}$$

where we have used Lemmas 3.1 and 3.4. Plugging (3.23) and (3.24) into (3.22) yields

$$\begin{aligned}\operatorname{tr}(D\eta\circ D\eta) = -\frac{2s}{Q(y,y)}&\Sigma_i Q(y,D_{e_i}\eta)Q\big(e_i,[y_0,y_1]\big)\\ &+ \psi\Sigma_i Q\big([e_i,y_0], \psi' D_{e_i}s\cdot[y_0,y_1] + \psi[y_0,e_i]\big)\end{aligned} \tag{3.25}$$

By (2.2), we see that $Q(y,[y_0,y_1]) = Q([y_0,y_0],y_1) - Q([y_1,y_1],y_0) = 0$ from which, together with Lemma 3.5, we obtain

$$\begin{aligned}Q(y,D_{e_i}\eta) &= Q\big(y, \psi' D_{e_i}s\cdot[y_0,y_1] + \psi[y_0,e_i]\big)\\ &= \psi' D_{e_i}s\cdot Q\big(y,[y_0,y_1]\big) + \psi Q\big(y,[y_0,e_i]\big)\\ &= -\psi Q\big([y_0,y],e_i]\big) = -\psi Q\big([y_0,y_1],e_i\big).\end{aligned}$$

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It follows that

$$\sum_i Q(y, D_{e_i}\eta)Q(e_i, [y_0, y_1]) \tag{3.26}$$
$$= -\psi\Sigma_i\big[Q([y_0, y_1], e_i)\big]^2 = -\psi Q\big([y_0, y_1], [y_0, y_1]\big)$$
$$= -\psi Q(y_0, y_0)Q(y_1, y_1) = -s(1-s)\psi\big[Q(y, y)\big]^2,$$

where we have used Lemma 2.1 and the second equation of (2.14). Note that $\dim_{\mathbb{R}} = 4n$. Combining with Lemma 2.1, we have

$$\Sigma_i Q\big([e_i, y_0], [y_0, e_i]\big) = -Q(y_0, y_0)\Sigma_i Q(e_i, e_i) = -4nQ(y_0, y_0). \tag{3.27}$$

A similar calculation of (3.24) yields $D_{e_i}s = -\frac{2s}{Q(y,y)}Q(y_1, e_i)$. It follows that

$$\sum_i Q\big([e_i, y_0], D_{e_i}s\cdot[y_0, y_1]\big) = \Sigma_i D_{e_i}s\cdot Q\big([e_i, y_0], [y_0, y_1]\big) \tag{3.28}$$
$$= -\frac{2s}{Q(y, y)}Q\big(y_1, [y_0, [y_0, y_1]]\big)$$
$$= \frac{2s}{Q(y, y)}Q\big([y_0, y_1], [y_0, y_1]\big)$$
$$= 2s^2(1-s)Q(y, y).$$

Plugging (3.26), (3.27), and (3.28) into (3.25) yields

$$\begin{aligned}\operatorname{tr}(D\eta\circ D\eta) &= 4s^2(1-s)\psi\psi' Q(y, y) - 4n\psi^2 Q(y_0, y_0)\\ &= 4s\psi\big[s(1-s)\psi' - n\psi\big]Q(y, y).\end{aligned}$$

■

***Lemma 3.8*** *Let $F$ be a Finsler metric on $S^{4n+3}$ defined in (2.12). Then its spray vector $\eta$ at $y$ satisfies*

$$\operatorname{tr} D\eta\circ \operatorname{ad}_{\mathfrak{m}}(y)) = \big[2s(s-1)\psi' + (3-3s-4ns)\psi\big]Q(y, y), \tag{3.29}$$

*where $\psi$ is defined in (3.10).*

**Proof** Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{m}_1$ with respect to the inner product $Q$ and $v_i = \operatorname{ad}_{\mathfrak{m}}(e_i))$. Then $Q(y, v_i) = Q(y, [y, e_i]) - Q(y, [y, e_i]_{\mathfrak{h}}) = Q([y, y], e_i) = 0$. Recall that $\operatorname{ad}_{\mathfrak{m}}(y)$ denotes the linear map sending $v\in\mathfrak{m}$ to $[y, v]_{\mathfrak{m}}$. Together with Lemma 3.1, we obtain

$$D_{v_i}s = \frac{2}{Q(y, y)}\big[Q(y_0, (v_i)_0) - sQ(y, v_i)\big] \tag{3.30}$$
$$= \frac{2}{Q(y, y)}Q\big(y_0, ([y, e_i]_{\mathfrak{m}})_0\big)$$
$$= \frac{2}{Q(y, y)}Q\big(y_0, ([y, e_i]\big) = \frac{2}{Q(y, y)}Q\big([y_0, y_1], e_i\big).$$

For any $v\in\mathfrak{m}$, we have $[D\eta\circ\operatorname{ad}_{\mathfrak{m}}(y)](v) = D\eta(\operatorname{ad}_{\mathfrak{m}}(y)v) = D_{\operatorname{ad}_{\mathfrak{m}}(y)v}\eta$. In particular,

$$\big[D\eta\circ\operatorname{ad}_{\mathfrak{m}}(y)\big](e_i) = D_{v_i}\eta.$$

It follows that the linear map $D\eta\circ \mathrm{ad}_{\mathfrak{m}}(y)$ maps $\mathfrak{m}$ into $\mathfrak{m}_1$, where we have made use of Lemma 3.5. Thus, we have

$$(3.31)\qquad \begin{aligned}&\operatorname{tr} D\eta\circ \mathrm{ad}_{\mathfrak{m}}(y))\\ &\quad=\Sigma_i Q(e_i, D\eta\circ\mathrm{ad}_{\mathfrak{m}}(y)(e_i))\\ &\quad=\Sigma_i Q(e_i,\psi' D_{v_i}s\cdot[y_0,y_1]+\psi[(v_i)_0,y_1]+\psi[y_0,(v_i)_1])\\ &\quad=(\mathrm{I})+(\mathrm{II})+(\mathrm{III}),\end{aligned}$$

where we have used (3.31) and Lemma 3.5, and

$$(3.32)\qquad \begin{aligned}(\mathrm{I}):=\Sigma_i Q(e_i,\psi' D_{v_i}s\cdot[y_0,y_1])&=\frac{2\psi'}{Q(y,y)}\Sigma_i\big[Q([y_0,y_1],e_i])\big]^2\\ &=\frac{2\psi'}{Q(y,y)}Q\big([y_0,y_1],[y_0,y_1]\big)=2s(1-s)\psi' Q(y,y),\end{aligned}$$

where we have made use of (3.26) and (3.30). Using (2.11), we obtain

$$(v_i)_0=[y_0,e_i]_{\mathfrak{m}_0}+[y_1,e_i]_{\mathfrak{m}_0}=[y_1,e_i]_{\mathfrak{m}_0}+[y_1,e_i]_{\mathfrak{m}_1}=[y_1,e_i]_{\mathfrak{m}}.$$

In (3.32),

$$(3.33)\qquad \begin{aligned}(\mathrm{II}):&=\Sigma_i Q\big(e_i,\psi[(v_i)_0,y_1]\big)\\ &=\psi\Sigma_i Q\big([y_1,e_i],[y_1,e_i]_{\mathfrak{m}}\big)\\ &=3\psi Q(y_1,y_1)=3s(1-s)\psi\cdot Q(y,y),\end{aligned}$$

where we have used (2.2), (3.27), and (3.33). By (2.11), we have

$$(3.34)\qquad (v_i)_1=[y,e_i]_{\mathfrak{m}}|_{\mathfrak{m}_1}=[y,e_i]_{\mathfrak{m}_1}=[y_0,e_i]_{\mathfrak{m}_1}+[y_1,e_i]_{\mathfrak{m}_1}=[y_0,e_i].$$

In (3.32),

$$(3.35)\qquad \begin{aligned}(\mathrm{III}):&=\Sigma_i Q(e_i,\psi[y_0,(v_i)_1])\\ &=\psi\Sigma_i Q([e_i,y_0],(v_i)_1)=-\psi\Sigma_i Q(y_0,y_0)Q(e_i,e_i)\\ &=-4ns\psi\cdot Q(y,y),\end{aligned}$$

where we have made use of (2.2), (3.34), Lemma 2.1 and the second equation of (2.14). Plugging (3.32), (3.33), and (3.35) into (3.31) yields (3.21). ∎

***Proposition 3.9*** *Let $F$ be a Finsler metric on $S^{4n+3}$ defined in* (2.12). *Then its Ricci curvature* Ric *at $y$ satisfies*

$$(3.36)\qquad \begin{aligned}\mathrm{Ric}(y)&=\frac{1}{2}Q(y,y)\\ &\times\big[4n+5+3\psi+(2n\psi^2-4n\psi-3\psi-2n-1)s+2s(1-s)(1-s\psi)\psi'\big].\end{aligned}$$

**Proof** By [7, Corollary 4.9], we have

$$(3.37)\qquad \mathrm{Ric}(y)=-\operatorname{tr}_{\mathfrak{m}}\big(\mathrm{ad}(y)\circ\mathrm{ad}_{\mathfrak{h}}(y)\big)+D_\eta\big(\operatorname{tr}(N)\big)-\operatorname{tr}(N^2).$$

From (3.14) and Proposition 3.6, we obtain $\mathrm{tr}(N) = S = 0$. Plugging this into (3.37) yields

$$\text{(3.38)}\quad \begin{aligned}\mathrm{Ric}(y) &= -\mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}(y)\circ \mathrm{ad}_{\mathfrak{h}}(y)\big) - \mathrm{tr}(N^2)\\ &= (\mathrm{I}) - \mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}(y)\circ \mathrm{ad}_{\mathfrak{h}}(y)\big) - \frac{1}{4}\mathrm{tr}\big(\mathrm{ad}_{\mathfrak{m}}(y)\circ \mathrm{ad}_{\mathfrak{m}}(y)\big),\end{aligned}$$

where

$$\begin{aligned}(\mathrm{I}) &= \frac{1}{2}\mathrm{tr}(D\eta\circ \mathrm{ad}_{\mathfrak{m}}(y)) - \frac{1}{4}\mathrm{tr}(D\eta\circ D\eta)\\ &= \Big[ns\psi^2 - s^2(1-s)\psi\psi' + s(1-s)\psi' + \frac{3}{2}(1-s)\psi - 2ns\psi\Big]Q(y,y),\end{aligned}$$

where we have used (3.12), (3.21), and (3.29). Substituting this into (3.38) yields

$$\text{(3.39)}\quad \begin{aligned}\mathrm{Ric}(y) = &-\mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}(y)\circ \mathrm{ad}_{\mathfrak{h}}(y)\big) - \frac{1}{4}\mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}_{\mathfrak{m}}(y)\circ \mathrm{ad}_{\mathfrak{m}}(y)\big)\\ &+ \Big[ns\psi^2 - s^2(1-s)\psi\psi' + s(1-s)\psi' + \frac{3}{2}(1-s)\psi - 2ns\psi\Big]Q(y,y).\end{aligned}$$

Let $\phi = \frac{1}{2}(1+s)$. Then $\phi' = \phi - s\phi' = \frac{1}{2}$. It follows that $\psi = -1, \psi' = 0$, where $\psi$ is defined in (3.10). Hence the Ricci curvature $\overline{\mathrm{Ric}}$ of $\overline{F} := \sqrt{Q(y,y)(1+s)/2}$ is given by

$$\overline{\mathrm{Ric}} = Q(y,y)\Big[3ns - \frac{3}{2}(1-s)\Big] - \mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}(y)\circ \mathrm{ad}_{\mathfrak{h}}(y)\big) - \frac{1}{4}\mathrm{tr}_{\mathfrak{m}}\big(\mathrm{ad}_{\mathfrak{m}}(y)\circ \mathrm{ad}_{\mathfrak{m}}(y)\big).$$

We know that $\overline{F}$ is the standard metric on $S^{4n+3}$ of constant sectional curvature 1. It follows that $\overline{\mathrm{Ric}} = (4n+2)\overline{F}^2 = (2n+1)(1+s)Q(y,y)$. Plugging this into (3.39) yields (3.36). ∎

***Remark*** When $n = 1$, (3.36) is equivalent to the following formula given in [8]:

$$\begin{aligned}\mathrm{Ric}(y) = &\frac{2\overline{g}(y,y)}{(\varphi - t\varphi')^3}\\ &\times\big[2t(t-1)\varphi\varphi'' + t^2(4t-5)\varphi'^3 + t(8-5t)\varphi\varphi'^2 - (2t+3)\varphi^2\varphi' + 3\varphi^3\big],\end{aligned}$$

where $s = \frac{t}{2-t}, \phi(s) = \frac{\varphi(t)}{2-t}$.

**Proof of Theorem 1.1** By (3.36) and the first equation of (2.14),

$$\begin{aligned}(8n+4)\phi &= \frac{2(4n+2)}{Q(y,y)}F^2\\ &= 4n+5+3\psi + (2n\psi^2 - 4n\psi - 3\psi - 2n - 1)s\\ &\quad + 2s(1-s)(1-s\psi)\psi'.\end{aligned}$$

Inspection shows that there are two solutions of (1.1) in the form $\phi(s) = \lambda + \mu s$, given by

$$\text{(3.40)}\qquad \phi(s) = \frac{1}{2}(1+s),\qquad \phi(s) = \frac{4n^2+14n+9}{2(2n+1)(2n+3)}\Big(1 - \frac{2n+1}{2n+3}s\Big).$$

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In fact, they are the only linear solutions of (1.1). When $n = 1$, (3.40) is equivalent to the following linear solutions of Einstein equation given in [8]: $\varphi(t) = 1, \varphi(t) = \frac{9}{5} - \frac{36}{25}s$, where $s = t/(2-t), \phi(s) = \varphi(t)/(2-t)$. ■

## 4 Regularity of Solutions

In this section we are going to investigate the regularity of solutions of the ordinary differential equation (1.1). Concretely, we will discuss the following two problems.

(i) Are there any nontrivial solutions of (1.1) that satisfy the regularity condition (2.17)?

(ii) How many regular solutions are there?

The first problem is easy to answer. Two linear solutions $\phi(s) = \lambda + \mu s$ are always regular. In this case, the four inequalities in (2.17) are all reduced to the inequality $\min\{\lambda, \lambda+\mu\} > 0$, because $\phi - s\phi' = \phi - s\phi' + 2s(1-s)\phi'' = \lambda$ and $\phi + (1-s)\phi' = \phi + (1-s)\phi' + 2s(1-s)\phi'' = \lambda + \mu$. Moreover, for the linear solutions (1.1),

$$\min\{\lambda, \lambda+\mu\} = \frac{1}{2} \quad \text{and} \quad \min\{\lambda, \lambda+\mu\} = \frac{4n^2+14n+9}{(2n+1)(2n+3)^2},$$

respectively. They correspond to the canonical metric on $S^{4n+3}$ of constant sectional curvature 1 and the Einstein metric of non-constant sectional curvature given by Jensen in 1973 [9,16,17], respectivel.

Observe that (3.10) is equivalent to the equation $\phi' = \frac{-\phi\psi}{1-s\phi}$. It follows that $F$ has constant Ricci curvature, $\mathrm{Ric} = (4n+2)F$, if and only if $(\phi, \varphi)$ satisfies

$$\begin{cases} \psi' = \dfrac{(8n+4)\phi - 4n - 5 - 3\psi - (2n\psi^2 - 4n\psi - 3\psi - 2n - 1)s}{2s(1-s)(1-s\psi)}, \\ \phi' = \dfrac{-\phi\psi}{1-s\psi}. \end{cases} \tag{4.1}$$

A solution of (1.1) gives rise to a curve $s \mapsto (s, \phi(s), \psi(s))$ in $\mathbb{R}^3$ with coordinate $(s, \phi, \psi)$. For instance, for the linear solutions (3.40), they correspond the following curves:

$$\Gamma_C : s \longmapsto \left(s, \frac{1+s}{2}, -1\right),$$

$$\Gamma_J : s \longmapsto \left(s, \frac{4n^2+14n+9}{2(2n+1)(2n+3)}\left(1 - \frac{2n+1}{2n+3}s\right), \frac{2n+1}{2n+3}\right).$$

By (3.10) and (4.1), $\phi$ satisfies the regularity conditions (2.15) if and only if

$$\frac{(1-\psi)\phi}{1-s\psi} > 0, \qquad \phi\frac{\Theta - (8n-4)\phi - (1-s\psi)^2\psi}{(1-s\psi)^3} > 0, \tag{4.2}$$

$$\frac{\phi}{1-s\psi} > 0, \qquad \phi\frac{\Theta - (8n-4)\phi}{(1-s\psi)^3} > 0,$$

where $\Theta := s^2\psi^2 + 2ns\psi^2 - s(4n+s)\psi - (2n+1)s + 3\psi + 4n + 6$. Note that $Q$ is a positive definite inner product. It follows that

$$\phi > 0, \qquad s \geq 0, \tag{4.3}$$

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where we have used (2.14). Furthermore, $Q(y,y) = Q(y_0,y_0)+Q(y_1,y_1) \geq Q(y_0,y_0)$. Hence, we have

$$0 \leq s \leq 1. \tag{4.4}$$

Together with the first equation of (4.3), we obtain that (4.2) is equivalent to (4.4) and

$$\psi < 1, \quad 0 < (8n+4)\phi < \min\left\{\Theta, \Theta - \psi(1-s\psi)^2\right\}.$$

Define $\Omega := \left\{(s,\phi,\psi) \in [0,1] \times (-\infty,1) \times (0, \frac{1}{8n+4}\min[\Theta, \Theta - \psi(1-s\psi)^2])\right\}$. Then $\Omega$ looks like a bottom-free box with one face bent. Define $X = (1, X_2, X_3)$, where

$$X_2 = \frac{-\phi\psi}{1-s\psi},$$
$$X_3 = \frac{(8n+4)\phi - 4n - 5 - 3\psi - (2n\psi^2 - 4n\psi - 3\psi - 2n - 1)s}{2s(1-s)(1-s\psi)}.$$

Then the vector field $X$ has no singularities in the interior of $\Omega$. Consequently, every integral curve will eventually cross the boundary of $\Omega$. It follows from (4.1) that $\frac{d}{ds}(s,\phi,\psi) = (1,\phi',\psi') = X$. Hence, the solutions of (4.1) can also be described as the integral curves of vector field $X$. It is easy to see that a solution $(\phi,\psi)$ of (4.1) is regular, if and only if the corresponding integral curve $(s,\phi(s),\psi(s))$ lies in $\Omega$ and it connects the two boundary plane $s = 0$ and $s = 1$. The first linear solution in (3.40) gives rise to a line segment that connects the two points $p_0 = (0, \frac{1}{2}, -1)$ and $p_1 = (1,1,-1)$. Since $p_0$ and $p_1$ are interior points in the corresponding boundary planes, we conclude that the nearby integral curves also connect the two planes; thus, they are also regular. Similarly, the second linear solution in (1.1) gives rise to a line segment that connects the two points

$$p_2 = \left(0, \frac{4n^2+14n+9}{2(2n+1)(2n+3)}, \frac{2n+1}{2n+3}\right),$$
$$p_3 = \left(1, \frac{4n^2+14n+9}{(2n+1)(2n+3)^2}, \frac{2n+1}{2n+3}\right).$$

Since $p_2$ and $p_3$ are interior points in the corresponding boundary planes, we conclude that the nearby integral curves also connect the two planes; thus, they are also regular. We have thus completed the proof of Theorem 1.2.

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