FREE AND INJECTIVE LIE MODULES*

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(received May 19, 1965)

We study free and injective Lie modules by investigating the relationship between Lie modules and (associative) modules. An important role is played by the universal enveloping ring of a Lie ring [4]. If L is an arbitrary Lie ring and W(L) its universal enveloping ring, we show that the category of Lie L-modules and the category of associative W(L)-modules are isomorphic (section 2). In section 3 we study free Lie modules and show how they may be obtained from free associative modules. A Lie module is free if and only if it is a direct sum of copies of the free Lie module on one generator. The existence of the injective hull for an associative module is well known [2]. In section 4 we show that a Lie module, too, possesses an injective hull. The free Lie module on one generator serves as a "test module" in the verification of injectivity for Lie modules. The first section gives some basic definitions.

1. Let L be a commutative (additive) group. L is said to be a <u>Lie ring</u> if there is defined a multiplication in L which is distributive and which satisfies:

(i) $a^2 = 0$,

(ii) $(ab)c + (bc)a + (ca)b = 0$ - the Jacobi identity - (for all $a, b, c \in L$) .

(i) implies that L is anticommutative; that is: $ab = -ba$ $(a, b \in L)$.

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* I wish to thank Professor J. Lambek for many helpful suggestions.

Note: The above definition as well as some of those that follow may be found in [1] (where, however, Lie algebras, not Lie rings, are considered). We include them here for the sake of completeness.

If L_1, L_2 are Lie rings, a Lie ring homomorphism from L_1 to L_2 is a map $f: L_1 \to L_2$ such that $f(a+a') = f(a) + f(a')$ and $f(aa') = f(a)f(a')$, $(a, a' \in L_1)$. Thus, a homomorphic image of a Lie ring is again a Lie ring.

A (right) Lie module over the Lie ring L (a Lie L-module) is a commutative (additive) group M together with a multiplication: $M \times L \to M$ such that

(i) $x(a+a') = xa + xa'$, $(x+x')a = xa + x'a$,

(ii) $x(aa') = (xa)a' - (xa')a$, $(x, x' \in M;\ a, a' \in L)$.

We denote the right Lie L-module M by M_L. Left Lie modules are defined in a similar manner. Thus, (ii) becomes: $(aa')x = a(a'x) - a'(ax)$.

Remarks:

(a) One can turn the right Lie module M_L into a left Lie module ${}_LM$ by defining $ax = -xa (x \in M, a \in L)$. Thus, when speaking of Lie modules one need not distinguish between right and left.

(b) The Lie ring L may be considered as a module over itself. The Jacobi identity and the anticommutative law imply condition (ii) above.

(c) A module M_R, where R is an associative ring, will be called an associative module.

If M_L, N_L are two Lie modules, a Lie module homomorphism (an L-homomorphism) from M_L to N_L is a map $f: M \to N$ such that $f(x+x') = f(x) + f(x')$ and $f(xa) = f(x)a$,

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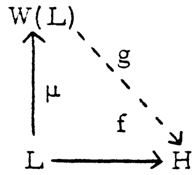
$(x, x' \in M,\ a \in L)$. Difference modules and submodules of Lie modules are defined in the obvious way (as for the associative case), and the fundamental homomorphism and isomorphism theorems (corresponding to those in the associative case) hold. It is easily verified that the class of Lie modules (over some arbitrary fixed Lie ring L) forms a category. In section 2 we shall show that this category is abelian.

If M_i ($i \in I$, where I is some index set) is a family of Lie L-modules, we form their direct sum $\sum_{i \in I} \oplus M_i$ as follows: the additive group of $\Sigma \oplus M_i$ is the direct sum of the additive groups of the modules M_i. Multiplication by the elements of L is defined componentwise: if $x = (x_i) \in \Sigma \oplus M_i$, $a \in L$, then $(xa)_i$, the i-th component of xa, is defined to be $x_i a$. It is easily verified that with this multiplication $\Sigma \oplus M_i$ becomes a Lie L-module. It then follows that a Lie-module M is a direct sum of its submodules $M_i (i \in I)$ if and only if $M = \Sigma M_i$ (that is, every element of M is a finite sum of elements of M_i) and $M_i \cap \sum_{j \neq i} M_j = 0$, for all $i, j \in I$. (This is so since the corresponding result is known to hold for groups.)

2. If R is an associative ring, we associate with it a Lie ring $C(R)$ as follows. The additive group of $C(R)$ is the same as that of R; multiplication in $C(R)$ is defined by the additive commutator: if $a, b \in C(R)$ we define $[a, b] = ab - ba$ (where the product on the right hand side is that in R). It is easily verified that with this multiplication $C(R)$ forms a Lie ring.

If R_1, R_2 are associative rings and $f: R_1 \to R_2$ a ring homomorphism, then f induces a Lie ring homomorphism $C(f)$: $C(R_1) \to C(R_2)$ in the obvious way: if $a \in C(R_1)$, we set $C(f)(a) = f(a)$. It is easily verified that C is, in fact, a functor from the category of associative rings to that of Lie rings. We shall now define a functor in the opposite direction.

Let L be a Lie ring. The pair $(W(L), \mu)$, (where $W(L)$ is an associative ring, μ a Lie ring homomorphism of L into $C(W(L))$) is called a universal enveloping ring (u. e. r.) of L if for any Lie ring homomorphism f of L into the Lie ring $C(H)$ (where H is any associative ring), there exists a unique (associative) ring homomorphism $g: W(L) \to H$ such that $g\mu = f$. That is, the accompanying diagram commutes.



(To be precise, $W(L)$, H and g should be replaced by $C(W(L))$, $C(H)$ and $C(g)$ respectively; however, no confusion will arise.)

PROPOSITION 2.1. Every Lie ring L possesses a universal enveloping ring $(W(L), \mu)$. Moreover, μ is one-one. (Thus, L may be considered as a subring of $C(W(L))$.)

Proof. It is not difficult to show that L possesses a u. e. r. Thus, to each $a \in L$ let there correspond (uniquely) some element (object) z_a. Let $F(L)$ be the free associative ring with identity generated by all the z_a $(a \in L)$. Let K be the ideal of $F(L)$ generated by all elements of the form $z_a + z_b - z_{a+b}$ and $z_a z_b - z_b z_a - z_{ab}$ $(a, b \in L)$. Let $W(L) = F(L)/K$ and define $\mu: L \to C(W(L))$ by $\mu(a) = z_a + K$. It is not difficult to verify that $(W(L), \mu)$ is a u. e. r. of L. The difficulty arises in showing that μ is one-one. The proof forms the contents of a paper by Witt [4] and will here be omitted.

Remarks:

(a) From the construction of $W(L)$ it is clear that it possesses an identity element.

(b) $W(L)$ is generated by $\mu(L)$. This is a standard proof [1, p. 152].

(c) A u. e. r. of L is unique (up to isomorphism) [1. p, 152].

The mapping W which associates with each Lie ring L its u. e. r. W(L) is a functor from the category of Lie rings to that of associative rings. This may be verified directly, but it also follows from Kan's theorem on adjoint functors [3, pp. 58, 59] (which also shows, incidentally, that W is an adjoint of C).

Our object is to show that the category $\mathfrak{M}(L)$ of Lie modules over any Lie ring L and the category $\mathfrak{N}(W(L))$ of associative modules over the u. e. r. W(L) of L are <u>isomorphic</u>: that is, there exist functors $F: \mathfrak{M}(L) \to \mathfrak{N}(W(L))$ and $G: \mathfrak{N}(W(L)) \to \mathfrak{M}(L)$ such that $GF = I$, $FG = I'$ (where I and I' are the identity functors of $\mathfrak{M}(L)$ and $\mathfrak{N}(W(L))$ respectively). To this end we need the following

LEMMA 2.2. To every Lie L-module M_L there corresponds an associative module $M_{W(L)}$ (the additive groups of the two being identical). Moreover, any L-homomorphism $h: M_L \to N_L$ (N_L an arbitrary Lie module) may be regarded as a W(L)-homomorphism: $M_{W(L)} \to N_{W(L)}$.

<u>Proof</u>. As mentioned, the additive group of $M_{W(L)}$ is the same as that of M_L. To define the multiplication: $M \times W(L) \to M$, set $H = \mathrm{Hom}_Z(M, M)$. If $u, v \in H$, define uv to be the mapping u followed by v. Then H is an associative ring and M becomes a right H-module. Define a map $f: L \to C(H)$ as follows: if $a \in L$, f(a) is a map: $M \to M$ given by $f(a)(x) = xa$ ($x \in M$). It is clear that $f(a) \in C(H)$. If $x \in M$, $a, b \in L$, then $f(a+b)(x) = x(a+b) = xa + xb = f(a)(x) + f(b)(x)$ $= [f(a)+f(b)](x)$, and $[f(ab)](x) = x(ab) = (xa)b - (xb)a$ $= f(a)f(b)(x) - f(b)f(a)(x) = [f(a)f(b) - f(b)f(a)](x)$.

Thus, f is a momomorphism of L into C(H). Hence there exists a unique homomorphism $g: W(L) \to H$ such that $g\mu = f$. We now turn M into a W(L)-module as follows: if $x \in M$, $w \in W(L)$, define $xw = g(w)(x)$. Since M is an H-module, it is easily verified that M is also a W(L)-module.

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Now suppose h is an L-homomorphism: $M_L \to N_L$. Let $K = \{ w \in W(L): h(xw) = h(x)w,$ for all $x \in M\}$. We want $K = W(L)$. Let $k_1, k_2 \in K$, $x \in M$. Then $h[x(k_1 - k_2)] = h(xk_1 - xk_2)$ $= h(xk_1) - h(xk_2) = h(x)k_1 - h(x)k_2 = h(x)(k_1 - k_2)$, and $h[x(k_1 k_2)] = h[(xk_1)k_2] = h(xk_1)k_2 = [h(x)k_1]k_2 = h(x)(k_1 k_2)$. This shows that K is a subring of $W(L)$. If $a \in L$, $x\mu(a) = g\mu(a)(x) = f(a)(x) = xa$. Hence K contains $\mu(L)$ (since h is an L-homomorphism). But $\mu(L)$ generates $W(L)$, hence $K = W(L)$. This completes the proof.

We now define a mapping F from the category $\mathfrak{M}(L)$ of Lie L-modules to the category $\mathfrak{N}(W(L))$ of associative $W(L)$-modules as follows: if $M_L \in \mathfrak{M}(L)$, set $F(M_L) = M_{W(L)}$, the $W(L)$-module as defined in the above lemma. If $h: M_L \to M'_L$ ($M'_L \in \mathfrak{M}(L)$) then, also by the above lemma, h may be regarded as a $W(L)$-homomorphism: $F(M_L) \to F(M'_L)$. We define this to be the mapping $F(h)$. Then it is clear that F is a functor.

We also define a mapping $G: \mathfrak{N}(W(L)) \to \mathfrak{M}(L)$ as follows: if $N_{W(L)} \in \mathfrak{N}(W(L))$ then N may be regarded as a $C(W(L))$-module in the obvious way (the multiplications $N \times W(L)$ and $N \times C(W(L))$ being identical). Since L (more precisely $\mu(L)$) is a subring of $C(W(L))$, we obtain an L-module N_L by restricting the scalar multiplication $N \times C(W(L))$ to $N \times L$. This is the definition of $G(N_{W(L)})$.

If $k: N_{W(L)} \to N'_{W(L)}$ ($N'_{W(L)} \in \mathfrak{N}(W(L))$), then k may be regarded, in the obvious way, as a mapping: $N_{C(W(L))} \to N'_{C(W(L))}$. Hence $k: N_L \to N'_L$ (where, of course, $N_L = G(N_{W(L)})$, $N'_L = G(N'_{W(L)})$). This is the definition of $G(k)$. Again, it is easy to verify that G is a functor.

THEOREM 2.3. The category $\mathfrak{M}(L)$ of Lie modules over any Lie ring L and the category $\mathfrak{N}(W(L))$ of associative

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modules over the universal enveloping ring $W(L)$ of L are isomorphic.

Proof. We have defined functors $F: \mathfrak{M}(L) \to \mathfrak{N}(W(L))$ and $G: \mathfrak{N}(W(L)) \to \mathfrak{M}(L)$ above. It remains to show that $GF = I$, $FG = I'$ (I and I' being the identity functors of $\mathfrak{M}(L)$ and $\mathfrak{N}(W(L))$ respectively). Let $M_L \in \mathfrak{M}(L)$, then $F(M_L) = M_{W(L)}$ (as in 2.2). If now $x \in G(M_{W(L)})$, $a \in L$, then $x\mu(a) = g\mu(a)(x)$ $= f(a)(x) = xa$, (the g and f as in 2.2). This shows that $G(M_{W(L)}) = M_L$ (after identification of $\mu(L)$ with L). That is, $GF(M_L) = M_L$. It is clear that $GF(h) = h$ for any $h: M_L \to M'_L$. Hence $GF = I$.

Let now $N_{W(L)} \in \mathfrak{N}(W(L))$ and set $H = \mathrm{Hom}_Z(N, N)$. It is easily verified that the $W(L)$-module structure of N induces a ring homomorphism $g': W(L) \to H$ given by $g'(w)(x) = xw$ ($w \in W(L)$, $x \in N$). Now, the module $N_L = G(N_{W(L)})$ is obtained by restricting the scalar multiplication to $N \times L$. Then $F(N_L)$ is obtained by extending the Lie ring homomorphism $f: L \to C(H)$, defined by $f(a)(x) = xa$ ($x \in N$, $a \in L$), to a ring homomorphism $g: W(L) \to H$, and then by defining $xw = g(w)(x)$. Since the extension g is unique, hence $g' = g$. Thus $FG(N_{W(L)}) = N_{W(L)}$. It is clear that $FG(k) = k$ for any $k: N_{W(L)} \to N'_{W(L)}$. Hence $FG = I'$ and the categories are isomorphic.

COROLLARY 2.4. The category of Lie modules (over any fixed Lie ring) is abelian.

Proof. The category of associative modules is abelian [3, p. 66].

We shall derive other consequences of the above theorem in the next two sections.

3. The module M_L is said to be a free Lie L-module generated by the set $X \subseteq M$ (or X is a free set of generators of M_L) if given any Lie module N_L and any map $f: X \to N$,

there exists a unique L-homomorphism $f': M_L \to N_L$ extending f.

We now show that under the correspondence of theorem 2.3 free Lie modules correspond to free associative modules.

PROPOSITION 3.1. If $M_L \to M_{W(L)}$ is the correspondence between Lie-L-modules and associative W(L)-modules under 2.3, then a subset $X \subseteq M$ is a free set of generators of M_L if and only if it is a free set of generators of $M_{W(L)}$.

Proof. Let $M_{W(L)}$ be the free associative W(L)-module generated by the set $X \subseteq M$. Let N_L be an arbitrary Lie D-L-module and $f: X \to N_L$ any map. Then clearly $f: X \to N_{W(L)}$ (where $N_{W(L)}$ corresponds to N_L under 2.3). Hence there exists a unique W(L)-homomorphism $f': M_{W(L)} \to N_{W(L)}$ extending f. Then also $f': M_L \to N_L$ (this latter f' is actually the G(f') of 2.3, but we denote it also by f' since the two maps are the same on the set M). Now, f' is unique as a W(L)-homomorphism, and it follows from the correspondence of maps under 2.3 that f' is also a unique extension of f as an L-homomorphism. That is, M_L is a free Lie L-module generated by X. To prove the converse, one just reverses the steps. Thus, M_L is a free Lie L-module if and only if $M_{W(L)}$ is a free associative W(L)-module.

Remarks:

(a) The free Lie-L-module generated by a set X is unique (up to isomorphism). This follows from the above proposition and the well known fact of the uniqueness of the free associative module on a set of generators.

(b) As in the associative case, the generators X of the free Lie L-module M_L are torsion free. That is, $xa = 0$ implies $a = 0$ $(x \in X,\ a \in L)$.

For, $xa = 0$ implies $x\mu(a) = o$ (since $xa = x\mu(a)$, - see proof of theorem 2.3). Since $\mu(a) \in W(L)$ and $M_{W(L)}$ is the free associative $W(L)$-module generated by X, it follows that $\mu(a) = 0$, hence $a = 0$ (since μ is one-one).

Of particular interest is the free Lie L-module on one generator. It corresponds, by 3.1, to the free associative $W(L)$-module on one generator. This may be taken to be $W(L)$ (considered as a module over itself), since $W(L)$ has an identity. Thus $W(L)$ (considered now as a Lie L-module) may be taken to be the free Lie-L-module on one generator. As in the associative case, we have

PROPOSITION 3.2. M_L is a free Lie L-module if and only if it is isomorphic to a direct sum of copies of the free Lie L-module on one generator.

<u>Proof</u>. Let $M_{W(L)}$ correspond to M_L as under 2.3. Then the following are equivalent:

(i) M_L is a free Lie L-module;

(ii) $M_{W(L)}$ is a free associative $W(L)$-module;

(iii) $M_{W(L)}$ is a direct sum of copies of $W(L)$ (considered as a module over itself);

(iv) M_L is a direct sum of copies of $W(L)$ (considered now as a Lie module over L).

The equivalence of (i) and (ii) follows by 3.1. It is well known that (ii) and (iii) are equivalent (these being associative modules - see, for example, [2, IV]). The equivalence of (iii) and (iv) is clear (since M_L and $M_{W(L)}$ have the same additive group). The proposition now follows.

4. The Lie L-module M_L is said to be <u>injective</u> if given any Lie modules A_L, B_L with $A_L \subseteq B_L$, and an L-homomorphism

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$f: A_L \to M_L$, there exists an L-homomorphism $f': B_L \to M_L$ extending f. An <u>injective hull</u> of the Lie module M_L is a minimal injective extension of M_L; that is, an injective extension I_L of M_L such that if J_L is another injective extension of M_L and $J \subseteq I$, then necessarily $J = I$.

PROPOSITION 4.1. Every Lie-module M_L possesses an injective hull.

<u>Proof.</u> Let $M_{W(L)}$ be the associative module corresponding to M_L under 2.3. Let $I_{W(L)}$ be its injective hull (that this exists for associative modules is well known - see, for example, [2, IV]). Again, let I_L correspond to $I_{W(L)}$ under 2.3. Then I_L is the injective hull of M_L. For, the W(L)-injectivity of $I_{W(L)}$ implies (by 2.3) the L-injectivity of I_L. The isomorphism of the categories under 2.3 also ensures that I_L is a minimal injective extension of M_L.

<u>Remarks:</u>

(a) Since injectivity is categorically defined, it follows from 2.3 that a Lie D-L-module is L-injective if and only if the corresponding associative W(L)-module (under 2.3) is W(L)-injective.

(b) An injective hull of a Lie module is unique (up to isomorphism). This follows from the above remark and the well known fact that the injective hull of an associative module is unique.

We now give a criterion for testing the injectivity of a Lie module internally.

PROPOSITION 4.2. The Lie module M_L is injective if and only if for every submodule K_L of F_L (where F_L is

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the free Lie L-module on one generator), every $f \in \mathrm{Hom}_L(K, M)$ can be extended to some $f' \in \mathrm{Hom}_L(F, M)$.

Proof. The necessity is obvious. If the above condition holds, let $M_{W(L)}$ be the associative module corresponding to M_L under 2.3. We show that $M_{W(L)}$ is injective, hence M_L will be injective (see remark (a) above). Thus, let A be a right ideal of $W(L)$ and suppose $f \in \mathrm{Hom}_{W(L)}(A, M)$. Let A_L be the Lie L-module corresponding to the W(L)-module $A_{W(L)}$ under 2.3. Then A_L is a submodule of F_L and we may consider f as mapping A_L into M_L; by hypothesis, f may be extended to some $f' : F_L \to M_L$. Then also $f' \in \mathrm{Hom}_{W(L)}(W(L), M)$ (by 2.2). Since the above criterion of injectivity is known to hold for associative modules [2, IV], this shows that $M_{W(L)}$ is injective and completes the proof.

For an arbitrary Lie ring L we have considered the functor: $M_L \to M_{W(L)}$ from the category of Lie L-modules to the category of associative W(L)-modules and shown that M_L is L-injective if and only if $M_{W(L)}$ is W(L)-injective. To conclude, we consider the functor: $M_R \to M_{C(R)}$ from the category of associative R-modules (R an arbitrary associative ring) to the category of Lie C(R)-modules, and ask the analogous question: is M_R R-injective if and only if $M_{C(R)}$ is C(R)-injective? To answer this we first need a further definition and result.

An extension E_L of M_L is said to be essential if for any non-zero L-submodule D of E we have $D \cap M \neq 0$.

LEMMA 4.3. Every essential extension of a Lie module is contained (up to isomorphism) in any injective extension.

Proof. The proof follows readily from 2.3 (and the known result of the above lemma for the associative case). However, since a direct proof is simple, we give it here. Thus, let E_L be an essential extension of M_L, I_L an injective extension. Let j be the injection mapping of M_L into I_L. Then j can be extended to some L-homomorphism $j' : E_L \to I_L$. If K is the kernel of j', then $K \cap M = 0$ (since j is one-one), hence $K = 0$ (since E is essential). Thus j' is one-one and E is (up to isomorphism) a submodule of I.

Let us now consider the problem mentioned above in the following form: let R be an associative ring, M_R an arbitrary R-module, I_R its injective hull. Consider now M as a Lie module $M_{C(R)}$ over the Lie ring C(R) and take its injective hull $J_{C(R)}$, say. What is the relationship between $I_{C(R)}$ and $J_{C(R)}$?

Let $A_{C(R)}$ be a submodule of $I_{C(R)}$ such that $A_{C(R)} \cap M_{C(R)} = 0$. Since for any $a \in A$ and $r \in R$, $ar \in A$ and I is an R-module, it follows that A is an R-submodule of I_R. Since also $A_R \cap M_R = 0$ and I_R is an essential extension of M_R (it is well known [2, IV] that the injective hull is an essential extension), it follows that $A_R = 0$. That is, $I_{C(R)}$ is an essential extension of $M_{C(R)}$, and hence by 4.3, $I_{C(R)}$ may be considered as a submodule of $J_{C(R)}$.

We now show that in general $I_{C(R)} \neq J_{C(R)}$.*

Thus, let $R = Z$, the ring of integers, and $M = Q$, the additive group of rational numbers made into a Z-module in the obvious way. It is well known that Q is a divisible group, and hence Q_Z is injective as a Z-module [2, IV]. Thus

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* For the example that follows, as well as for other helpful comments, I wish to thank Professor B. Banaschewski.

$I_R = Q_Z$ and $I_{C(R)} = Q_{C(Z)}$. It will follow that $I_{C(R)} \neq J_{C(R)}$ if we show that $Q_{C(Z)}$ is not injective (as a Lie module). By the remark following 4.1 this is so if and only if $Q_{W(C(Z))}$ is not injective (as an associative module). We shall show that this is in fact the case, by showing that $Q_{W(C(Z))}$ possesses a proper essential extension (this will do it, since an injective extension has no proper essential extensions [2, IV]).

First, we determine the u.e.r. of $C(Z)$. Let $Z[x]$ be the polynomial ring over Z in one indeterminate x, and define a mapping $\mu: Z \rightarrow Z[x]$ by $\mu(z) = zx$ $(z \in Z)$. It then follows that μ is a Lie ring homomorphism: $C(Z) \rightarrow C(Z[x])$. We show that $(Z[x], \mu)$ is the u.e.r. of $C(Z)$. Thus, let H be an arbitrary associative ring and $f: C(Z) \rightarrow C(H)$ a Lie ring homomorphism. Define a mapping $g: Z[x] \rightarrow H$ by setting $g(x) = f(1)$ (1 being the identity element of Z). Since $Z[x]$ is the free associative ring generated by x, g can be extended uniquely to a ring homomorphism: $Z[x] \rightarrow H$. Then, for $z \in Z$, $g\mu(z) = g(zx) = zg(x) = zf(1) = f(z)$. Hence $g\mu = f$. If there exists a homomorphism $g': Z[x] \rightarrow H$ such that $g'\mu = f$, then $g'\mu(1) = f(1)$. That is, $g'(x) = f(1)$. Hence g' and g coincide on the generator x of the free ring $Z[x]$ and must therefore be identical. This shows that $Z[x] = W(C(Z))$.

By the isomorphism theorem 2.3 we now have the correspondence $Q_{C(Z)} \rightarrow Q_{Z[x]}$. If $q \in Q$, $z \in Z$, define a multiplication $Q \times Z[x] \rightarrow Q$ by setting $q(zx^i) = qz$ $(i \geq 0$ with $x^o = 1)$, extending it to all of $Z[x]$ in the obvious way. This makes Q into a $Z[x]$-module. In fact, this is the module $Q_{Z[x]}$ (corresponding to $Q_{C(Z)}$) since the restriction of the above multiplication to $Q \times C(Z)$ gives the module structure of $Q_{C(Z)}$.

Consider now the polynomial ring $Q[y]$ in the indeterminate y. On the additive group of this ring define a $Z[x]$-module structure as follows: for $q \in Q$, $z \in Z$, put $(qy^k)(zx) = qz(y^k + y^{k-1})$, $(k \geq 0$, with $y^o = 1$, $y^{-1} = 0)$, and define inductively $(qy^k)(zx^i) = [(qy^k)(zx^{i-1})]x$, $(i \geq 2)$.

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Extending by linearity to all of $Q[y]$ and $Z[x]$, one easily verifies that $Q[y]$ becomes a $Z[x]$-module; in fact $Q[y]$ is a proper extension of the module $Q_{Z[x]}$.

It remains to show that $Q[y]_{Z[x]}$ is an essential extension of $Q_{Z[x]}$. Thus, let $K_{Z[x]}$ be a non-zero D-submodule of $Q[y]_{Z[x]}$. Let $0 \neq q \in K$. Then we may take

$q = \sum_{k=0}^{n} q_k y^k$, where $q_k \in Q$ and $q_n \neq 0$. Since

$(x-1)^n \in Z[x]$, hence $q(x-1)^n \in K$. But

$q(x-1)^n = (\sum_{k=0}^{n} q_k y^k)(x-1)^n = q_n$ (this may be easily verified by induction on n), hence $K \cap Q \neq 0$. This completes the proof.

Thus, we have shown that if M_R is an R-module (R an arbitrary associative ring), then the injectivity of M_R (as an associative module) does not imply the injectivity of $M_{C(R)}$ (as a Lie module). However, $M_{C(R)}$ injective implies M_R injective.

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