THE TERM AND STOCHASTIC RANKS OF A MATRIX

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1. Introduction. The *term rank* ρ of a matrix is the order of the largest minor which has a non-zero term in the expansion of its determinant. In a recent paper **(1)**, the authors made the following conjecture. If S is the sum of all the entries in a square matrix of non-negative real numbers and if M is the maximum row or column sum, then the term rank ρ of the matrix is greater than or equal to the least integer which is greater than or equal to S/M. A generalization of this conjecture is proved in § 2.

The term *doubly stochastic* has been used to describe a matrix of non-negative entries in which the row and column sums are all equal to one. In this paper, by a doubly stochastic matrix, the authors mean a matrix of non-negative entries in which the row and column sums are all equal to the same real number T. If an $n \times n$ matrix A is embedded by the addition to A of r rows and columns in an $(n + r) \times (n + r)$ matrix B with row and column sums equal to T, we say that B is an (r, T) *doubly stochastic* (*abbreviated as* (r, T) *d.s.*) *extension of* A. In **(1)**, the authors made use of a d.s. extension of a matrix A to obtain an estimate of the term rank of A. In this paper, the authors describe all such extensions and give a necessary and sufficient condition that a matrix B be a vertex matrix of the convex set of all (r, T) d.s. extensions of A.

For a square matrix of non-negative entries, the concept of *stochastic rank* is introduced. Some results concerning this rank are obtained and the connection between it and term rank is noted.

In the final section, the problem of finding all d.s. extensions of a matrix A is formulated as a linear programming problem.

2. A lower bound for term rank. Let I and J be arbitrary sets and let $f(i, j)$, $i \in I, j \in J$, be a real-valued non-negative function on $I \times J$ which is not identically zero. The concept of term rank can be extended to such a function $f(i, j)$ as follows. A finite set of pairs (i_1, j_1), $(i_2, j_2), \ldots, (i_r, j_r)$ is *disjoint* if $i_p = i_q$ only if $p = q$ and if $j_p = j_q$ only if $p = q$. A function $f(i, j)$ has term rank ρ if, and only if, there exists a disjoint set of pairs (i_1, j_1), $(i_2, j_2), \ldots, (i_\rho, j_\rho)$ such that $f(i_r, j_r) > 0$ for $r = 1, 2, \ldots, \rho$ but for any disjoint set consisting of $\rho + 1$ pairs, $f(i, j) = 0$ for at least one pair of the set. If no such maximal disjoint set exists, the term rank is infinite.

Let σ be the collection of finite subsets of I and τ the collection of finite subsets of J. In this setting, we have the following theorem.

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THEOREM 1. *If $f(i, j)$ satisfies the conditions*

(i) $$R_i = \sup_{B \epsilon \tau} \Big[\sum_{j \epsilon B} f(i, j) \Big] \quad \textit{is finite for all } i \in I$$

and

(ii) $$C_j = \sup_{A \epsilon \sigma} \Big[\sum_{i \epsilon A} f(i, j) \Big] \quad \textit{is finite for all } j \in J$$

then either the term rank ρ of the function $f(i, j)$ is infinite or

$$S = \sup_{\substack{A \epsilon \sigma \\ B \epsilon \tau}} \Big[\sum_{i \epsilon A} \sum_{J \epsilon B} f(i, j) \Big] \textit{ and } M = \sup_{\substack{i \epsilon I \\ j \epsilon J}} [R_i, C_j]$$

are finite and ρ is greater than or equal to the least integer which is greater than or equal to S/M.

Proof. Let K be the graph of which the edges are the pairs (i, j) for which $f(i, j) > 0$. The vertex sets of this (bipartite) graph K are I and J. If ρ is finite, the exterior dimension (see **(3)**) of K is equal to ρ. If $[P, Q]$ is a minimal exterior pair for K and if U, V is any pair of finite subsets of I and J, then, since $f(i, j) = 0$ for $i \in \bar{P}$ and $j \in \bar{Q}$, we have

$$\begin{aligned}
\sum_{i \epsilon U} \sum_{j \epsilon V} f(i, j) &= \sum_{i \epsilon U \cap P} \sum_{j \epsilon V \cap Q} f(i, j) \\
&\quad + \sum_{i \epsilon U \cap \bar{P}} \sum_{j \epsilon V \cap Q} f(i, j) + \sum_{i \epsilon U \cap P} \sum_{j \epsilon V \cap \bar{Q}} f(i, j) \\
&\leqslant \sum_{i \epsilon U} \sum_{j \epsilon V \cap Q} f(i, j) + \sum_{i \epsilon U \cap P} \sum_{j \epsilon V} f(i, j) \\
&\leqslant \sum_{i \epsilon P} R_i + \sum_{j \epsilon Q} C_j
\end{aligned}$$

which is finite and independent of U and V.

Now

$$S = \sup_{\substack{U \epsilon \sigma \\ V \epsilon \tau}} \sum_{i \epsilon U} \sum_{j \epsilon V} f(i, j) \leqslant \sum_{i \epsilon P} R_i + \sum_{j \epsilon Q} C_j$$

so that S is finite. Further

$$\begin{aligned}
R_i &= \sup_{V \epsilon \sigma} \Big[\sum_{J \epsilon V} f(i, j) \Big] \\
&\leqslant \sup_{\substack{U \epsilon \sigma \\ V \epsilon \tau}} \Big[\sum_{i \epsilon U} \sum_{j \epsilon V} f(i, j) \Big] = S,
\end{aligned}$$

for all i. Similarly $C_j \leqslant S$ for all j. Thus,

$$M = \sup_{\substack{i \epsilon I \\ j \epsilon J}} [R_i, C_j] \leqslant S \quad \text{so that } M \text{ is finite.}$$

Now, let t be the unique integer such that $t - 1 < S/M \leqslant t$. We must show $\rho \geqslant t$. If $\rho < t$ then, since ρ is integral, we have $\rho \leqslant t - 1$. If $[P, Q]$ is a minimal exterior pair for K and $\nu(P)$ denotes the number of elements in P then $\rho = \nu(P) + \nu(Q)$ (**3**, Theorem 2). It follows that

$$\rho M = (\nu(P) + \nu(Q))M \geqslant \sum_{i \epsilon P} R_i + \sum_{j \epsilon Q} C_j \geqslant S.$$

Thus $S/M \leqslant \rho \leqslant t - 1$, a contradiction.

If the sets I and J in Theorem 1 are finite sets of orders n and m, ρ becomes the term rank of an $n \times m$ matrix a_{ij} in which $a_{ij} = f(i, j)$, M becomes the maximum row or column sum and S is the sum of all the entries in the matrix. If, in addition, $n = m$, Theorem 1 reduces to the conjecture in (**1**) referred to in the Introduction.

3. The stochastic rank of a matrix. Let A be an $n \times n$ matrix with non-negative entries a_{ij}. If M is the maximum row or column sum in A, then, for every $T \geqslant M$, and for every integer $r \geqslant n$, there exists a matrix B which is an (r, T) d.s. extension of A. In fact, if

$$R_i = \sum_{j=1}^{n} a_{ij}$$

and

$$C_j = \sum_{i=1}^{n} a_{ij}$$

for $i, j = 1, 2, \ldots, n$, the matrix $B = (b_{ij})$ may be defined as follows

$$b_{ij} = a_{ij} \text{ for } i \leqslant n,\ j \leqslant n,$$

$$\begin{aligned}
b_{ij} &= a_{2n+1-j,\, 2n+1-i}, && \text{for } n + 1 \leqslant i \leqslant 2n, \quad n + 1 \leqslant j \leqslant 2n. \\
b_{ij} &= 0 && \text{for } i \leqslant n, n + 1 \leqslant j \leqslant 2n \text{ provided } i + j \neq 2n + 1, \\
&= T - R_i; && \text{for } i + j = 2n + 1,\ i \leqslant n, \\
b_{ij} &= 0 && \text{for } n + 1 \leqslant i \leqslant 2n,\ j \leqslant n \text{ provided } i + j \neq 2n + 1, \\
&= T - C_j && \text{for } i + j = 2n + 1,\ j \leqslant n, \\
b_{ij} &= 0 && \text{for } 2n + 1 \leqslant i \leqslant n + r, \text{or } 2n + 1 \leqslant j \leqslant n + r, \text{provided } i \neq j \\
&= T && \text{for } 2n + 1 \leqslant i = j \leqslant n + r.
\end{aligned}$$

The question naturally arises, for what $r \leqslant n - 1$ and $T \geqslant M$ is an (r, T) d.s. extension of A possible? In Theorem 2, we have a complete answer to this question. Its proof will make use of the following lemma.

LEMMA 1. *Let B be an $m \times m$ doubly stochastic matrix with row and column sums equal to T. Let A be a $u \times v$ submatrix of B and let B be partitioned into submatrices A, A_1, A_2, A_3 as in Figure 1. If S is the sum of all the elements in A and S_2 is the sum of all the elements in A_2, then*

$$S - S_2 = (u + v - m)T.$$

Proof. Let S_1 and S_3 be the sums of the elements in A_1 and A_3 respectively. We have

$$\begin{aligned} S + S_1 &= uT \\ S + S_3 &= vT \\ S + S_1 + S_2 + S_3 &= mT, \end{aligned}$$

from which the result follows.

THEOREM 2. *Let $A = (a_{ij})$ be an $n \times n$ matrix of non-negative real numbers and let M be the maximum row or column sum and S the sum of all the entries. If $r \leqslant n - 1$, the necessary and sufficient condition that there exist a matrix B which is an (r, T) d.s. extension of A is that $M \leqslant T \leqslant S/(n - r)$.*

Proof. If B is an (r, T) d.s. extension of A, we apply Lemma 1 to B. We have $S - S_2 = (n - r)T$. Since $0 \leqslant S_2$, it follows that

$$M \leqslant T = \frac{S - S_2}{n - r} \leqslant \frac{S}{n - r}.$$

Clearly, $T = S/(n - r)$ if, and only if, $S_2 = 0$ and $T = M$ if and only if $S_2 = S - (n - r)M$. To show the possibility of such extreme d.s. extensions we construct the appropriate matrices. We first construct a matrix $C = (c_{ij})$ which is an $(r, S/(n - r))$ d.s. extension of A. Let

$$c_{ij} = a_{ij} \qquad \text{for } i \leqslant n,\ j \leqslant n,$$

$$c_{ij} = \frac{\dfrac{S}{n - r} - R_i}{r} \qquad \text{for } i \leqslant n, n + 1 \leqslant j \leqslant n + r,$$

$$c_{ij} = \frac{\dfrac{S}{n - r} - C_j}{r} \qquad \text{for } n + 1 \leqslant i \leqslant n + r, j \leqslant n,$$

$$c_{ij} = 0 \qquad \text{for } n + 1 \leqslant i \leqslant n + r,\ n + 1 \leqslant j \leqslant n + r.$$

We next construct a matrix $D = (d_{ij})$ which is an (r, M) d.s. extension of A. Let

$$d_{ij} = a_{ij} \qquad \text{for } i \leqslant n, j \leqslant n,$$

$$d_{ij} = \frac{M - R_i}{r} \qquad \text{for } i \leqslant n, n + 1 \leqslant j \leqslant n + r,$$

$$d_{ij} = \frac{M - C_j}{r} \qquad \text{for } n + 1 \leqslant i \leqslant n + r, j \leqslant n,$$

$$d_{ij} = \frac{S - M\,(n - r)}{r^2} \qquad \text{for } n + 1 \leqslant i \leqslant n + r \text{ and } n + 1 \leqslant j \leqslant n + r.$$

For any T, $M \leqslant T \leqslant S/(n - r)$, let p be the unique real number $0 \leqslant p \leqslant 1$ defined by $pS/(n - r) + (1 - p)M = T$. The matrix $B = pC + (1 - p)D$ is an (r, T) d.s. extension of A.

We now define *stochastic rank*. An $n \times n$ matrix A with non-negative entries has stochastic rank σ if A can be embedded in a d.s. matrix B formed from A by the addition of $n - \sigma$ rows and columns and if A cannot be embedded in a d.s. matrix B by the addition of fewer than $n - \sigma$ rows and columns. By Theorem 2, the least r for which A can be embedded in an $(n + r) \times (n + r)$ d.s. matrix B is the minimum r for which $M/S \leqslant 1/(n - r)$. This minimum r is $n - [S/M]$. It follows that $\sigma = [S/M]$.

An *$n \times n$ sub-permutation matrix of rank r* is an $n \times n$ matrix consisting of r ones, no two of which are in the same row or column, and $n^2 - r$ zeros. The convex hull of the sub-permutation matrices $P_k^{(r)}$ of rank r consists of all matrices A expressible in the form $A = \sum_k \lambda_k P_k^{(r)}$ where $\sum_k \lambda_k = 1$ and $\lambda_k \geqslant 0$ for all k. The convex polyhedral cone generated by the sub-permutation matrices $P_k^{(r)}$ of rank r consists of all matrices A expressible in the form $A = \sum_k \mu_k P_k^{(r)}$ where $\mu_k \geqslant 0$ for all k. In **(2)**, the authors showed that the necessary and sufficient condition that a matrix A of non-negative entries is in the convex hull of sub-permutation matrices of rank $n - r$ is that $S = n - r$ and $M \leqslant 1$. A simple restatement of this theorem is that the necessary and sufficient condition that a matrix of non-negative entries is in the convex polyhedral cone generated by the sub-permutation matrices of rank $n - r$ is that $M/S \leqslant 1/(n - r)$. Hence, the maximum rank $n - r$ satisfying this inequality is $[S/M]$ and this is equal to the stochastic rank σ of A. Thus, we have the following corollary to Theorem 2.

COROLLARY. *The stochastic rank of an $n \times n$ matrix A of non-negative entries is σ if, and only if, A is in the convex polyhedral cone of the $n \times m$ sub-permutation matrices of rank σ but is not in the convex polyhedral cone of the $n \times n$ sub-permutation matrices of rank $\sigma + 1$.*

4. Vertices of a set of doubly stochastic extensions. If we consider each (r, T) d.s. extension of a matrix A as a point in a space of dimension $(n + r)^2$, it is apparent the set α of all such matrices is convex. An *extreme or vertex matrix* for the convex set α is an (r, T) d.s. extension of A which is not expressible in the form $pC + (1 - p)D$ in which C and D belong to α, $C \neq D$ and $0 < p < 1$.

We may define the bipartite graph K_A of an $n \times m$ matrix A of non-negative entries to be the graph in which the vertex sets are the set of indices of the n rows and m columns and the edges are the places of the matrix in which the entries are positive. A graph is *disjoint* if no two of its edges have a vertex in common. A graph K_1 is a subgraph of K_2 if every edge of K_1 is an edge of K_2.

A *cycle* in a bipartite graph K is a finite subgraph K^1 with the following properties. Let I and J be the vertex sets. If (i_1, j_1) is any edge of K^1 then there exists exactly one vertex $i_2 \in I$, $i_2 \neq i_1$, such that (i_2, j_1) is an edge of K^1, and there exists exactly one vertex $j_2 \in J$, $j_2 \neq j_1$, such that (i_2, j_2) is an edge of K^1, and there exists exactly one vertex $i_3 \in I$, $i_3 \neq i_2$, such that (i_3, j_2) is an edge of K^1, etc. If after $2k - 1$ such steps, $k \geqslant 2$, we find that

(i_1, j_1), (i_2, j_1), (i_2, j_2), ... , (i_k, j_k), (i_1, j_k) are distinct and are exactly the edges of K^1, then K^1 is a cycle. It follows that for a cycle K^1 in the bipartite graph of a matrix, there exists no row or column which contains exactly one edge of the cycle.

In **(1)** the core of an R and C marking of an incidence matrix consists of the union of a number of cycles no two of which have an edge in common.

In Theorem 3 we require the following lemma.

LEMMA 2. *For a bipartite graph K, a necessary and sufficient condition that there exist a subgraph of K which is a cycle is that there exist a finite subgraph of K in which no vertex of either vertex set is edge connected to exactly one vertex of the other vertex set.*

Proof. The necessity is immediate.

To establish the sufficiency, we show that any finite subgraph K^1 of K in which no vertex of either vertex set is edge connected to exactly one vertex of the other, contains a subgraph which is a cycle of K. Let the vertex sets of K be I and J. If (i_1, j_1) is an edge of K^1, $i_1 \in I$ and $j_1 \in J$, there exists $i_2 \neq i_1$, such that (i_2, j_1) is an edge of K^1. Similarly, there exists $j_2 \neq j_1$ such that (i_2, j_2) is an edge of K^1. Continuing this process, since K^1 is a finite graph, it follows that in the sequence (i_1, j_1), (i_2, j_1), $(i_2, j_2,)$ ... , there must exist a first edge E_1 in which either the i is identical with the i of a previous edge or the j is identical with the j of some previous edge. In either case, let this previous edge be E_0. The sequence of edges beginning with E_0 and ending with E_1 is a cycle.

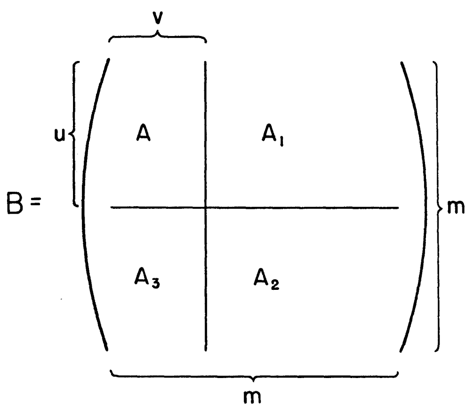

FIGURE 1

Now, consider any (r, T) d.s. extension B of a matrix A of non-negative elements and let B be partitioned into submatrices A, A_1, A_2, and A_3 as in Figure 1. Let

$$K_{A_1},\ K_{A_2},\ K_{A_3},$$

be the bipartite graphs of A_1, A_2, and A_3 and let L_B be the union of

$$K_{A1},\ K_{A2},\ K_{A3},$$

so that K_B is the union of K_A and L_B. We are now in a position to state the main theorem of this section.

THEOREM 3. *Let α be the convex set of all (r, T) d.s. extensions of a matrix A. A necessary and sufficient condition that a matrix $B \in \alpha$ be a vertex matrix of the convex set α is that no subgraph of L_B is a cycle.*

Proof. If a subgraph $L_B{}^1$ of L_B is a cycle, let the edges of the cycle be (i_1, j_1), (i_2, j_1), (i_2, j_2), ..., (i_k, j_k), (i_1, j_k).

Let $\epsilon = \frac{1}{2} \min b_{ij}$ taken over all edges (i, j) of $L_B{}^1$. Now, if $C = (c_{ij})$ is defined by

$$\begin{aligned} c_{ij} &= b_{ij} && \text{if } (i,j) \text{ is not an edge of } L_B{}^1, \\ c_{ij} &= b_{ij} + \epsilon && \text{if } (i,j) \text{ is } (i_1, j_1), (i_2, j_2), \ldots, \text{or } (i_k, j_k), \\ c_{ij} &= b_{ij} - \epsilon && \text{if } (i,j) \text{ is } (i_2, j_1), (i_3, j_2), \ldots, (i_1, j_k), \end{aligned}$$

and if $D = (d_{ij})$ is defined by

$$\begin{aligned} d_{ij} &= b_{ij} && \text{if } (i,j) \text{ is not an edge of } L_B{}^1 \\ &= b_{ij} - \epsilon && \text{if } (i,j) \text{ is } (i_1, j_1), (i_2, j_2), \ldots, \text{or } (i_k, j_k) \\ &= b_{ij} + \epsilon && \text{if } (i,j) \text{ is } (i_2, j_1), (i_3, j_2), \ldots, (i_1, j_k), \end{aligned}$$

clearly C and D belong to α. Since $B = \frac{1}{2}C + \frac{1}{2}D$, B is not a vertex matrix of the set α.

We now show that if B and C are (r, T) d.s. extensions of A such that $B \neq C$ and $K_B = K_c$ then L_B $(= L_C)$ has a subgraph $L_B{}^1$ which is a cycle. Indeed, let $L_B{}^*$ be the subgraph consisting of the edges (i, j) at which $0 < b_{ij}$, $0 < c_{ij}$ and $c_{ij} \neq b_{ij}$. Since $b_{ij} = c_{ij}$ for all (i, j) in K_A, $L_B{}^*$ is a subgraph of L_B and since $B \neq C$, $L_B{}^*$ has at least one edge. Since the matrices B and C are doubly stochastic with row and column sums equal to T, they cannot differ at exactly one place in a row or column. By Lemma 2, L_B (in fact $L_B{}^*$) contains a subgraph $L_B{}^1$ which is a cycle.

Next, suppose that $B \in \alpha$ is not a vertex, so that B is expressible in the form $B = pC + (1 - p)D$ where $0 < p < 1$, $C \neq D$ and C and D belong to α. Now L_C and L_D are subgraphs of L_B, but we cannot say $L_C = L_D = L_B$, for we might have $c_{ij} = 0$, $b_{ij} \neq 0$, and $d_{ij} \neq 0$. However, if $q \neq p$, $0 < q < 1$, then $E = qC + (1 - q)D$ belongs to α, $B \neq E$ and $K_B = K_E$. Hence L_B contains a subgraph $L_B{}^1$ which is a cycle. This completes the proof of Theorem 3.

COROLLARY. *Let α be the convex set of all (r, T) d.s. extensions of a matrix A. A necessary and sufficient condition that a matrix $B \in \alpha$ be a vertex matrix of the convex set α, is that there exist no matrix $C \in \alpha$ such that $B \neq C$ and $K_B = K_C$.*

LEMMA 3. *If P is an $r \times r$ matrix of non-negative elements with at least two non-zero elements in every row then the bipartite graph K_P contains a subgraph $K_P{}^1$ which is a cycle.*

Proof. Delete from P all the columns containing no non-zero elements and let the deleted $r \times s$ matrix ($s \leqslant r$) be Q. If there are at least two non-zero elements in every column of Q then the required cycle exists by Lemma 2. If a column contains exactly one non-zero element q_{ij}, delete the ith row and jth column of Q and denote the deleted matrix by Q_1. Continue this process. If we find Q_t such that every column of Q_t contains 2 non-zero elements, the cycle exists by Lemma 2. If no such Q_t exists for $t = 1, 2, \ldots, s - 3$, then, Q_{s-2} is an $(r - s + 2) \times 2$ matrix with two columns and with two non-zero elements in every row. Since $r - s + 2 \geqslant 2$ the graph of Q_{t-2} contains a cycle.

Let B be an (r, T) d.s. extension of A. Let the rows and columns of B be rearranged as in Figure 1. If in the ith row of B ($i = 1, 2, \ldots, n$) there is at most one $j > n$ such that the element $b_{ij} > 0$ then the ith row of A is *simply extended.* Similarly, if in the jth column of B there is at most one $i > n$ such that $b_{ij} > 0$, then the jth column of A is *simply* extended.

THEOREM 4. *If B is a vertex of the convex set α of all (r, T) d.s. extensions of A, then at least $n - r + 1$ of the rows and at least $n - r + 1$ of the columns of A are simply extended.*

Proof. Suppose that r rows of B are not simply extended. In each of these rows we have at least two elements

$$b_{ij_1} > 0 \qquad \text{and} \qquad b_{ij_2} > 0, \qquad j_1 > n, j_2 > n, j_1 \neq j_2.$$

Thus the $n \times r$ matrix A_1 (see Figure 1) contains an $r \times r$ submatrix A_1 in every row of which there are two non-zero elements.

Hence, by Lemma 3, the graph of

$$K_{A_1}$$

contains a subgraph which is a cycle and, by Theorem 3, B is not a vertex of α.

The proof when r columns of A are not simply extended is similar.

5. The connection between term rank and stochastic rank.

Since ρ is greater than or equal to ρ the least integer which is greater than or equal to S/M and since $\sigma = [S/M]$, we have the following result. If S/M is an integer, $\rho \geqslant \sigma$, and if S/M is not an integer, $\rho \geqslant \sigma + 1$.

For an $n \times n$ doubly stochastic matrix, $\rho = \sigma = n$, and for a sub-permutation matrix of rank r, $\rho = \sigma = r$. However, there are $n \times n$ matrices for which $\rho - \sigma = n - 1$. In fact, the matrix $A = (a_{ij})$ in which $a_{11} = n$, $a_{22} = a_{33} = \ldots = a_{nn} = 1$, $a_{ij} = 0$ for $i \neq j$ is such a matrix. We have $S/M = 2 - 1/n$. Thus $\sigma = 1$ and $\rho = n$.

For a matrix of zeros and ones, Ryser **(4;5)** has considered the transformation which replaces a minor

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The effect of this transformation is that the term rank varies between limits which Ryser finds. It is interesting to note that the stochastic rank of a matrix of zeros and ones is invariant under Ryser's transformation.

If $M < S/(n - r)$, then, since $\rho \geqslant \sigma \geqslant n - r$ there exist integers t such that $\rho \geqslant t \geqslant n - r$. We have the following theorem.

THEOREM 5. *Let A be an $n \times n$ matrix of non-negative elements. If $M < S/(n - r)$ and $M \leqslant T \leqslant S/(n - r)$ and if $K_A{}^1$ is any disjoint subgraph of K_A consisting of t edges ($\rho \geqslant t \geqslant n - r$) then there exists an (r, T) d.s. extension B of A with the property that the graph K_B contains a disjoint subgraph $K_B{}^1$ consisting of $n + r$ edges such that the edges common to $K_B{}^1$ and K_A are exactly those of $K_A{}^1$.*

Proof. If we select any p such that $0 < p < 1$, then since $M < S/(n - r)$, the matrix $B = pC + (1 - p)D$ of Theorem 2 is an (r, T) d.s. extension of A in which every element of A_1, A_2, and A_3 (Figure 1) is positive. Thus all the places of A_1, A_2, and A_3 are edges of K_B. Rearrange the rows and columns of B so that $K_A{}^1$ consists of the places $(1, 1)$ $(2, 2)$ $(3, 3)$ $\ldots$ (t, t). Now consider the disjoint graph L which has as its edges the places (i, j) of B defined by $i + j = n + t + r + 1$. Since $t \geqslant n - r$, we have $i + j \geqslant 2n + 1$ and hence every edge in L is a place in A_1, A_2, and A_3 and L is a subgraph of K_B. The number of edges in L is $n + r - t$. For an edge (i, j) of L we cannot have $i \leqslant t$, for this would imply $j \geqslant n + r + 1$ and similarly we cannot have $j \leqslant t$. Thus the edges of L and $K_A{}^1$ have no vertices in common. Clearly, the graph $K_B{}^1$ defined as the union of L and $K_A{}^1$ is the required disjoint subgraph of K_B.

Let K be a bipartite graph whose edges are a set of places in an $n \times n$ array and let A be a matrix formed by putting positive entries in the places of K and zeros elsewhere. For a given graph K, the term rank ρ of all such matrices A is the same and is equal to the exterior dimension **(3)** of the graph. Thus, term rank is really a graphical concept. On the other hand, for a given graph K, the stochastic rank σ of such matrices A will vary between 1 and an attainable maximum which we denote by σ_K. We now show that $\sigma_K \leqslant \rho \leqslant \sigma_K + 1$. The inequality on the left is a consequence of Theorems 1 and 2. To establish the inequality on the right, consider the matrix A formed by placing 1 in each of the ρ places of a maximal disjoint subgraph of K and ϵ in the other places of K. If a is the maximum number of places of K in any row or column of the $n \times n$ array and if b is the number of places in K, then

$$\frac{S}{M} = \frac{\rho + (b - \rho)\epsilon}{1 + (a - 1)\epsilon}.$$

If $a = 1$, then $b = \rho$ and $\sigma_K = \rho$. In other cases, ϵ can be chosen small enough that $\sigma = [S/M] \geqslant \rho - 1$. Hence, $\sigma_K \geqslant \sigma \geqslant \rho - 1$ or $\rho \leqslant \sigma_K + 1$. The inequality $\sigma_K \leqslant \rho \leqslant \sigma_K + 1$ is best possible in the sense that there exist graphs K for which $\sigma_K = \rho$ and others for which $\rho = \sigma_K + 1$. The graph K consisting of the places on a main diagonal in an $n \times n$ array is a graph in which $\sigma_K = \rho$. The graph K consisting of 3 of the 4 places in a 2×2 array is such that $\rho = 2$. But any matrix A with non-zero elements in the places of K and a zero in the fourth place of the array lies in the convex polyhedral cone of sub-permutation matrices of rank 1 and does not lie in the convex polyhedral cone of sub-permutation matrices of rank 2. Hence $\sigma_K = 1$. Consider a graph K for which the maximum σ_K is attained in a matrix A in which S/m is non integral. We have $\rho \leqslant \sigma_K + 1$ and $\rho \geqslant \sigma_K + 1$ so that $\rho = \sigma_K + 1$.

The result just proved may be reformulated as the following theorem.

THEOREM 6. *Let K be a bipartite graph whose edges are the places in an $n \times n$ array. Let α be the set of all matrices A with positive entries in the places of K and zeros elsewhere. Let σ_K be the maximum stochastic rank attainable by a matrix of the set α. Then every matrix A of α has the same term rank ρ. Furthermore, if S_A and M_A represent the entry sum and maximal row or column sum of A respectively then*

$$\rho = \sup_{A \epsilon \alpha} \left(\frac{S_A}{M_A} \right).$$

Also if this supremum is attained by some matrix A, then $\rho = \sigma_K$, otherwise $\rho = \sigma_K + 1$.

6. Linear programming formulation. Some of the theorems concerning (r, T) d.s. extensions of an $n \times n$ matrix A may be reformulated as problems in the language of linear programming. In these reformulations the restrictions on A to non-negative entries may be relaxed somewhat. The only requirement is that A satisfy the condition $S \geqslant (n - r)M \geqslant 0$. Two such formulations follow.

PROBLEM 1. *Let A be an $n \times n$ matrix having $S \geqslant (n - r)M \geqslant 0$, and let T be any number. Find a set of numbers χ_{ij} ($i = 1, 2, \ldots, n + r; j = 1, 2, \ldots, n + r$; at least one of i and j is greater than n), subject to the following conditions.*

$$(1) \qquad \chi_{ij} \geqslant 0 \qquad \text{for all } i, j.$$

$$(2) \qquad \sum_{j=1}^{n} a_{ij} + \sum_{j=n+1}^{n+r} \chi_{ij} = T \qquad \text{for } i = 1, 2, \ldots, n.$$

$$(3) \qquad \sum_{j=1}^{n+r} \chi_{ij} = T \qquad \text{for } i = n + 1, n + 2, \ldots, n + r.$$

$$(4) \qquad \sum_{i=1}^{n} a_{ij} + \sum_{i=n+1}^{n+r} \chi_{ij} = T \qquad \text{for } j = 1, 2, \ldots, n.$$

$$(5) \qquad \sum_{i=1}^{n+r} \chi_{ij} = T \qquad \text{for } j = n + 1, n + 2, \ldots, n + r.$$

Theorem 2 states that the inequalities have solutions if and only if $M \leqslant T \leqslant S/(n-r)$ and exhibits some of these solutions. If now each set of values of χ_{ij} satisfying (1), (2), ..., (5) is considered as a point in a space of $(n+r)^2 - n^2$ dimensions, the set of all such points is convex and Theorem 3 gives a graphical characterization of the vertices of this set.

PROBLEM 2. *Let A be an $n \times n$ matrix having $S \geqslant (n-r)M \geqslant 0$. Find a set of numbers χ_{ij} $(i = 1, 2, \ldots, n+r;\ j = 1, 2, \ldots, n+r$; at least one of i and j is greater than n), subject to the following conditions:*

(1) $$\chi_{ij} \geqslant 0 \qquad \text{for all } i, j.$$

(2) $$\sum_{j=1}^{n} a_{ij} + \sum_{j=n+1}^{n+r} \chi_{ij} = \sum_{j=1}^{n+r} \chi_{n+r,j} \qquad \text{for } i = 1, 2, \ldots, n.$$

(3) $$\sum_{j=1}^{n+r} \chi_{ij} = \sum_{j=1}^{n+r} \chi_{n+r,j} \qquad \text{for } i = n+1, n+2, \ldots, n+r-1.$$

(4) $$\sum_{i=1}^{n} a_{ij} + \sum_{i=n+1}^{n+r} \chi_{ij} = \sum_{i=1}^{n+r} \chi_{i,n+r} \qquad \text{for } j = 1, 2, \ldots, n.$$

(5) $$\sum_{i=1}^{n+r} \chi_{ij} = \sum_{i=1}^{n+r} \chi_{i,n+r} \qquad \text{for } j = n+1, n+2, \ldots, n+r-1.$$

The sum

$$\sum_{i=1}^{n+r} \chi_{i,n+r}$$

is to be maximized or minimized.

In this formulation our theories state that feasible solutions always exist for both the maximum and minimum problems. They also exhibit solutions at which the maximum and minimum are attained and state that the maximum value is $S/(n-r)$ and the minimum value is M. Our graphical theorems characterize the sets of all maximal and minimal solutions.

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