Inhomogeneous Helmholtz equations in wave guides – existence and uniqueness results with energy methods

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The Helmholtz equation $-\nabla \cdot (a\nabla u) - \omega^2 u = f$ is considered in an unbounded wave guide $\Omega := \mathbb{R} \times S \subset \mathbb{R}^d$, $S \subset \mathbb{R}^{d-1}$ a bounded domain. The coefficient a is strictly elliptic and either periodic in the unbounded direction $x_1 \in \mathbb{R}$ or periodic outside a compact subset; in the latter case, two different periodic media can be used in the two unbounded directions. For non-singular frequencies ω, we show the existence of a solution u. While previous proofs of such results were based on analyticity arguments within operator theory, here, only energy methods are used.

Keywords: Helmholtz equation, wave guide, periodic media, Fredholm alternative

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1 Introduction

We investigate the existence and uniqueness of solutions to the Helmholtz equation

$$-\nabla \cdot (a\nabla u) - \omega^2 u = f \tag{1.1}$$

in an infinite wave guide $\Omega := \mathbb{R} \times S$. The cross-section S is given by a bounded Lipschitz domain $S \subset \mathbb{R}^{d-1}$, the right hand side $f \in H^{-1}(\Omega)$ has compact support; below, the frequency $\omega > 0$ is assumed to be non-singular. The differential operator $Au := -\nabla \cdot (a\nabla u)$ is given by coefficients $a : \Omega \to \mathbb{R}^{d \times d}$ of class $L^\infty(\Omega)$ with $a(x)$ symmetric and positive for every x, satisfying $\lambda |\xi|^2 \leq \xi \cdot a(x)\xi \leq \Lambda |\xi|^2$ for some $0 < \lambda < \Lambda < \infty$ and all $\xi \in \mathbb{R}^d$, $x \in \Omega$. We treat two settings: Periodic coefficients, that is, coefficients a that satisfy $a(x + e_1) = a(x)$ for every $x \in \Omega$, see Figure 1. The other setting is that of coefficients a that are periodic at the far left and at the far right and arbitrary in a central region: There exists $R_0 > 0$ such that $a(x + e_1) = a(x)$ for every $x \in \Omega$ with $|x_1| > R_0$, see Figure 2. We impose a Neumann condition on $\partial\Omega$; Dirichlet conditions can be treated in the same way.

We say that a function $u \in H^1_{\text{loc}}(\Omega)$ solves the *radiation problem* if the following three conditions are met:

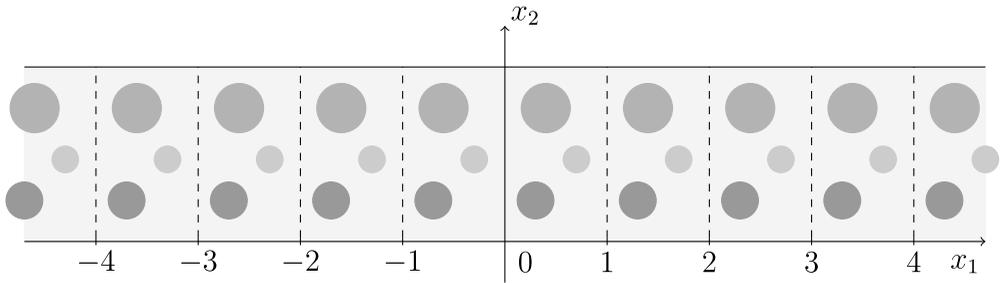(i) u solves (1.1) in Ω in the weak sense.

FIGURE 1. The wave guide geometry in two dimensions. The coefficient *a* is indicated by different levels of grey. It is 1-periodic in x_1-direction.

(ii) $\sup_{r \in \mathbb{Z}} \|u\|_{L^2((r,r+1) \times S)} < \infty$.

(iii) the radiation condition of Definition 2.4 is satisfied.

Regarding the radiation condition, we note that Definition 2.4 is equivalent to a more standard condition, see Lemma 3.2. One of our main results is the following existence and uniqueness statement.

Theorem 1.1 (Existence and uniqueness result for periodic media). *Let the data* Ω, *f,* ω, *and a be as above, the coefficients* 1*–periodic in* x_1, *that is:* $a(x + e_1) = a(x)$ *for every* $x \in \Omega$. *Let* ω *be non-singular in the sense of Definition 2.3 below. Then, there exists one and only one solution u to the radiation problem (i)–(iii).*

The statement of Theorem 1.1 is not new, but contained, for example, in [9] (Theorem 1.2 below is the new result of this article). The decisive difference between existing literature and the paper at hand regards the methods of proof. The proof in [9] uses operator theory (just as the proofs of similar results in [10 13, 20]): One constructs families of operators in subsets of the complex plane, sketches specific curves in the complex plane and evaluates corresponding line integrals of operators. The constructions provide bounded families of operators and thus, as a result, an inverse to the Helmholtz operator. The proofs rely on analyticity properties and exploit Kato's perturbation theory for operators.

By contrast, our proof uses only energy methods and is self-contained. Using only energy methods means here that (a) we only use L^2-based function spaces, (b) the existence result is obtained from a priori estimates, (c) the a priori estimates are obtained with the help of appropriately chosen test functions for the equations. Energy conservation of the physical system is reflected by the mathematical fact that the energy flux is independent of the position, see Remark 2.2. The flux equality in the form of Lemmas 2.1 and A.1 is the central tool in the proofs of the main results, Theorems 1.1 and 1.2. This article is self-contained in the sense that the proofs of the two main theorems use only standard theory of partial differential equations. The further analysis of the non-singularity assumption for the frequency ω is not the aim of this contribution.

Our results have the character of a Fredholm alternative. The assumption that the frequency ω is non-singular implies that the homogeneous problem has only the trivial solution. From this uniqueness property, we obtain the existence result. In order to obtain the existence, we introduce an approximate problem which is easy to solve. If the approximate solutions are bounded,
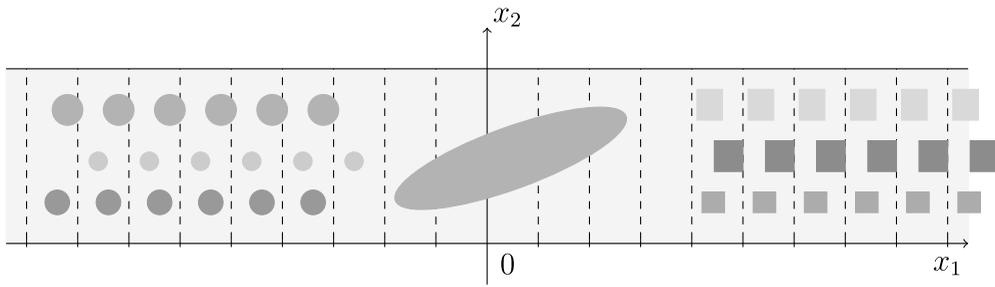
FIGURE 2. A non-periodic coefficient a as in Theorem 1.2. The coefficient is periodic as $x_1 \to \infty$ and as $x_1 \to -\infty$. The medium satisfies $a(x + e_1) = a(x)$ for every $x \in \Omega$ with $|x_1| > R_0$ for $R_0 = 5$. The two periodic media (far left and far right) can be different.

then any limit is the desired solution to the original problem. If the approximate solutions are unbounded, we normalise them and obtain, in the limit, a nontrivial solution to the homogeneous problem – in contradiction to the uniqueness property.

From the above description of the proof, it is clear that the approach is very direct. The two difficulties are (1) the construction of a useful approximate problem and (2) the verification of the radiation condition for limits. Our choice is inspired by constructions of [5] and [18]. We work with truncated domains and radiation boxes to formulate boundary conditions. We demand that approximate solutions look like outgoing waves in the radiation boxes. The proofs rely on the flux equality for solutions: In every cross-section of the wave guide, the solution has the same energy flux.

Our methods are very flexible and provide also new results. The next theorem treats a medium that is arbitrary on a compact subdomain and periodic outside of that subdomain. Our assumption on the medium can also be formulated as follows: There are two periodic fields $a^{\text{left}}, a^{\text{right}} : \Omega \to \mathbb{R}^{d \times d}$, $a^{\text{left}}(x + e_1) = a^{\text{left}}(x)$ and $a^{\text{right}}(x + e_1) = a^{\text{right}}(x)$ for every $x \in \Omega$. The coefficient a is of class $L^\infty(\Omega)$, and it is pointwise symmetric and positive and has the ellipticity bounds $\Lambda > \lambda > 0$. It satisfies, for some $R_0 > 0$:

$$a(x) = a^{\text{left}}(x) \qquad \text{if } x_1 < -R_0 \,,$$
$$a(x) = a^{\text{right}}(x) \qquad \text{if } x_1 > R_0 \,.$$

For such a medium, even if ω is non-singular for a^{left} and a^{right}, the number ω^2 can be an eigenvalue of the elliptic operator A. We therefore have to assume the uniqueness property for the homogeneous problem.

Theorem 1.2 (Media that are periodic at infinity). *Let the wave guide $\Omega = \mathbb{R} \times S$ be as above. Let $a : \Omega \to \mathbb{R}^{d \times d}$ be essentially bounded, symmetric and positive as above. We assume that, for some $R_0 > 0$,*

$$a(x + e_1) = a(x) \quad \text{for every } x \in \Omega \text{ with } |x_1| > R_0 \,,$$

which implies that a coincides with a^{left} and a^{right} outside a compact region for some periodic coefficients a^{left} and a^{right}. Let $\omega > 0$ be a non-singular frequency in the sense of Definition 2.3 below for the two periodic media given by a^{left} and a^{right}. If the radiation problem (i)–(iii) with

f = 0 possesses only the trivial solution, then the radiation problem (i)–(iii) has a unique solution u for arbitrary $f \in H^{-1}(\Omega)$ *with compact support.*

We emphasise that the medium in the above theorem is not necessarily a compact perturbation of a periodic medium. Our proof works also when a compact perturbation of the underlying geometry $\Omega = \mathbb{R} \times S$ is considered, for example, when the channel is thicker in a central region. Moreover, the unbounded half-channels could have also different cross-sections or they could lead in different directions.

Regarding literature, we mention [15] for classical methods. We note that some of the references below treat the problem on the whole space $\mathbb{R}^d$; some results on that problem can be interpreted as statements on the wave guide problem with periodicity boundary condition in the bounded direction. In this sense, the line defect analysis of [11] provides a uniqueness result for a local perturbation of a periodic medium. In general, uniqueness does not hold in the situation of Theorem 1.2, see [1, 6, 7]. For another form of a radiation condition, we mention [13]. The work [4] treats a similar problem and makes a connection to a Lippmann-Schwinger equation; uniqueness is obtained there from a positive absorption parameter. An interface with a metamaterial is considered in [2].

Regarding the spectral properties of the Helmholtz operator, we mention [19]; Theorem 3.2 in Chapter 5 of that reference provides in certain settings that all frequencies $\omega > 0$ are non-singular in the sense of Definition 2.3. The existence of Floquet modes is shown in [12].

We mention that the Fredholm alternative for a limiting absorption principle was also exploited in [22] in order to improve the existence statement of [5] with a vanishing absorption principle. Similar methods are also used in [2]. The analysis of guided modes in a wave guide with purely harmonic dependence in the unbounded direction was treated in [3]. In the work [8], the solution to half-space problems is used for the computation of guided modes, which is further exploited in [14].

It is worth noting at this point that our results have the drawback that they do not include a limiting absorption principle for the unbounded domain. This fact is related to the method of proof since we construct approximate solutions with truncated problems and not with a small absorption parameter.

2 Preliminaries

In this section, we discuss various properties of the system and specify the setting for our results. We start with the conservation of fluxes. This is a fundamental property of the Helmholtz equation, and our existence result is built on it. We recall the concept of propagating modes and introduce the non-singularity assumption on ω, which allows also to introduce a useful radiation condition in Definition 2.4. We furthermore show some results on orthogonality and the equivalence of our radiation condition with a more standard formulation.

2.1 Conservation of fluxes and the form Q

During the entire approach, we will work with a number $l \in \mathbb{N}$ that gives the width of a so called 'radiation box'. The number is arbitrary, but fixed throughout this article; accordingly, we nowhere mark the dependence of l by a sub- or superscript. The reader might want to think
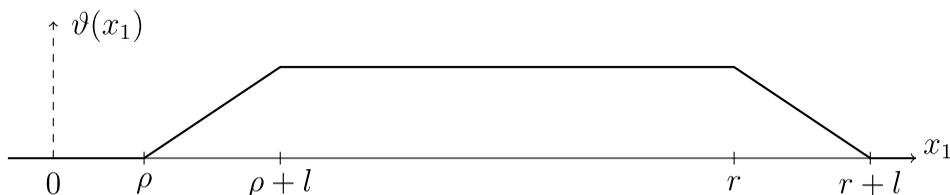
FIGURE 3. The cut-off function ϑ.

of $l = 1$ in the whole text. Given $l > 0$, we consider the central box $W_0 := (0, l) \times S$ and, for arbitrary $r \in \mathbb{Z}$, the shifted boxes $W_r := (r, r + l) \times S$.

We describe the situation first for a periodic medium a. We can identify a function $u : W_r \to \mathbb{C}$ with the function $\tilde{u} : W_0 \to \mathbb{C}$, which is obtained with a shift: $\tilde{u}(x) := u(x + re_1)$.

Of crucial importance in our approach will be the following sesquilinear form Q. For $u \in H^1(W_0)$ and $v \in L^2(W_0)$, we define

$$Q(u, v) := \frac{1}{l} \int_{W_0} a \nabla u \cdot e_1 \bar{v}, \quad \text{and} \quad \mathcal{Q}(u) := Q(u, u), \tag{2.1}$$

where the overbar denotes complex conjugation. The sesquilinear form Q and the quadratic form $\mathcal{Q}$ are used to measure the energy flux of solutions. We also consider an (anti-)symmetrised variant of Q,

$$Q^s(u, v) := \frac{1}{2} \left(Q(u, v) - \overline{Q(v, u)} \right). \tag{2.2}$$

The symmetrised variant satisfies $Q^s(u, v) = -\overline{Q^s(v, u)}$ and $Q^s(u, u) = i \operatorname{Im} \mathcal{Q}(u)$. We mention already here that a more standard description of the flux $\operatorname{Im} \mathcal{Q}(u)$ as a surface integral is given with formula (2.5) of Remark 2.2 below.

We will repeatedly use piecewise affine cut-off functions ϑ that are 1 in an interior interval and 0 outside a larger interval. More precisely, given four consecutive points $(\rho, \rho + l, r, r + l)$, we set: $\vartheta(s) = 0$ for $s \leq \rho$ and for $s \geq r + l$, $\vartheta(s) = 1$ for $\rho + l \leq s \leq r$, and ϑ affine linear in the two remaining intervals, compare Figure 3. By slight abuse of notation, we identify ϑ with a cut-off function on Ω by setting $\vartheta(x) := \vartheta(x_1)$ for $x \in \Omega$.

The basis for our approach is the energy flux equality. In its simplest form, it states: For a homogeneous solution ϕ, the energy flux quantity $\operatorname{Im} \mathcal{Q}(\phi|_{W_r})$ is independent of the position r. We use the notation $\Omega_{r_1, r_2} := (r_1, r_2) \times S$.

Lemma 2.1 (Simple flux equality) *Let $\phi, \psi \in H^1_{\text{loc}}(\Omega)$ be two solutions to $A\phi = \omega^2 \phi$ on Ω. Then, for arbitrary $\rho, r \in \mathbb{R}$ with $\rho + l \leq r$, there holds the flux equality*

$$\operatorname{Im} \mathcal{Q}(\phi|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(\phi|_{W_r}). \tag{2.3}$$

The sesquilinear form Q^s satisfies

$$Q^s(\phi|_{W_\rho}, \psi|_{W_\rho}) = Q^s(\phi|_{W_r}, \psi|_{W_r}). \tag{2.4}$$

Proof. Equality (2.4) implies (2.3) by $Q^s(u, u) = i \operatorname{Im} \mathcal{Q}(u)$.

We use the piecewise affine cut-off function ϑ corresponding to the four points $(\rho, \rho + l, r, r + l)$. Multiplication of the equation $A\phi = \omega^2\phi$ with $\vartheta\bar{\psi}$ and an integration over Ω yields

$$
\begin{aligned}
0 &= \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla(\bar{\psi}\vartheta) - \int_{\Omega_{\rho,r+l}} \omega^2\,\phi\,\bar{\psi}\,\vartheta \\
&= \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla\bar{\psi}\,\vartheta - \int_{\Omega_{\rho,r+l}} \omega^2\,\phi\,\bar{\psi}\,\vartheta - \int_{W_r} a\nabla\phi\,\bar{\psi}\cdot\frac{1}{l}e_1 + \int_{W_\rho} a\nabla\phi\,\bar{\psi}\cdot\frac{1}{l}e_1 \\
&= \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla\bar{\psi}\,\vartheta - \int_{\Omega_{\rho,r+l}} \omega^2\,\phi\,\bar{\psi}\,\vartheta - Q(\phi|_{W_r}, \psi|_{W_r}) + Q(\phi|_{W_\rho}, \psi|_{W_\rho}).
\end{aligned}
$$

The same expression can be written with ϕ and ψ exchanged; performing additionally a complex conjugation, we have

$$
0 = \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla\bar{\psi}\,\vartheta - \int_{\Omega_{\rho,r+l}} \omega^2\,\phi\,\bar{\psi}\,\vartheta - \overline{Q(\psi|_{W_r}, \phi|_{W_r})} + \overline{Q(\psi|_{W_\rho}, \phi|_{W_\rho})}.
$$

Subtracting the two results, the first two integrals cancel and we obtain (2.4). $\qquad\square$

Remark 2.2 (The flux through an interface). *Let $\phi \in H^1_{\text{loc}}(\Omega)$ be a solution of $A\phi = \omega^2\phi$ on Ω. Multiplication with $\bar{\phi}$, integration over $(\rho, r) \times S \subset \Omega$ and taking the imaginary part yields*

$$
\text{Im} \int_{\{\rho\}\times S} a\nabla\phi \cdot e_1\,\bar{\phi} = \text{Im} \int_{\{r\}\times S} a\nabla\phi \cdot e_1\,\bar{\phi} \tag{2.5}
$$

in the sense of traces. This shows that the expression on the right does not depend on the position r.

The fact that the surface integral is independent of r implies that every volume integral $\text{Im}\,\mathcal{Q}(\phi|_{W_r})$ of (2.3) actually coincides with the expression in (2.5).

2.2 Propagating modes and radiation condition

We next study solutions to the radiation problem. We are interested in solutions u that do not decay at infinity. Regarding regularity, on the other hand, we cannot expect that solutions are locally of class L^∞. We therefore introduce a new norm to measure functions. For $u : \Omega \to \mathbb{C}$ we set

$$
\|u\|_{sL} := \sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)}. \tag{2.6}
$$

We have chosen the subscript sL for the norm to recall that a *supremum* over L^2-*norms* is taken. We study the following subspace of $H^1(W_0)$:

$$
X := \left\{ u|_{W_0} \,\middle|\, u \in H^1_{\text{loc}}(\Omega),\ \|u\|_{sL} < \infty,\ Au = \omega^2 u \text{ in } \Omega \right\}. \tag{2.7}
$$

In many situations (see the text below), it is known that all functions u in (2.7) are *quasiperiodic functions*. We say that a function $u : \Omega \to \mathbb{C}$ is quasiperiodic if there exists a real number $\xi \in [0, 2\pi)$ such that $u(x + e_1) = e^{i\xi}u(x)$ for every $x \in \Omega$. The number ξ is called the *quasimoment* and we also say that u is ξ-quasiperiodic. Note that, when u is ξ-quasiperiodic, u can be reconstructed from $u|_{W_0}$.

Let us recall some spectral analysis facts. In the next paragraph, we describe a generic situation and typical methods. We do not cite the rigorous mathematical results. Our intention is rather to clarify that the assumptions in our main theorems are reasonable. The proofs of our theorems do not use any of the results sketched here. For more material on the spectral analysis, we refer to [9, 16, 17, 21].

For a fixed number $\xi \in [0, 2\pi)$, one can consider the space of all ξ-quasiperiodic functions v. The Floquet-Bloch transform allows to decompose the partial differential equation $Au = \omega^2 u$ into a family of problems $A(\xi)u = \omega^2 u$, where $\xi \in [0, 2\pi)$ is the quasimoment and $A(\xi)$ is the operator A restricted to ξ-quasiperiodic functions. Every operator $A(\xi)$ has a compact resolvent and hence a pure point spectrum. The eigenvalues of $A(\xi)$ depend continuously on ξ and have lower bounds which imply that, for generic values of ω, the number ω^2 coincides with eigenvalues of $A(\xi)$ only for finitely many values of ξ. The corresponding finitely many eigenfunctions then form a basis of the space X of (2.7). In particular, the space X is finite-dimensional.

We will assume this situation and one additional property: For the frequency ω, the imaginary part of the form $\mathcal{Q}$ does not vanish in basis functions. If, for every ξ, ω^2 is not a multiple eigenvalue of $A(\xi)$, then our definition of non-singular frequencies coincides with the one in [9], see the set σ_0 in (33) of [9]. We emphasise that our space X is, a priori, not identical with the space F of propagating Floquet modes; instead, X could be larger and we essentially say that a frequency is non-singular when no mode of X is propagating. It is an interesting task for future research to check under what conditions $X = F$ holds.

Definition 2.3 (Non-singular frequency). Let $A = -\nabla \cdot (a\nabla)$ be an x_1-periodic elliptic operator. A number $\omega > 0$ is called a *non-singular frequency* (for the periodic medium a) if the following holds:

(a) *Finite dimension:* The space X of (2.7) has a finite dimension $M \in \mathbb{N}$. There exists a basis $(\varphi_j)_{1 \leq j \leq M}$ and quasimoments $\xi_j \in [0, 2\pi)$ such that each φ_j possesses a ξ_j-quasiperiodic extension satisfying $A\varphi_j = \omega^2 \varphi_j$ in Ω.

(b) *Non-vanishing flux:* For every quasiperiodic function $u \in H^1_{\text{loc}}(\Omega)$ with $Au = \omega^2 u$, the restriction $\varphi = u|_{W_0} \in X$ has the property

$$\text{Im } \mathcal{Q}(\varphi) \neq 0 \,. \tag{2.8}$$

Whenever $\omega > 0$ is a non-singular frequency, we can actually achieve the following situation: There holds $M = 2N$ for some $N \in \mathbb{N}$, there exists a basis $(\phi_1^+, ..., \phi_N^+, \phi_1^-, ..., \phi_N^-)$ of X and corresponding quasimoments $\xi_j^{\pm} \in [0, 2\pi)$ such that the quasiperiodic extensions with $\phi_j^{\pm}(. + e_1) = e^{i\xi_j^{\pm}} \phi_j^{\pm}(.)$ solve $A\phi = \omega^2 \phi$ in Ω. For two basis functions ϕ and $\tilde{\phi}$ for the same quasimoment ξ, there holds the orthogonality $\mathcal{Q}^s(\phi, \tilde{\phi}) = 0$. Furthermore, the fluxes have a sign as indicated by the superscript: For every j holds

$$\text{Im } \mathcal{Q}(\phi_j^+) > 0 \quad \text{and} \quad \text{Im } \mathcal{Q}(\phi_j^-) < 0 \,. \tag{2.9}$$

Let us indicate how to obtain the basis $(\phi_1^+, ..., \phi_N^+, \phi_1^-, ..., \phi_N^-)$. We fix a quasimoment ξ and consider only the basis functions φ_j having that quasimoment, say $\varphi_1, ..., \varphi_m$. We

perform a standard diagonalisation procedure with respect to Q^s: In the first step, we choose $\phi_1 = \varphi_1$. In the second step, we set $\alpha_1 = Q^s(\phi_1, \varphi_2)/Q^s(\phi_1, \phi_1) \in \mathbb{C}$, which is possible by condition (b) and $Q^s(u, u) = i \, \mathrm{Im} \, \mathcal{Q}(u)$, and define $\phi_2 = \varphi_2 - \alpha_1 \phi_1$. This guarantees $Q^s(\phi_1, \phi_2) = Q^s(\phi_1, \varphi_2) - \alpha_1 Q^s(\phi_1, \phi_1) = 0$. The (anti-)symmetry of Q^s yields additionally $Q^s(\phi_2, \phi_1) = 0$. The process can be continued until $\varphi_1, ..., \varphi_m$ are Q^s-orthogonalised. We exploit in the process the symmetry of Q^s and the fact that, for fixed $\xi \in \mathbb{R}$, ξ-quasiperiodic functions form a vector space. Relabelling the functions $\phi_1, ..., \phi_M$ and exploiting once more (b), we obtain (2.9).

Below, we will obtain additionally the orthogonality of all basis functions $\phi_j^\pm$ with respect to the form Q^s. Given a function ϕ_j^+ with $\mathrm{Im} \, \mathcal{Q}(\phi_j^+) > 0$, the complex conjugate function $\overline{\phi_j^+}$ is also contained in the space X of (2.7), it satisfies $\mathcal{Q}(\overline{\phi_j^+}) = \overline{\mathcal{Q}(\phi_j^+)}$ and hence $\mathrm{Im} \, \mathcal{Q}(\overline{\phi_j^+}) < 0$. This argument shows that the number of modes ϕ_j^+ is identical to the number of modes ϕ_j^-. We can even choose the basis such that $\phi_j^- = \overline{\phi_j^+}$ and $\xi_j^- = -\xi_j^+$ for all $j \leq N$. In the following, we will always work with a basis $(\phi_1^+, ..., \phi_N^+, \phi_1^-, ..., \phi_N^-)$ as described before and in (2.9).

We already noted that Property (a) of Definition 2.3 is generically satisfied. Property (b) demands that there is no nontrivial wave with vanishing flux. The existence of a wave with vanishing flux implies the existence of a nontrivial solution to the homogeneous radiation problem, which contradicts the uniqueness statement of Theorem 1.1. In this sense, Definition 2.3 (b) is a necessary condition for Theorem 1.1. The non-singularity of all frequencies ω is verified in certain settings. Theorem 5.3.2 of [19] provides also the orthogonality with respect to Q^s. The orthogonality is derived also in [9], see their Theorem 3.

Projections and radiation condition.

We proceed with the construction of function spaces and projections. For a non-singular frequency ω and basis functions as above, we define the following two subspaces of $H^1(W_0)$,

$$X_+ := \mathrm{span}\{\phi_j^+ \mid 1 \leq j \leq N\}, \quad X_- := \mathrm{span}\{\phi_j^- \mid 1 \leq j \leq N\}. \tag{2.10}$$

Every function $\phi^+ \in X_+$ can be extended to a solution of the homogeneous problem. Here, some care should be taken, since a function in X_+ is, in general, not quasiperiodic for any ξ. We construct as follows: Every basis function $\phi_j^+ \in X_+$ has a ξ_j^+-quasiperiodic extension. We denote the extension as $E_j^+ \phi_j^+$, it is characterised by the property $E_j^+ \phi_j^+(x + m e_1) = e^{i m \xi_j^+} \phi_j^+(x)$ for every $m \in \mathbb{Z}$. An arbitrary element $\phi^+ \in X_+$ is a linear combination of basis elements, $\phi^+(x) := \sum_j \alpha_j \phi_j^+(x)$ for some coefficients $\alpha_j \in \mathbb{C}$, and we have to extend every basis function with the appropriate quasiperiodicity. More precisely, the extension of $\phi^+(x)$ as above is given by

$$(E\phi^+)(x) := \sum_j \alpha_j (E_j^+ \phi_j^+)(x).$$

This defines an extension operator E which maps elements of X_+ to solutions of the homogeneous problem $A\phi = \omega^2 \phi$ in Ω. Later on, we oftentimes simplify the notation and write again ϕ^+ for the extension $E\phi^+$. Analogously, extensions are defined for $\phi^- \in X_-$.

Since the basis functions are linearly independent, there holds $X = X_+ \oplus X_-$. An arbitrary element $u \in X$ can be written uniquely as $u = \sum_{j=1}^N \alpha_j \phi_j^+ + \sum_{j=1}^N \beta_j \phi_j^-$. The natural projections

$\Pi_{X,+} : X \to X_+ \subset X$ and $\Pi_{X,-} : X \to X_- \subset X$ are given by

$$\Pi_{X,+}(u) = \sum_{j=1}^{N} \alpha_j \phi_j^+ \quad \text{and} \quad \Pi_{X,-}(u) = \sum_{j=1}^{N} \beta_j \phi_j^- \ .$$

We emphasise that the basis function $\phi_j^\pm$ is, in general, not L^2-orthogonal. Accordingly, the projections are not necessarily L^2-orthogonal projections.

The $L^2(W_0)$-orthogonal projection onto the subspace X is denoted as $\Pi_X : L^2(W_0) \to L^2(W_0)$. With the help of Π_X, we define the two projections $\Pi_+ := \Pi_{X,+} \circ \Pi_X : L^2(W_0) \to L^2(W_0)$ onto X_+ and $\Pi_- := \Pi_{X,-} \circ \Pi_X : L^2(W_0) \to L^2(W_0)$ onto X_-.

The case of two different periodic media.

Let us now discuss how the above concepts can be applied when a is given by a^{left} for $x_1 < -R_0$ and by a^{right} for $x_1 > R_0$. In this case, the spaces $X_\pm$ can be determined for a^{right}, which defines $X_\pm^{\text{right}}$, and for a^{left}, which defines $X_\pm^{\text{left}}$. Accordingly, we can define $\Pi_\pm^{\text{right}}$ and $\Pi_\pm^{\text{left}}$, Q^{right} and Q^{left}, $\mathcal{Q}^{\text{right}}$ and $\mathcal{Q}^{\text{left}}$. In order to take limits, it is still convenient to identify a function in $L^2(W_r)$ with a function in $L^2(W_0)$. Nevertheless, one has to be careful in the definition of the forms Q, since we have no information on a in W_0: The form $\mathcal{Q}^{\text{right}}$ is defined as in (2.1), but with a replaced by a^{right}. Accordingly, on the left, $\mathcal{Q}^{\text{left}}$ is defined as in (2.1) with the coefficient function a^{left}.

Radiation condition.

The projections allow to introduce the radiation condition that is used in this work. We remark that the equivalence with the usual radiation condition is established in Lemma 3.2. The norm $\|u\|_{sL}$ was introduced in (2.6).

Definition 2.4 (Radiation condition) Let ω be non-singular in the sense of Definition 2.3. For $r \in \mathbb{N}$ we consider boxes $W_{\pm r}$ and the corresponding projections $\Pi_\pm$. We say that $u : \Omega \to \mathbb{C}$ with $\|u\|_{sL} < \infty$ satisfies the radiation condition if

$$\Pi_-(u|_{W_r}) \to 0 \quad \text{and} \quad \Pi_+(u|_{W_{-r}}) \to 0 \quad \text{as } r \to +\infty \ . \tag{2.11}$$

In this formula, we identify a function on W_r with a function on W_0 via a shift. The convergence is that of $L^2(W_0)$.

In the case of two different media at infinity, a^{right} and a^{left}, the condition (2.11) is modified in the natural way to

$$\Pi_-^{\text{right}}(u|_{W_r}) \to 0 \quad \text{and} \quad \Pi_+^{\text{left}}(u|_{W_{-r}}) \to 0 \quad \text{as } r \to +\infty \ . \tag{2.12}$$

2.3 Orthogonality

For notational convenience, we return to the discussion of one periodic medium a. The spaces X_+ and X_- are not orthogonal in $L^2(W_0)$. Nevertheless, we will obtain orthogonality with respect to the form Q^s.

We have announced that the width $l \in \mathbb{N}$ of the boxes can be chosen arbitrarily. In this section, in order to derive the orthogonality, we will use l as a variable parameter. Here and below, we

normalise the basis functions such that

$$\frac{1}{l} \int_{W_0} |\phi_j^\pm|^2 = 1 \,. \tag{2.13}$$

A normalised basis function remains normalised when l is changed. This follows immediately from quasiperiodicity, which provides by its definition that $|\phi_j^\pm|^2$ is 1-periodic in x_1.

Proposition 2.5 (Orthogonality). *Every two different elements u and v of the set $\{\phi_1^+, ..., \phi_N^+, \phi_1^-, ..., \phi_N^-\}$ satisfy*

$$Q^s(u, v) = 0 \,. \tag{2.14}$$

Proof. In the case that u and v have the same quasimoment ξ, there holds $Q^s(u, v) = 0$ by the orthogonalisation of the basis functions. The construction was performed after Definition 2.3.

It remains to consider the case that u has the quasimoment ξ and v has the quasimoment $\zeta \neq \xi$. We want to calculate the expression

$$Q^s(u, v) = \frac{1}{2}\left(Q(u, v) - \overline{Q(v, u)}\right) = \frac{1}{2l}\int_{W_0} a\nabla u \cdot e_1 \bar{v} - \frac{1}{2l}\int_{W_0} a\nabla \bar{v} \cdot e_1 u \,.$$

Equation (2.4) of Lemma 2.1 shows that the sesquilinear form Q^s is independent of the x_1-position. This implies that Q^s is independent of l.

With the notation $\Omega_{r_1, r_2} := (r_1, r_2) \times S$, we calculate for the first expression

$$\frac{1}{2l}\int_{W_0} a\nabla u \cdot e_1 \bar{v} = \frac{1}{2l}\sum_{k=0}^{l-1}\int_{\Omega_{k,k+1}} a\nabla u \cdot e_1 \bar{v}$$

$$= \frac{1}{2l}\sum_{k=0}^{l-1}\int_{\Omega_{0,1}} e^{ik\xi} a\nabla u \cdot e_1 \, e^{-ik\zeta} \bar{v} = \frac{1}{2l}\sum_{k=0}^{l-1} e^{ik(\xi-\zeta)}\int_{\Omega_{0,1}} a\nabla u \cdot e_1 \, \bar{v} \,.$$

The integral on the right is independent of l. With $\beta := e^{i(\xi-\zeta)}$ we write the factor in front of the integral as $\frac{1}{2l}\sum_{k=0}^{l-1} e^{ik(\xi-\zeta)} = \frac{1}{2l}\sum_{k=0}^{l-1}\beta^k = \frac{1}{2l}\frac{1-\beta^l}{1-\beta} \to 0$ for $\mathbb{N} \ni l \to \infty$. Note that $\beta \neq 1$ is satisfied because of $\xi \neq \zeta$. The other term is treated in the same way and we obtain (2.14).   □

Corollary 2.6 (Sign of the sesquilinear form). *For some $\gamma > 0$ holds*

$$\pm \operatorname{Im} Q(u, u) \geq \frac{\gamma}{l}\|u\|_{L^2(W_0)}^2 \quad \forall u \in X_\pm \,. \tag{2.15}$$

Proof. Let $u \in X_+$ be arbitrary, $u = \sum_{j=1}^N \alpha_j \phi_j^+$. We use $\gamma_+ := \min_j \operatorname{Im} \mathcal{Q}(\phi_j^+)$, which is positive by (2.9). The orthogonality (2.14) allows to calculate

$$\operatorname{Im} Q(u, u) = \operatorname{Im} Q^s(u, u) = \sum_{j=1}^N |\alpha_j|^2 \operatorname{Im} Q^s(\phi_j^+, \phi_j^+) \geq \gamma_+ \sum_{j=1}^N |\alpha_j|^2 \,. \tag{2.16}$$

The immediate inequality $\|u\|_{L^2(W_0)}^2 \leq Cl \sum_{j=1}^N |\alpha_j|^2$ for a constant $C > 0$ provides the claim in X_+ for $\gamma = \gamma_+/C > 0$. The argument for X_- is analogous.   □

We defined X to consist of restrictions of homogeneous solutions u on Ω. We now turn to a more quantitative version of this fact: If u is a homogeneous solution on a large subdomain, then its restriction is close to an element of X.

Lemma 2.7 (Outside a compact region, solutions are close to X). *Let $l \in \mathbb{N}$ be fixed and let $\eta > 0$ be an arbitrary error quantifier. There exists a large number $r_0 \in \mathbb{N}$ such that, for every $\mathbb{N} \ni r > r_0$, there holds: Every function $u_r \in H^1_{\mathrm{loc}}(\Omega)$ with the properties*

$$A u_r = \omega^2 u_r \text{ in } \Omega_{-r,r} \quad and \quad \|u_r\|_{sL} \leq 1 \tag{2.17}$$

satisfies

$$\left\| u_r|_{W_0} - \Pi_X(u_r|_{W_0}) \right\|_{H^1(W_0)} \leq \eta. \tag{2.18}$$

We will later use repeatedly the following immediate consequence of (2.18), which exploits $\Pi_X = \Pi_+ + \Pi_-$:

$$\left\| u_r|_{W_0} - \Pi_+(u_r|_{W_0}) - \Pi_-(u_r|_{W_0}) \right\|_{L^2} \leq \eta. \tag{2.19}$$

Proof. The aim is to show that $u_r|_{W_0}$ is near an element of X. We recall that l and η are fixed. We want to show the existence of r_0 and argue by contradiction. If there is no r_0 with the desired property, then there exists a sequence $r \to \infty$ and a sequence of functions u_r, which satisfy (2.17), but not (2.18). In the following, we work with this sequence and our aim is to derive a contradiction.

The boundedness of (2.17) allows to select a subsequence and to find a limit function u such that $u_r \to u$ converges weakly in $H^1(K)$ for every bounded subset of the form $K = (-l_0, l_0) \times S \subset \Omega$, $l_0 > 0$. From now on, we work with this subsequence. As a limit function, u also satisfies both properties of (2.17), the solution property and the boundedness. Locally, the sequence u_r converges even strongly in H^1, as can be shown easily by testing the equation for $u_r - u$ with $(u_r - u)\,\theta$, where θ is a cut-off function. The strong convergence $u_r \to u$ in $H^1(W_0)$ implies that the limit u satisfies the same inequality as the approximate functions:

$$\left\| u|_{W_0} - \Pi_X(u|_{W_0}) \right\|_{H^1} \geq \eta. \tag{2.20}$$

This provides a contradiction: $u|_{W_0} \in X$ holds by definition of X in (2.7), so the left hand side of (2.20) vanishes. $\qquad \square$

We will later also exploit the H^1-regularity of the elements $\phi \in X$, where X is either X^{left} or X^{right}: For some constant C, there holds

$$\|\phi\|_{H^1(W_0)} \leq C \|\phi\|_{L^2(W_0)} \tag{2.21}$$

for all elements $\phi \in X$. The constant $C = C(\omega, \lambda)$ depends only on the frequency ω and on the ellipticity constant λ of the coefficients. The property (2.21) can be obtained by testing the equation with the solution and a cut-off function, but it can be concluded for general C also immediately from the fact that the basis functions are of class $H^1(W_0)$ and that the space X is finite-dimensional.

3 Uniqueness

As mentioned in the introduction, we show uniqueness and existence with energy methods, using the conservation of fluxes. The essential proofs rely on a simple trick that we want to describe here in loose terms. We explain the trick for the uniqueness proof, it is very similar in the existence proof.

Let u be a solution to the radiation problem with $f = 0$, and our aim is to show that u vanishes. We use a contradiction argument and assume that for a position $\rho \in \mathbb{N}$ the function $u|_{W_\rho}$ does not vanish. The radiation condition yields that for a large number $r \in \mathbb{N}$ the function $u|_{W_r}$ is close to a right-going wave.

If we use the flux equality for u, we conclude that the flux of u in W_ρ coincides with the flux in W_r – but this information in itself is not very helpful, since $u|_{W_\rho}$ can consist of right-going and left-going waves.

The trick is to consider the following: Let ϕ be the projection of $u|_{W_\rho}$ to right-going waves. We extend ϕ to all of Ω and set $w := u - \phi$. The properties of w are the following: (a) w is a solution, since u and ϕ are. (b) w is (approximately) right-going in W_r, since u and ϕ are. (c) w is left-going in W_ρ, since we subtracted the right-going part from u. The flux equality for w yields that the fluxes in W_ρ and W_r coincide. This is a valuable information, since the two fluxes have opposite sign (up to small errors). We conclude that all fluxes are small, which implies that w is small in W_ρ, from which it follows that u has a small left-going component in W_ρ. In the same way, choosing r to the left of ρ, one concludes that u has a small right-going component in W_ρ. This yields that u is small in W_ρ. We find the desired contradiction to the choice of ρ.

We now turn to the rigorous proofs and make the above ideas precise. We recall the norm of (2.6) for $u : \Omega \to \mathbb{C}$, $\|u\|_{sL} := \sup_{r \in \mathbb{Z}} \|u\|_{L^2((r,r+1) \times S)}$.

Proposition 3.1 (Uniqueness on the unbounded domain). *For non-singular frequencies $\omega > 0$, the problem of Theorem 1.1 has at most one solution. More precisely, every solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of $Au = \omega^2 u$ with $\|u\|_{sL} < \infty$ that satisfies the radiation conditions of Definition 2.4 vanishes identically.*

Proof. Let us assume that u is a non-vanishing solution to the homogeneous problem. Our aim is to arrive at a contradiction.

Step 1: Preparations. We normalise u such that $\sup_{r \in \mathbb{Z}} \|u|_{W_r}\|_{L^2(W_r)} = 1$. Let $\rho \in \mathbb{Z}$ be a number with $\|u|_{W_\rho}\|_{L^2(W_\rho)} \geq 1/2$.

We choose a small quantifier $1 \geq \varepsilon > 0$, and the choice will be specified below after inequality (3.5). The radiation condition (2.11) allows to choose $r \in \mathbb{N}$, $r \geq |\rho|$ large, so that the smallness $\|\Pi_-(u|_{W_r})\|_{L^2} + \|\Pi_+(u|_{W_{-r}})\|_{L^2} \leq \varepsilon$ is satisfied (and remains satisfied for every larger r). Using the H^1-regularity property $\|\phi_\pm\|_{H^1(W_0)} \leq C \|\phi_\pm\|_{L^2(W_0)}$ of (2.21), we can improve the regularity to

$$\|\Pi_-(u|_{W_r})\|_{H^1} + \|\Pi_+(u|_{W_{-r}})\|_{H^1} \leq C\varepsilon . \tag{3.1}$$

We consider $\phi(x) := \sum_j \alpha_j \phi_j^+(x)$ with $\Pi_+((u - \phi)|_{W_\rho}) = 0$ and set $w = u - \phi$. There holds

$$\Pi_-(w|_{W_r}) = \Pi_-(u|_{W_r}) - \Pi_-(\phi|_{W_r}) = \Pi_-(u|_{W_r}),$$

hence $\|\Pi_-(w|_{W_r})\|_{L^2} \leq \varepsilon$ and $\|\Pi_-(w|_{W_r})\|_{H^1} \leq C\varepsilon$. This quantifies the fact that w is approximately right-going in W_r.

Regarding boundedness, we observe that $\sup_{r \in \mathbb{Z}} \|\phi\|_{L^2(W_r)} \leq C$ holds, since ϕ is obtained by a projection of u. As a difference, w satisfies $\sup_{r \in \mathbb{Z}} \|w\|_{L^2(W_r)} \leq 1 + C$.

Step 2: Flux equality. We use a cut-off function which is similar to that of Figure 3: We choose ϑ_ρ corresponding to the four points $(\rho, \rho + l, r, r + l)$. Multiplication of $Aw = \omega^2 w$ with $\bar{w}\vartheta_\rho$ yields

$$0 = \int_{\Omega_{\rho,r+l}} a\nabla w \cdot \nabla\bar{w}\vartheta_\rho - \int_{\Omega_{\rho,r+l}} \omega^2\, w\bar{w}\vartheta_\rho - \mathcal{Q}(w|_{W_r}) + \mathcal{Q}(w|_{W_\rho})\,.$$

Taking the imaginary part provides the flux equality

$$\operatorname{Im} \mathcal{Q}(w|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(w|_{W_r})\,. \tag{3.2}$$

Step 3: Conclusion. The fact that w is a solution on Ω implies that $w|_{W_r}$ is an element of X, we can write $w|_{W_r} = \Pi_+(w|_{W_r}) + \Pi_-(w|_{W_r})$. The smallness of $\Pi_-(w|_{W_r})$ therefore yields

$$\left\| w|_{W_r} - \Pi_+(w|_{W_r}) \right\|_{H^1} \leq C_0\varepsilon\,. \tag{3.3}$$

This allows to calculate the quadratic form on the right hand side of (3.2) as

$$\operatorname{Im} \mathcal{Q}(w|_{W_r}) = \operatorname{Im} \mathcal{Q}\left(\Pi_+(w|_{W_r}) + [w|_{W_r} - \Pi_+(w|_{W_r})]\right)\,.$$

Inserting the definition of the quadratic form $\mathcal{Q}$, using $\|w\|_{L^2(W_r)} \leq 1 + C$ and $\|w\|_{H^1(W_r)} \leq C_1$, we find from (3.3) and the definition of $\mathcal{Q}$ in (2.1), for some constant C_2 that depends on C, C_0, and C_1,

$$\operatorname{Im} \mathcal{Q}(w|_{W_r}) \geq \operatorname{Im} \mathcal{Q}(\Pi_+(w|_{W_r})) - \frac{C_2\Lambda\varepsilon}{l} \geq -\frac{C_2\Lambda\varepsilon}{l}\,, \tag{3.4}$$

where we used the positivity of $\mathcal{Q}$ on X_+ of (2.15) in the second inequality. The flux equality (3.2) transfers this lower bound to the domain W_ρ.

Since also $u|_{W_\rho}$ is an element of X, there holds $w|_{W_\rho} = u|_{W_\rho} - \phi|_{W_\rho} = u|_{W_\rho} - \Pi_+(u|_{W_\rho}) = \Pi_-(u|_{W_\rho}) \in X_-$. We calculate with (3.4), (3.2), and (2.15):

$$\frac{C_2\Lambda\varepsilon}{l} \geq -\operatorname{Im} \mathcal{Q}(w|_{W_r}) = -\operatorname{Im} \mathcal{Q}(w|_{W_\rho}) \geq \frac{\gamma}{l}\|\Pi_-(u|_{W_\rho})\|_{L^2}^2\,. \tag{3.5}$$

Choosing $\varepsilon > 0$ so small that $\sqrt{C_2\Lambda\varepsilon/\gamma} \leq 1/6$ holds, we find $\|\Pi_-(u|_{W_\rho})\|_{L^2} \leq 1/6$.

The argument can be repeated with the left-going wave $\phi_- = \Pi_-(u|_{W_\rho})$, which yields the same estimate for $\Pi_+(u|_{W_\rho})$. Together, we obtain

$$\|u|_{W_\rho}\|_{L^2} \leq \|\Pi_-(u|_{W_\rho})\|_{L^2} + \|\Pi_+(u|_{W_\rho})\|_{L^2} \leq \frac{1}{3}\,, \tag{3.6}$$

in contradiction to the choice of ρ. $\qquad\square$

3.1 An equivalent radiation condition

We next provide another formulation of the radiation condition. Indeed, in the literature, usually variants of (3.7) are used; one has some liberty in the choice of the precise convergence condition.

We describe the equivalence here in the setting of Theorem 1.1. Since the two directions $x_1 \to \pm\infty$ can be treated independently, the equivalence follows with the same proof also in the setting of Theorem 1.2.

Lemma 3.2 (Equivalent radiation condition). *Let the coefficient a be as in Theorem 1.1, let $\omega > 0$ be a non-singular frequency, let $u \in H^1_{\text{loc}}(\Omega)$ be a function that satisfies (i) and (ii) of our solution concept, that is: u solves (1.1) in the sense of distributions and $\sup_{r \in \mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)} < \infty$.*

The function u satisfies the radiation condition of Definition 2.4 if and only if the following holds: There exist $\phi_+ \in X_+$ and $\phi_- \in X_-$, which we identify with their extensions $E\phi_+$ and $E\phi_-$, such that, as $r \to \infty$,

$$\|(u - \phi_+)|_{W_r}\|_{L^2(W_r)} \to 0 \ \text{ and } \ \|(u - \phi_-)|_{W_{-r}}\|_{L^2(W_{-r})} \to 0 . \tag{3.7}$$

Proof. *The 'if'-part.* Let u satisfy (3.7) with $\phi_+ \in X_+$ and $\phi_- \in X_-$. Using a triangle inequality, boundedness of projections, and $\Pi_-(\phi_+|_{W_r}) = 0$, we find

$$\|\Pi_-(u|_{W_r})\|_{L^2(W_r)} \le \|\Pi_-((u - \phi_+)|_{W_r})\|_{L^2(W_r)} + \|\Pi_-(\phi_+|_{W_r})\|_{L^2(W_r)}$$
$$\le C\|(u - \phi_+)|_{W_r}\|_{L^2(W_r)} \to 0$$

by (3.7). This shows one part of (2.11) the calculation for $\Pi_+(u|_{W_{-r}})$ is analogous.

The 'only-if'-part. Vice versa, let u satisfy the radiation condition (2.11). We consider the right boundary. For any sequence $R \to \infty$, the sequences $u|_{W_R}$ are bounded in $L^2(W_0)$ by the assumption on the boundedness of the sL-norm. As a preparation of the proof, we subtract the right-going part: With the help of a projection, we define $\phi^R = \Pi_+(u|_{W_R}) \in X_+$ (and we identify, as usual, ϕ^R with its extension as a solution on all of Ω, the extension operator is now the one that extends functions on W_R). We obtain $\Pi_+((u - \phi^R)|_{W_R}) = 0$.

The sequence of functions $\phi^R|_{W_0}$ is bounded in $H^1(W_0)$. We select a subsequence $R \to \infty$ (not relabelled) and a limit function ϕ_+ with $\phi^R|_{W_0} \to \phi_+$, weakly in $H^1(W_0)$ and strongly in $L^2(W_0)$. The space X_+ is finite-dimensional and hence closed; we therefore have $\phi_+ \in X_+$. Once more, we identify ϕ_+ with its extension $E\phi_+$ to a homogeneous solution on Ω. In particular, $\phi^R \to \phi_+$ holds locally on all of Ω. Our aim is to show, for an arbitrary sequence $r \to \infty$, that $(u - \phi_+)|_{W_r} \to 0$ holds in L^2. With that, (3.7) for the right half-axis is shown.

We fix an arbitrary error quantifier $\eta > 0$. A large number $r \in \mathbb{N}$ is chosen in dependence of η, the choice is specified below.

We define ϕ^r corresponding to $u|_{W_r}$ as above, with $\Pi_+((u - \phi^r)|_{W_r}) = 0$, and consider $w := w_r := u - \phi^r$. We assume that the number r is sufficiently large such that the support of f is contained in $\{x_1 < r\}$. Then, w solves the homogeneous problem on $\Omega \cap \{x_1 > r\}$ and hence satisfies the flux equality

$$\text{Im } \mathcal{Q}(w|_{W_R}) = \text{Im } \mathcal{Q}(w|_{W_r}) \tag{3.8}$$

for every number $R \in \mathbb{N}$, $R > r + l$.

Upper bound for the right hand side of (3.8). The difference $w|_{W_r} - \Pi_-(u|_{W_r}) = u|_{W_r} - \phi^r|_{W_r} - \Pi_-(u|_{W_r}) = u|_{W_r} - \Pi_X(u|_{W_r})$ is small by Lemma 2.7; more precisely, we achieve $\|w|_{W_r} - \Pi_-(u|_{W_r})\|_{H^1} \le \eta$ when r is sufficiently large (large distance to the support of f).

Together with the definition of $\mathcal{Q}$, this allows to calculate the right hand side of (3.8) as

$$\text{Im } \mathcal{Q}(w|_{W_r}) = \text{Im } \mathcal{Q}(\Pi_-(u|_{W_r}) + [w|_{W_r} - \Pi_-(u|_{W_r})])$$

$$\leq \text{Im } \mathcal{Q}(\Pi_-(u|_{W_r})) + C\eta \leq C\eta . \tag{3.9}$$

In this calculation, we inserted in a trivial way the function $\Pi_-(u|_{W_r})$. We then used the definition of $\mathcal{Q}$, in particular, its continuity on H^1 as a bilinear form, treating the squared bracket as an error term. In the last inequality we used $\text{Im } \mathcal{Q}(\Pi_-(u|_{W_r})) \leq 0$ of Corollary 2.6.

Lower bound for the left hand side of (3.8). We exploit that, for large R, the function $w|_{W_R}$ is close to an element of X_+, which follows from the radiation condition. Let us make this fact precise: By definition of w, using $\Pi_+(\phi^r|_{W_R}) = \phi^r|_{W_R}$, we find

$$w|_{W_R} - \Pi_+(w|_{W_R}) = u|_{W_R} - \phi^r|_{W_R} - \Pi_+(u|_{W_R}) + \Pi_+(\phi^r|_{W_R})$$

$$= \left(u|_{W_R} - \Pi_X(u|_{W_R})\right) + \Pi_-(u|_{W_R}) \to 0$$

as $R \to \infty$. The smallness of the first bracket follows from Lemma 2.7 and the smallness of the last term from the radiation condition (2.11). The convergence is in L^2, but the solution property allows once more to lift the regularity order and we obtain convergence also in H^1. This can be used to calculate the left hand side of (3.8):

$$\text{Im } \mathcal{Q}(w|_{W_R}) = \text{Im } \mathcal{Q}(\Pi_+(w|_{W_R}) + [w|_{W_R} - \Pi_+(w|_{W_R})])$$

$$\geq \text{Im } \mathcal{Q}(\Pi_+(w|_{W_R})) + o(1) \geq \frac{\gamma}{l}\|\Pi_+(w|_{W_R})\|_{L^2}^2 + o(1)$$

as $R \to \infty$, where we used the quantitative estimate (2.15) in the last step.

Combining the calculations for the two sides of (3.8), we have obtained the smallness result

$$\|\Pi_+(w|_{W_R})\|_{L^2}^2 \leq \frac{Cl}{\gamma}\eta + o(1) \tag{3.10}$$

as $R \to \infty$. We evaluate the left hand side:

$$\Pi_+(w|_{W_R}) = \Pi_+((u - \phi^r)|_{W_R}) = (\phi^R - \phi^r)|_{W_R} .$$

The functions ϕ^R and ϕ^r are both extensions of elements in X_+. This implies that also the difference $\phi^R - \phi^r$ is an extension of an element in X_+, we may write it with the basis functions ϕ_j^+ as $\phi^R - \phi^r = \sum_j \alpha_j \phi_j^+$. For every ρ, the norm of the restriction $(\phi^R - \phi^r)|_{W_\rho}$ is equivalent to the norm defined by $\left(\sum_{j \leq N} |\alpha_j|^2\right)^{1/2}$. In particular, $\|(\phi^R - \phi^r)|_{W_r}\|_{L^2}^2 \leq C\|(\phi^R - \phi^r)|_{W_R}\|_{L^2}^2$ for some constant C. Quasiperiodicity and relation (3.10) allow to calculate for the limit $R \to \infty$

$$\|(\phi_+ - \phi^r)|_{W_r}\|_{L^2}^2 \leftarrow \|(\phi^R - \phi^r)|_{W_r}\|_{L^2}^2 \leq C\|(\phi^R - \phi^r)|_{W_R}\|_{L^2}^2 \leq \frac{Cl}{\gamma}\eta + o(1) .$$

It remains to exploit once more the radiation condition (2.11) and Lemma 2.7 to find $\|(u - \phi^r)|_{W_r}\|_{L^2}^2 = \|u|_{W_r} - \Pi_+(u|_{W_r})\|_{L^2}^2 \leq \eta$ for sufficiently large r. Since we have shown that ϕ^r is close to ϕ_+, and since $\eta > 0$ was arbitrary, we have shown the desired result $(u - \phi_+)|_{W_r} \to 0$.

The condition on the left follows in the same way. □

4 The truncated problem

The definition of the truncated problem is at the heart of our analysis. We recall the overall strategy: (1) Define truncated problems with a domain width parameter m. The truncated problems should be easy to solve. Denote the solutions by u_m. (2) Take the limit $m \to \infty$. Any limit u of the sequence, $u_m \to u$, should be a solution of the original problem. The problem is to define a truncated problem with these two properties. If, for example, one solves a Dirichlet problem on truncated domains, then the limit solution will in general not satisfy the radiation condition.

In order to obtain the radiation condition in the limit, our approach is to demand that u_m looks like an outgoing wave in a neighbourhood of the artificial boundary. Indeed, we enforce this property in a hard way in our function space $V_{L,R}$, see (4.1). Once this space is defined, it is quite natural how to encode the equation with a bilinear form. Testing the equation $Au = \omega^2 u$ with an element $v \in V_{L,R}$ is not helpful since the integration by parts leads to unwanted boundary integrals. We therefore test the equation with $v\vartheta$ where ϑ is a cut-off function. In this way, no boundary integrals occur; they are replaced by integrals in the radiation boxes, which are exactly those of the bilinear form Q. Testing $Au = \omega^2 u$ with $v\vartheta$ provides exactly the bilinear form β of (4.2). The truncated problem has strong similarities with that in [5].

Let us now turn to the precise construction. We use truncated domains of the form $\Omega_{-L,R} := (-L, R) \times S$ with two natural numbers $R, L > 0$. With the four consecutive points on the real line $(-L - l, -L, R, R + l)$, we define (as before) a cut-off function $\vartheta : \mathbb{R} \to \mathbb{R}$ as the piecewise affine function which vanishes to the left of the first and to the right of the fourth point, and which is 1 between the second and the third point. The length parameter $l \in \mathbb{N}$ is the same as in the first sections and suppressed in everything that follows.

We will study solutions u to $Au = \omega^2 u$ on different subdomains W_r. In this process, we exploit the following fact: For every function $\phi \in X_+$, we can identify ϕ with its extension $E\phi$. We therefore identify $\phi \in X_+$ with a solution of $A\phi = \omega^2\phi$ on Ω. This extension satisfies $\phi|_{W_r} \in X_+$ for every $r \in \mathbb{Z}$. This follows immediately from the extension process, see the text after (2.10). The same is true for X_-.

In this section, we consider the situation of Theorem 1.2, a must only describe a periodic medium for x_1 large and a (possibly different) periodic medium for $-x_1$ large. We denote the spaces $X_\pm$ for the periodic medium of the far right by $X_\pm^{\text{right}}$ and for the periodic medium of the far left by $X_\pm^{\text{left}}$. Projections, bilinear and quadratic forms are defined for the two media separately, $\Pi_\pm^{\text{right}}$, $\Pi_\pm^{\text{left}}$, Q^{right}, Q^{left}, $\mathcal{Q}^{\text{right}}$, and $\mathcal{Q}^{\text{left}}$.

Definition 4.1 (Function space and sesquilinear form). For $R, L \in \mathbb{N}$ we use the function space

$$V_{L,R} := \left\{ u \in H^1(\Omega_{-L-l,R+l}) \,\big|\, u|_{W_R} \in X_+^{\text{right}} ,\ u|_{W_{-L-l}} \in X_-^{\text{left}} \right\} . \tag{4.1}$$

With ϑ as above, corresponding to the points $(-L - l, -L, R, R + l)$, we introduce the sesquilinear form

$$\beta(u, v) := \int_{\Omega_{-L-l,R+l}} a\nabla u\nabla\bar{v}\vartheta - \int_{\Omega_{-L-l,R+l}} \omega^2\, u\bar{v}\vartheta$$
$$- Q^{\text{right}}(u|_{W_R}, v|_{W_R}) + Q^{\text{left}}(u|_{W_{-L-l}}, v|_{W_{-L-l}}) . \tag{4.2}$$

We define the following approximate problem.

Definition 4.2 (Truncated problem). Given $f \in H^{-1}(\Omega)$ with support in $\Omega_{-M,M}$ for some $M > 0$, we say that a function u solves the truncated problem for $\mathbb{N} \ni R, L > M$, if

$$u \in V_{L,R} \quad and \quad \beta(u, v) = \langle f, v \rangle \quad \forall v \in V_{L,R}. \tag{4.3}$$

The main line of our existence proof is to solve truncated problems and to take limits. We show the existence of solutions for the truncated problems in an indirect way, namely by concluding existence from uniqueness, see Lemma 4.4. Accordingly, later on, we need a uniqueness property, at least along some sequences $\mathbb{N} \ni R_k, L_k \to \infty$. We derive this property in Corollary 4.7.

First observations on uniqueness.

Let us observe already here that, for most choices of L and R, uniqueness can be expected. Non-uniqueness occurs only if ω^2 is an eigenvalue of A to Dirichlet boundary conditions at the lateral boundaries. For notational convenience, we use here the setting of Theorem 1.1 with a single periodic medium.

Remark 4.3 (On non-uniqueness in the truncated problem). *Let $L, R \in \mathbb{N}$ be two parameters such that the truncated problem has a nontrivial solution to $f = 0$. Then there exists a function u that satisfies $Au = \omega^2 u$ in $\Omega_{-L,R}$ and homogeneous Dirichlet conditions on $\{-L\} \times S$ and $\{R\} \times S$.*

Proof. Let $u = u_{L,R} \neq 0$ be a solution to the truncated problem with $f = 0$. We use the test function $v := u \in V_{L,R}$ in the sesquilinear form β. With ϑ as in Definition 4.1, we find

$$0 = \beta(u, u)$$
$$= \int_{\Omega_{-L-l,R+l}} a\nabla u \cdot \nabla u\, \vartheta - \int_{\Omega_{-L-l,R+l}} \omega^2 |u|^2\, \vartheta - \mathcal{Q}^{\text{right}}(u|_{W_R}) + \mathcal{Q}^{\text{left}}(u|_{W_{-L-l}}).$$

Taking the imaginary part yields the flux equality

$$\text{Im}\, \mathcal{Q}^{\text{right}}(u|_{W_R}) = \text{Im}\, \mathcal{Q}^{\text{left}}(u|_{W_{-L-l}}).$$

By definition of $V_{L,R}$, there holds $u|_{W_R} \in X_+^{\text{right}}$ and $u|_{W_{-L-l}} \in X_-^{\text{left}}$. The sign property (2.15) can be used to conclude that both flux terms vanish. Since the functions are elements of the appropriate spaces $X_\pm$, the sign property (2.15) implies additionally $u|_{W_R} = 0$ and $u|_{W_{-L-l}} = 0$.

By the $H^1(\Omega_{-L-l,R+l})$-property of $V_{L,R}$ in (4.1) we see that u is a solution on $\Omega_{-L,R}$ satisfying homogeneous Dirichlet conditions on $\{-L\} \times S$ and $\{R\} \times S$. □

4.1 Conditional existence for the truncated problem

We now turn to existence properties. As in a Fredholm alternative, the uniqueness property can imply an existence result. We use a limiting absorption principle to derive this fact.

Lemma 4.4 (Conditional existence for the truncated problem). *Let the setting be either that of Theorem 1.1 or that of Theorem 1.2, in particular: $\Omega = \mathbb{R} \times S$, $f \in H^{-1}(\Omega)$ with support in $\Omega_{-M,M}$ for some $M > 0$, the coefficient $a : \Omega \to \mathbb{R}$ periodic in e_1-direction outside a compact set, $\omega > 0$ a non-singular frequency for the one or the two periodic media, respectively. We consider parameters $\mathbb{N} \ni R, L \geq M$.*

If the truncated problem of Definition 4.2 has at most one solution, then there exists a solution to the truncated problem of Definition 4.2.

Proof. We use a limiting absorption principle. For every $\delta > 0$, we define a modified sesquilinear form by setting

$$\beta_\delta(u,v) := \int_{\Omega_{-L-l,R+l}} a\nabla u \nabla \bar{v}\vartheta - \int_{\Omega_{-L-l,R+l}} (\omega^2 + i\delta)\, u\bar{v}\vartheta$$
$$- Q^{\text{right}}(u|_{W_R}, v|_{W_R}) + Q^{\text{left}}(u|_{W_{-L-l}}, v|_{W_{-L-l}}). \tag{4.4}$$

Step 1: Solution for $\delta > 0$. We claim that the sesquilinear form β_δ is coercive on $V_{L,R}$. For $u \in V_{L,R}$, we first calculate

$$\text{Im}\, \beta_\delta(u,u) = \text{Im} \int_{\Omega_{-L-l,R+l}} a\nabla u \cdot \nabla u\vartheta - \text{Im} \int_{\Omega_{-L-l,R+l}} (\omega^2 + i\delta)\, |u|^2\vartheta$$
$$- \text{Im}\, Q^{\text{right}}(u|_{W_R}) + \text{Im}\, Q^{\text{left}}(u|_{W_{-L-l}})$$
$$\leq - \int_{\Omega_{-L-l,R+l}} \delta\, |u|^2\vartheta - \frac{\gamma}{l}\|u|_{W_R}\|^2_{L^2} - \frac{\gamma}{l}\|u|_{W_{-L-l}}\|^2_{L^2},$$

where we have used (2.15). This shows the coercivity inequality

$$- \text{Im}\, \beta_\delta(u,u) \geq \gamma(\delta)\|u\|^2_{L^2(\Omega_{-L-l,R+l})} \tag{4.5}$$

for $\gamma(\delta) := \min\{\delta, \gamma/l\} > 0$. Let us now consider the real part of $\beta_\delta(u,u)$. From (4.4) with $v = u$ we obtain, for a constant $C_0 > 0$ that depends only on λ, Λ and ω,

$$\text{Re}\, \beta_\delta(u,u) \geq \lambda\|\nabla u\|^2_{L^2(\Omega_{-L,R})} - C_0\left(\|u\|^2_{L^2(\Omega_{-L-l,R+l})} + \|u|_{W_R}\|^2_{H^1} + \|u|_{W_{-L-l}}\|^2_{H^1}\right)$$
$$\geq \lambda\|\nabla u\|^2_{L^2(\Omega_{-L,R})} - C\|u\|^2_{L^2(\Omega_{-L-l,R+l})}, \tag{4.6}$$

where we used the definitions of Q^{right} and Q^{left} in the first inequality and the regularity property $\|\phi|_{W_0}\|_{H^1} \leq C\|\phi|_{W_0}\|_{L^2}$ for $\phi \in X$ of (2.21) in the second inequality; for the latter we recall that, by definition of the space $V_{L,R}$, the functions $u|_{W_R}$ and $u|_{W_{-L-l}}$ are in the spaces X^{left} and X^{right}.

We next calculate, using first (4.6) and $C_1 = C_1(R, L, C_0, \lambda)$, then once more the regularity property of (2.21) and a corresponding constant $C_2 = C_2(C_1)$, and finally (4.5):

$$\lambda\|u\|^2_{H^1(\Omega_{-L-l,R+l})}$$
$$\leq C_1\left(\text{Re}\, \beta_\delta(u,u) + \|u\|^2_{L^2(\Omega_{-L-l,R+l})} + \|u|_{W_R}\|^2_{H^1} + \|u|_{W_{-L-l}}\|^2_{H^1}\right)$$
$$\leq C_2\left(\text{Re}\, \beta_\delta(u,u) + \|u\|^2_{L^2(\Omega_{-L-l,R+l})}\right)$$
$$\leq C_2\, \text{Re}\, \beta_\delta(u,u) - \frac{C_2}{\gamma(\delta)}\text{Im}\, \beta_\delta(u,u)$$
$$= \text{Re}\left[C_2\left(1 + i\gamma(\delta)^{-1}\right)\beta_\delta(u,u)\right].$$

This inequality ensures coercivity of β_δ on $V_{L,R}$. We can apply the Lax-Milgram lemma and obtain that the equation $\beta_\delta(u_\delta, .) = \langle f, . \rangle$ can be solved with $u_\delta \in V_{L,R}$ for every $\delta > 0$. The solution satisfies

$$\|u_\delta\|_{H^1(\Omega_{-L-l,R+l})} \le C(\delta)\|f\|_{H^{-1}(\Omega)} \,. \tag{4.7}$$

We note that this estimate is not helpful for the limit process $\delta \to 0$ since $C(\delta) \sim 1/\gamma(\delta) \to \infty$ for $\delta \to 0$.

Step 2: Limit $\delta \to 0$. In order to perform the limit, we distinguish two cases. The distinction regards the numbers

$$N_\delta := \|u_\delta\|_{L^2(W_{-L-l,R+l})} \,. \tag{4.8}$$

Case 1: N_δ bounded along a subsequence. If N_δ is bounded along a subsequence, then we choose this subsequence $\delta \to 0$. The sequence u_δ is not only bounded in L^2, but also in H^1; this can be concluded by taking the real part of $\beta_\delta(u_\delta, u_\delta) = \langle f, u_\delta \rangle$, and using (2.21); we refer to the detailed argument in (4.6). We therefore find a limit function u and a further subsequence $\delta \to 0$ such that $u_\delta \rightharpoonup u$ weakly in $H^1(W_{-L-l,R+l})$.

The properties $u|_{W_R} \in X_+^{\text{right}}$ and $u|_{W_{-L-l}} \in X_-^{\text{left}}$ are satisfied, since all u_δ satisfy these properties. This shows $u \in V_{L,R}$. The weak convergence $u_\delta \rightharpoonup u$ is sufficient to take the limit $\delta \to 0$ in the relation $\beta_\delta(u_\delta, \varphi) = \langle f, \varphi \rangle$, and we obtain $\beta(u, \varphi) = \langle f, \varphi \rangle$. This shows that u is a solution of the truncated problem, and the existence statement is shown.

Case 2: $N_\delta \to \infty$. In this case, we study the normalised functions $v_\delta := N_\delta^{-1} u_\delta$. The sequence v_δ has all the properties of u_δ in the first case: The boundedness implies the existence of a limit function v (weak limit in H^1 and strong limit in L^2). Since v_δ solves the truncated problem with $f_\delta = N_\delta^{-1} f$, the limit v solves $\beta(v, \varphi) = \langle 0, \varphi \rangle$. Uniqueness for the truncated problem implies $v = 0$. We find a contradiction since v_δ has L^2-norm 1 and converges strongly to $v = 0$. Case 2 cannot occur. $\qquad\square$

4.2 Radiation conditions for limits

We have constructed a truncated problem that can be solved with standard methods. The test whether or not the truncated problem was chosen appropriately lies in the question: *Do limits of solutions to the truncated problem satisfy the radiation condition?*

In the proof below, we use a flux equality which was, at least formally, not yet observed: Differences also satisfies a flux equality. We speak here of differences of two solutions, one to the truncated problem and one to the original problem. The proof is along the same lines as in the other flux equalities, and we therefore moved this elementary observation to the appendix, see Lemma A.1.

Proposition 4.5 (Radiation conditions for limits). *Let the setting be either that of Theorem 1.1 or that of Theorem 1.2. For sequences $R_k, L_k \to \infty$, let u_k be a sequence of solutions to the truncated problems with right hand side f, we assume that the sequence $\sup\{\|u_k|_{W_r}\|_{L^2(W_r)} \mid r \in \mathbb{Z}, -L_k - l \le r \le R_k\}$ is bounded. Let $u \in H^1_{\text{loc}}(\Omega)$ be locally the weak H^1-limit of the solutions u_k. Then u satisfies the radiation conditions.*

Proof. In this proof, we treat only the right boundary and the left boundary can be treated analogously. Correspondingly, we only use the notation Q and $\mathcal{Q}$ of the setting of Theorem 1.1. In the setting of Theorem 1.2, we should everywhere write Q^{right} and $\mathcal{Q}^{\text{right}}$.

We suppress the subscript k in the following and write R and L instead of R_k and L_k. As solutions to the truncated problems, the functions u_k satisfy $u_k \in V_{L,R}$, in particular $u_k \in H^1(\Omega_{-L-l,R+l})$, and $\beta(u_k, v) = \langle f, v \rangle$ for every $v \in V_{L,R}$. It is clear that the local limit u solves the Helmholtz equation with source term f. Our aim is to verify the radiation condition.

A crucial step will be to derive the following property. Let $(r_k)_k$ be a sequence in $\mathbb{N}$ such that $r_k \to \infty$ with $R_k - r_k \to \infty$. We will suppress the subscript k also in the sequence r_k and claim that there holds, in $L^2(W_r)$,

$$\Pi_-(u_k|_{W_r}) \to 0. \tag{4.9}$$

Step 1: Verification of (4.9). We choose an error quantifier $\eta > 0$.

As in other proofs, we use $\phi(x) := \sum_j \alpha_j \phi_j^+(x)$ with $\Pi_+((u_k - \phi)|_{W_r}) = 0$ (ϕ is the projection onto the right-going part). We subtract this function from u_k and consider in the following $w_k := u_k - \phi$. We use the flux equality of Lemma A.1 with the positions r and R; this is possible for large k since f has compact support and r has the property $r \to \infty$. Together with the positivity of (2.15), we have

$$\operatorname{Im} \mathcal{Q}(w_k|_{W_r}) = \operatorname{Im} \mathcal{Q}(w_k|_{W_R}) \geq 0. \tag{4.10}$$

We used that both u_k and ϕ (and hence w) are right-going waves in W_R.

We now study $w_k|_{W_r}$ and the left hand side of (4.10). Because of $r \to \infty$ and $R - r \to \infty$, the function w_k is a solution of the homogeneous problem on a large domain with centre in r. This allows to use inequality (2.18) with the result that $\left\| w_k|_{W_r} - \Pi_+(w_k|_{W_r}) - \Pi_-(w_k|_{W_r}) \right\|_{H^1} \leq \eta$ for all $k \geq k_0(\eta)$. We observe that $\Pi_+(w_k|_{W_r}) = \Pi_+(u_k|_{W_r}) - \Pi_+(\phi|_{W_r}) = 0$ vanishes. We are therefore in the situation that $\left\| w_k|_{W_r} - \Pi_-(w_k|_{W_r}) \right\|_{H^1} \leq \eta$ is small, $w_k|_{W_r}$ is close to a left-going wave.

After these preparations, we next obtain a further smallness condition. The subsequent calculation starts with the positivity of the quadratic form, obtained in (4.10). The equality is a trivial insertion of the projection of w_k. The first inequality of the second line exploits the definition of $\mathcal{Q}$ as an integral and the η-smallness of the error term; we assume here $\eta \leq 1$ in order to absorb the quadratic term into the linear term. The estimate (2.15) for $\mathcal{Q}$ on X_- is used in the last inequality.

$$0 \leq \operatorname{Im} \mathcal{Q}(w_k|_{W_r}) = \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_r}) + [w_k|_{W_r} - \Pi_-(w_k|_{W_r})])$$

$$\leq \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_r})) + \frac{C\Lambda}{l}\eta \leq -\frac{\gamma}{l}\|\Pi_-(w_k|_{W_r})\|_{L^2}^2 + \frac{C\Lambda}{l}\eta.$$

The constant depends, among others, on the bound for u_k. The calculation provides the smallness

$$\|\Pi_-(u_k|_{W_r})\|_{L^2}^2 = \|\Pi_-(w_k|_{W_r})\|_{L^2}^2 \leq \frac{C\Lambda}{\gamma}\eta.$$

Since $\eta > 0$ was arbitrary, this provides the claim of (4.9).

Step 2: The radiation condition for u. We fix a sequence $r_m \to \infty$ as $m \to \infty$. We will verify the radiation condition along this sequence.

Given the sequence $(r_m)_m$, we can choose a subsequence of indices $(k_m)_m$ with $k_m \to \infty$ as $m \to \infty$, satisfying the following properties: (a) $L_k, R_k \to \infty$ and $R_k - r_k \to \infty$ along the subsequence $k = k_m$. (b) $\|(u_k - u)|_{W_r}\|_{L^2(W_r)} \to 0$ as $m \to \infty$ along the sequences $k = k_m$ and $r = r_m$. Property (b) can be satisfied because of the local convergence $u_k \to u$: For fixed r_m, we can choose $k = k_m$ such that the error is smaller than $1/m$.

In the following, we fix the subsequence $(k_m)_m$ and omit the subscript m. The triangle inequality provides

$$\left\| \Pi_-(u|_{W_r}) \right\|_{L^2(W_r)} \le \left\| \Pi_-(u_k|_{W_r}) \right\|_{L^2(W_r)} + \left\| \Pi_-(u|_{W_r}) - \Pi_-(u_k|_{W_r}) \right\|_{L^2(W_r)} .$$

The first term vanishes by (4.9) as $k \to \infty$. The second term is small by choice of the subsequence. This shows the smallness of the left hand side for large k and thus the radiation condition for u. $\qquad \square$

4.3 Limits of normalised solutions

We continue our investigation of limits of solutions to truncated problems. In the last section, we have seen that limits satisfy the radiation condition. In this section, we consider normalised solution sequences; our goal is to show that the limit of a normalised solution sequence is necessarily nontrivial. Technically, the lemma is used to show, with a contradiction argument, the uniqueness (and hence the existence) of solutions to truncated problems, see Corollary 4.7 below.

Lemma 4.6 (Limits of solutions to homogeneous truncated problems).

Let the setting be either that of Theorem 1.1 or that of Theorem 1.2. Let $u_k : \Omega \to \mathbb{C}$ be a normalised sequence of solutions to the truncated problems with L_k, R_k and $f = 0$ that converges locally:

$$\sup_\rho \|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1 , \quad u_k \to u \text{ in } H^1((-l_0, l_0) \times S) \text{ for every } l_0 > 0 , \qquad (4.11)$$

the supremum is taken over all integers ρ with $-L_k - l \le \rho \le R_k$. Then, the limit u is a nontrivial radiating solution to $Au = \omega^2 u$.

Proof. Because of the local convergence, there holds $\|u\|_{sL} \le 1$. Since locally u_k are solutions, also u is a distributional solution to $Au = \omega^2 u$. Proposition 4.5 provides that u satisfies the radiation condition. The important information of Lemma 4.6 is $u \ne 0$. We argue by contradiction and assume $u = 0$. We furthermore select a sequence $\rho = \rho(k)$ with $\|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1$ (the supremum is a maximum since u_k is defined only on a bounded domain). We have to distinguish three cases.

Case 1: $\rho(k)$ bounded. If there exists a bounded subsequence $\rho(k)$, then a further subsequence is constant: We find $\rho_0 \in \mathbb{Z}$ such that $\rho(k) = \rho_0$ for all k along the new subsequence. Since $u_k \to u$ strongly in L^2 locally, the limit function u satisfies $\|u|_{W_{\rho_0}}\|_{L^2} = 1$, and we find a contradiction to $u = 0$.

Case 2: $\rho(k)$ unbounded. Since Case 1 is excluded, we know $|\rho(k)| \to \infty$. Without loss of generality, we assume that $\rho(k) \to \infty$. Accordingly, in the following, Q and $\mathcal{Q}$ stand for Q^{right} and $\mathcal{Q}^{\text{right}}$.

To prepare the further arguments, we note that u_k can be used as a test function for the truncated problem. Because of $f = 0$ there holds $\beta(u_k, u_k) = \langle f, u_k \rangle = 0$. Taking the imaginary part and exploiting the sign properties of $\mathcal{Q}$ yields $u_k|_{W_{R_k}} = u_k|_{W_{-L_k-l}} = 0$.

Case 2a: Interior points. We assume that $R_k - \rho(k) \to \infty$, which means that the critical point $\rho(k)$ has a large distance to both 0 and R_k.

As in earlier proofs, we consider $\phi \in X_+$, extended as a solution to all of Ω, with $\Pi_+((u_k - \phi)|_{W_\rho}) = 0$. The difference $w_k := u_k - \phi$ satisfies the flux equality of Lemma A.1,

$$\operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(w_k|_{W_{R_k}}). \tag{4.12}$$

The right hand side is positive because of $u_k|_{W_{R_k}}, \phi|_{W_{R_k}} \in X_+$. Actually, $u_k|_{W_{R_k}}$ even vanishes (see the arguments in the beginning of Case 2); hence, only $\phi|_{W_{R_k}}$ remains in the argument. For the left hand side of (4.12), we use that, by choice of ϕ, there holds $\Pi_+(w_k|_{W_\rho}) = 0$; this implies the smallness of $w_k|_{W_\rho} - \Pi_-(w_k|_{W_\rho})$ by Lemma 2.7. The left hand side of (4.12) therefore can be calculated to satisfy $\operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) \leq \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_\rho})) + \eta$ for any small error quantifier $\eta > 0$. Equation (4.12) yields

$$\operatorname{Im} \mathcal{Q}(\phi|_{W_{R_k}}) - \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_\rho})) \leq \eta.$$

We obtain that both $\phi|_{W_{R_k}}$ (and hence $\Pi_+(u_k|_{W_\rho})$) and $\Pi_-(w_k|_{W_\rho}) = \Pi_-(u_k|_{W_\rho})$ are small. Using Lemma 2.7 again, we find a contradiction to $\|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1$.

Case 2b: Large values close to boundaries. For a subsequence, the shifted sequence $\tilde{u}_k$ (shifted by R_k) consists of solutions to the homogeneous problem with vanishing Dirichlet data on $\{0\} \times S$, but not vanishing in some subdomain, $\|\tilde{u}_k|_{W_{-D}}\|_{L^2(W_{-D})} = 1$ for all k and for some $D \in \mathbb{N}$. We find a local limit function $\tilde{u}$. The limit shares the above properties of $\tilde{u}_k$.

For every L_0, the solutions $\tilde{u}_k|_{W_{-R_k+L_0}}$ converge to 0 by assumption (local convergence of u_k to $u = 0$). The usual argument (subtracting the right-going part ϕ from $\tilde{u}_k$) we find that $\tilde{u}_k|_{W_{r_k}}$ converges to 0 for any sequence r_k with $r_k \to \infty$ and $R_k - r_k \to \infty$.

This implies that the limit $\tilde{u}$ vanishes as $x_1 \to -\infty$, but is different from 0 on W_{-D}. With an odd extension of $\tilde{u}$, we see that this is in contradiction to (a) of Definition 2.3. □

We mention that the above proof, in particular regarding the Cases 2a and 2b, has strong similarities with the proof of Lemma 5.1 and its Cases 2a and 2b, see below. Since the arguments are slightly more quantitative in the proof of Lemma 5.1, the reader might want to compare that proof for more details.

4.4 Uniqueness for truncated problems along a subsequence

This section is devoted to a uniqueness result for truncated problems. This is particularly useful since, by Lemma 4.4, uniqueness implies also existence of solutions. The following result improves Remark 4.3 in the direction that we do not have to choose L_k and R_k in an appropriate way to obtain uniqueness, but all sequences will guarantee uniqueness, at least for all sufficiently large indices k.

Corollary 4.7 (Uniqueness for the truncated problem). *Let the setting be either that of Theorem 1.1 or that of Theorem 1.2. Let $\mathbb{N} \ni R_k, L_k \to \infty$ be two sequences. Then there exists $k_0 \in \mathbb{N}$ such that for every pair $(L, R) = (L_k, R_k)$ with $k \geq k_0$ there exists at most one solution to the truncated problem of Definition 4.2. In particular, we also have existence of a solution to the truncated problem for all $k \geq k_0$.*

Proof. We argue by contradiction. The sequences $\mathbb{N} \ni R_k, L_k \to \infty$ are fixed, and we want to obtain the existence of k_0. Assuming that no k_0 with the desired properties exists, there must exist a subsequence $k \to \infty$ such that the truncated problems for parameters (L_k, R_k) possess a nontrivial solution u_k to $f = 0$. We normalise u_k to have $\sup_\rho \|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1$ as in Lemma 4.6. We can extract a further subsequence $k \to \infty$ and a limit function u such that $u_k \to u$ locally in H^1 (at first, one concludes weak convergence in H^1 on every subset of the form $(-l_0, l_0) \times S \subset \Omega$, then, by regularity of solutions, also strong convergence in H^1 on such sets). Lemma 4.6 provides that the limit function u is a nontrivial radiating solution to $Au = \omega^2 u$.

In the situation of Theorem 1.1, Proposition 3.1 yields that the radiation problem on Ω for $f = 0$ has only the trivial solution. This provides the desired contradiction. In the situation of Theorem 1.2, we have excluded the existence of such a function in the assumptions.

In both situations, we find the desired contradiction and conclude that there cannot exist sequences $L_k, R_k \to \infty$ with non-uniqueness.

The existence statement follows from Lemma 4.4. $\qquad\square$

Let us illustrate with an example that Corollary 4.7 improves the observation of Remark 4.3. We consider the elementary case of $A = -\Delta$ in dimension $d = 1$. Every $\omega > 0$ is non-singular in the sense of Definition 2.3: The two basis functions are $\phi_1^\pm = e^{\pm i\omega x}$ and, with $m := \max\{n \in \mathbb{N} | 2\pi n \le \omega\}$, the quasimoments are $\xi_1^+ = \omega - 2\pi m$ and $\xi_1^- = -\xi_1^+$. We check that $\mathcal{Q}(\phi_1^+) = \frac{1}{l} \int_0^l \partial_x \phi_1^+ \overline{\phi_1^+} = i\omega$ and $\mathcal{Q}(\phi_1^-) = -i\omega$.

In the above setting, we study the linear combination $v(x) := \phi_1^+(x) + \phi_1^-(x) = 2\cos(\omega x)$. For $R = L \in (\pi/\omega)(\mathbb{N} + \frac{1}{2})$, the function v solves the Dirichlet problem that was obtained in Remark 4.3. We see that resonances can occur even for large numbers R and L.

Our example makes clear that Corollary 4.7 is a surprising and strong result: Even if the values of R and L allow for some resonance phenomenon, the solution to the truncated problem is nevertheless unique. The point in the above construction is that v is a solution of a Dirichlet problem, but neither v nor any extension of it is a solution to the truncated domain problem.

5 Existence on unbounded domains

The aim of this section is to prove the two main theorems. It remains to derive existence in both settings, which is the content of the next two lemmas. Theorem 1.1 is a consequence of Lemma 5.1, and Theorem 1.2 is a consequence of Lemma 5.2.

Lemma 5.1 (Existence for the periodic problem). *Let the assumptions of Theorem 1.1 be satisfied. Then there exists a solution u to the radiation problem.*

Proof. We use a sequence of solutions u_k to the truncated problems with $L_k, R_k \to \infty$, which exist by Lemma 4.4. We consider the sequence of real numbers

$$N_k := \sup \left\{ \|u_k\|_{L^2(W_\rho)} \big| \rho \in \mathbb{Z}, -L_k - l \le \rho \le R_k \right\}. \tag{5.1}$$

We distinguish two cases.

Case 1: The sequence N_k is bounded. In this case, the sequence u_k is locally bounded in H^1. It therefore possesses (up to choosing a subsequence) a limit u such that $u_k \to u$ in H^1 for subdomains of the form $(-l_0, l_0) \times S \subset \Omega$. As a local limit of solutions,

u satisfies $Au = \omega^2 u + f$ in Ω. With the local limit u, the sequence u_k satisfies all assumptions of Proposition 4.5, which yields that u satisfies the radiation condition. The function u is the desired solution and the existence assertion is shown.

Case 2: Along a subsequence, there holds $N_k \to \infty$.

We choose such a subsequence and assume from now on $N_k \to \infty$. Our aim is to arrive at a contradiction. We study the normalised functions $v_k := N_k^{-1} u_k$. The sequence v_k has all the properties of u_k of Case 1: The local boundedness implies the existence of a local limit function v. Since v_k solves $Av_k = \omega^2 v_k + N_k^{-1} f$ in the sense of the truncated problem, the limit solves $Av = \omega^2 v$ in Ω. Proposition 4.5 implies that v satisfies the radiation condition. Uniqueness for this problem was shown in Proposition 3.1, we therefore obtain $v = 0$.

Another property of v_k is the following. Using v_k as a test function in the equation for v_k (with right hand side $N_k^{-1} f$), taking the imaginary part and exploiting that v_k is locally bounded, we find

$$\left| \operatorname{Im} \mathcal{Q}(v_k|_{W_{-L_k-l}}) - \operatorname{Im} \mathcal{Q}(v_k|_{v_{R_k}}) \right| \leq C_0 N_k^{-1} .$$

Since $v_k|_{W_{R_k}}$ is in X_+ and $v_k|_{W_{-L_k-l}}$ is in X_-, by Corollary 2.6, the two flux expressions have opposite signs. Moreover, the corollary provides bounds for the two arguments, which yields

$$\left\| v_k|_{W_{-L_k-l}} \right\|_{L^2}^2 + \left\| v_k|_{W_{R_k}} \right\|_{L^2}^2 \leq C_1 N_k^{-1} . \tag{5.2}$$

The definition of N_k implies that there is a position $\rho = \rho(k) \in \mathbb{Z}$ such that $\|v_k\|_{L^2(W_\rho)} \geq 1/2$. We observe that there holds $|\rho(k)| \to \infty$. Indeed, in the opposite case, we find a number $\rho_0 \in \mathbb{Z}$ and a constant subsequence, $\rho(k) = \rho_0$ along the subsequence. This is in contradiction with the local convergence $v_k|_{W_{\rho_0}} \to v|_{W_{\rho_0}} = 0$. We distinguish once more two cases.

Case 2a: Interior points. The first case is that a sequence $\rho = \rho(k)$ can be found with $R_k - \rho(k) \to \infty$ and $\rho(k) - (-L_k) \to \infty$.

We argue as in Proposition 4.5. We fix an error quantifier $\eta > 0$. From the function v_k, we want to subtract the right-going part in W_ρ: We consider $\phi \in X_+$, extended as a solution to all of Ω, with $\Pi_+((v_k - \phi)|_{W_\rho}) = 0$ (loosely speaking, $\phi = \Pi_+(v_k|_{W_\rho})$). We study the difference $w_k := v_k - \phi$. The function w_k satisfies $Aw_k = \omega^2 w_k + N_k^{-1} f$, and hence a flux equality as in Lemma A.1, now with an error term introduced by the right hand side $N_k^{-1} f$. The flux inequality is obtained by testing the equation for w_k with the (locally bounded) function w_k, we hence find

$$\left| \operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) - \operatorname{Im} \mathcal{Q}(w_k|_{W_{R_k}}) \right| \leq C N_k^{-1} . \tag{5.3}$$

We note that $\operatorname{Im} \mathcal{Q}(w_k|_{W_{R_k}}) \geq 0$ holds since $w_k|_{W_{R_k}}$ is in X_+.

Regarding the first flux term, we note that $w_k|_{W_\rho} - \Pi_-(w_k|_{W_\rho}) = w_k|_{W_\rho} - \Pi(w_k|_{W_\rho})$ is small because of the fact that $w_k|_{W_\rho}$ is close to the subspace X by Lemma 2.7 (we exploit here $|\rho(k)| \to \infty$ and $R_k - \rho(k) \to \infty$ and $\rho(k) + L_k \to \infty$).

This allows to evaluate (up to small error) the first term in (5.3). For arbitrary $\eta > 0$, exploiting the sign property of Corollary 2.6, we obtain

$$-C N_k^{-1} \leq \operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) \leq \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_\rho})) + \eta \leq -\frac{\gamma}{l} \|\Pi_-(w_k|_{W_\rho})\|_{L^2}^2 + \eta$$

$$= -\frac{\gamma}{l} \|\Pi_-(v_k|_{W_\rho})\|_{L^2}^2 + \eta$$

for k sufficiently large. We therefore have the smallness

$$\|\Pi_-(v_k|_{W_\rho})\|_{L^2}^2 \le \frac{l}{\gamma}(CN_k^{-1} + \eta) \tag{5.4}$$

for k sufficiently large. With the same arguments, exchanging Π_+ with Π_-, we find the same estimate for $\Pi_+(v_k|_{W_\rho})$. Invoking Lemma 2.7 once more (which is possible since $\rho(k)$ has an increasing distance to boundary points), we know that $\|v_k|_{W_\rho} - \Pi(v_k|_{W_\rho})\|_{L^2}$ is small. This is in contradiction with the normalisation $\|v_k\|_{L^2(W_\rho)} \ge 1/2$. We conclude that Case 2a cannot occur.

Case 2b: Large values near boundaries. It remains to treat the case that we cannot find points $\rho = \rho(k)$ with $R_k - \rho(k) \to \infty$ and $\rho(k) + L_k \to \infty$ satisfying $\|v_k\|_{L^2(W_\rho)} \ge 1/2$. In this case, along the sequence, the distance of $\rho(k)$ to a boundary point remains bounded. Without loss of generality, let this be R_k; hence, $R_k - \rho(k) \ge 0$ remains bounded. We can select a constant subsequence: Without loss of generality, we can assume for $D \in \mathbb{N}$ that $R_k - \rho(k) = D$ for all k.

We consider shifted versions of the sequence v_k, defined by $\tilde{v}_k(x) = v_k(x + R_k e_1)$. The functions $\tilde{v}_k$ are defined on domains $\Omega_{-L_k - R_k - l, l}$ and have the following properties. In the subdomain W_0, the solution $\tilde{v}_k$ is outgoing, but even more is true: By (5.2), $\tilde{v}_k|_{W_0}$ is vanishing in L^2 as $k \to \infty$. In contrast, the L^2-norm in the subdomain W_{-D} is bounded from below by a positive number by the choice of ρ. For arbitrary $L_0 \in \mathbb{Z}$, in the domains $W_{-R_k + L_0}$, the solutions $\tilde{v}_k$ converge to 0 by local convergence of v_k to $v = 0$. As in Case 2a regarding interior points, we can conclude that v_k is small on any sequence of domains W_σ with $\sigma = \sigma(k)$ satisfying $\mathbb{Z} \ni \sigma(k) \to -\infty$ and $\sigma(k) + R_k \to \infty$.

The local boundedness of the functions $\tilde{v}_k$ allows to find a local limit $\tilde{v}$. The limit function solves the homogeneous problem $A\tilde{v} = \omega^2 \tilde{v}$ in $\Omega_{-\infty, 0}$ and has vanishing Dirichlet data on $\{0\} \times S$. We can extend $\tilde{v}$ as an odd function to all of Ω. This provides a solution to the homogeneous problem in all of Ω, vanishing at $x_1 \to \pm\infty$, but different from 0 on W_{-D}. This is a contradiction to (a) of Definition 2.3. □

Lemma 5.2 (Existence for the problem with periodicity at infinity). *Let the assumptions of Theorem 1.2 be satisfied. Then there exists a solution u to the radiation problem.*

Proof. We choose two sequences $L_k, R_k \to \infty$. By Corollary 4.7, we can select a subsequence such that the truncated problems possess solutions. This provides a sequence of solutions u_k to truncated problems. The same arguments as in Lemma 5.1 provide that the approximate solutions are necessarily sL-bounded. This yields that, locally, a limit u exists. As a limit, it also satisfies $Au = \omega^2 u + f$ in Ω.

The radiation condition for the limit u is shown in Proposition 4.5. □

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A Another form of a flux equality

The flux equality is the central tool of our approach to existence results. It was formulated in Lemma 2.1 for solutions, but it actually holds also for solutions to the truncated problem. Here, we will show something more general: The difference of a solution and a solution of the truncated problem also satisfies the flux equality.

Lemma A.1 (Flux equality). *Let u be a solution to the truncated problem of Definition 4.2 to parameters $R, L \in \mathbb{N}$. Let $\phi \in X_+$ be extended to a quasiperiodic solution of $A\phi = \omega^2\phi$ on Ω. Then, for every $\rho \in \mathbb{N}$, $-L \le \rho \le R - l$, the difference $w = u - \phi$ satisfies the flux equality*

$$\operatorname{Im} \mathcal{Q}(w|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(w|_{W_R}). \tag{A.1}$$

Proof. As a solution of the truncated problem, the function $u \in V_{L,R}$ satisfies, with the cut-off function ϑ corresponding to the four points $(-L - l, -L, R, R + l)$,

$$0 = \beta(u, v) = \int_{\Omega_{-L-l, R+l}} a\nabla u \cdot \nabla \bar{v}\vartheta - \int_{\Omega_{-L-l, R+l}} \omega^2 u\bar{v}\vartheta$$
$$- Q(u|_{W_R}, v|_{W_R}) + Q(u|_{W_{-L-l}}, v|_{W_{-L-l}})$$

for every $v \in V_{L,R}$.

We introduce the cut-off function θ corresponding to the points $(\rho, \rho + l, R + l, R + 2l)$. We claim that $v := w\theta \in V_{L,R}$. Indeed, both u and ϕ are in X_+ on the right radiation box W_R; hence, also w is ($\theta = 1$ in W_R). In the left radiation box W_{-L-l}, the function θ vanishes; hence, $v := w\theta$ is trivially in X_-.

Due to these considerations, we can use $v = w\theta$ as a test function. We note that the product of cut-off functions provides a new cut-off function: $\vartheta_\rho := \theta\vartheta$ is the piecewise affine cut-off function which corresponds to the four points $(\rho, \rho + l, R, R + l)$. Inserting $v = w\theta$ above yields

$$0 = \int_{\Omega_{\rho, R+l}} a\nabla u \cdot \nabla \bar{w}\vartheta_\rho + \frac{1}{l}\int_{W_\rho} a\nabla u \cdot e_1 \bar{w} - \int_{\Omega_{\rho, R+l}} \omega^2 u\bar{w}\vartheta_\rho - Q(u|_{W_R}, w|_{W_R})$$
$$= \int_{\Omega_{\rho, R+l}} a\nabla u \cdot \nabla \bar{w}\vartheta_\rho - \int_{\Omega_{\rho, R+l}} \omega^2 u\bar{w}\vartheta_\rho - Q(u|_{W_R}, w|_{W_R}) + Q(u|_{W_\rho}, w|_{W_\rho}).$$

Regarding the solution ϕ of $A\phi = \omega^2\phi$, we can proceed as in Lemma 2.1. The equation for ϕ is multiplied with $\bar{v} = \vartheta_\rho \bar{w}$ and integrated. We find essentially the same expressions as above,

$$0 = \int_{\Omega_{\rho, R+l}} a\nabla\phi \cdot \nabla \bar{w}\vartheta_\rho - \int_{\Omega_{\rho, R+l}} \omega^2 \phi\bar{w}\vartheta_\rho - Q(\phi|_{W_R}, w|_{W_R}) + Q(\phi|_{W_\rho}, w|_{W_\rho}).$$

We can now subtract the relation for ϕ from the relation for u and obtain

$$0 = \int_{\Omega_{\rho, R+l}} a\nabla w \cdot \nabla \bar{w}\vartheta_\rho - \int_{\Omega_{\rho, R+l}} \omega^2 w\bar{w}\vartheta_\rho - Q(w|_{W_R}, w|_{W_R}) + Q(w|_{W_\rho}, w|_{W_\rho}).$$

Taking the imaginary part, we find the flux equality (A.1). □