

SOME MULTIVARIABLE GAUSSIAN HYPERGEOMETRIC EXTENSIONS OF THE PREECE THEOREM

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Abstract

Some generalisations of the Preece theorem involving the product of two Kummer's functions ${}_1F_1$ are obtained using Dixon's theorem and some well-known identities. Its special cases yield various new transformations and reduction formulae involving Pathan's quadruple hypergeometric function $F_p^{(4)}$ and Srivastava's quadruple hypergeometric function $F^{(4)}$ and triple hypergeometric function $F^{(3)}$. Some known results of Preece, Pathan and Bailey are also obtained as special cases.

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1. Introduction

Let (a_A) denote the sequence of A parameters given by a_1, \dots, a_A and $[(a_A)]_n$ denote the product of A Pochhammer symbols defined by

$$(b)_n = \frac{\Gamma(b+n)}{\Gamma(b)} = \begin{cases} 1, & \text{if } n = 0, \\ b(b+1) \cdots (b+n-1), & \text{if } n = 1, 2, \dots \end{cases}$$

In 1969, Srivastava and Daoust ([11, page 454], see also [12, page 37 (21,22)])

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gave the following multivariable hypergeometric function:

$$\begin{aligned}
 &F_{D:E^{(1)};\dots;E^{(n)}}^{A:B^{(1)};\dots;B^{(n)}} \left[[(a_A) : \theta^{(1)}, \dots, \theta^{(n)}] : [(b_{B^{(1)}}^{(1)}) : \Phi^{(1)}]; \dots; [(b_{B^{(n)}}^{(n)}) : \Phi^{(n)}] \right. \\
 &\quad \left. [(d_D) : \Psi^{(1)}, \dots, \Psi^{(n)}] : [(e_{E^{(1)}}^{(1)}) : \delta^{(1)}]; \dots; [(e_{E^{(n)}}^{(n)}) : \delta^{(n)}] \right. \quad z_1, \dots, z_n \Big] \\
 &= \sum_{m_1, \dots, m_n=0}^{\infty} \Xi(\mathbf{m}_1, \dots, \mathbf{m}_n) \frac{z_1^{m_1}}{(m_1)!} \dots \frac{z_n^{m_n}}{(m_n)!},
 \end{aligned}$$

where, for convenience,

$$\Xi(\mathbf{m}_1, \dots, \mathbf{m}_n) = \frac{\prod_{j=1}^A (a_j)_{m_1 \theta_j^{(1)} + \dots + m_n \theta_j^{(n)}} \prod_{j=1}^{B^{(1)}} (b_j^{(1)})_{m_1 \Phi_j^{(1)}} \dots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n \Phi_j^{(n)}}}{\prod_{j=1}^D (d_j)_{m_1 \Psi_j^{(1)} + \dots + m_n \Psi_j^{(n)}} \prod_{j=1}^{E^{(1)}} (e_j^{(1)})_{m_1 \delta_j^{(1)}} \dots \prod_{j=1}^{E^{(n)}} (e_j^{(n)})_{m_n \delta_j^{(n)}}},$$

the coefficients $\theta_j^{(k)}$, $j = 1, \dots, A$; $\Phi_j^{(k)}$, $j = 1, \dots, B^{(k)}$; $\Psi_j^{(k)}$, $j = 1, \dots, D$; $\delta_j^{(k)}$, $j = 1, \dots, E^{(k)}$; for all $k \in \{1, \dots, n\}$ are zero and real constants (positive, negative) [12, pages 270–272 (Equations 5–9, 19–21)] and $(b_{B^{(k)}}^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j = 1, \dots, B^{(k)}$; for all $k \in \{1, \dots, n\}$, with similar interpretations for others.

Preece [7, page 378 (11)] gave the following formula for the product of two hypergeometric functions:

$$\begin{aligned}
 &{}_1F_1 \left[\begin{matrix} c \\ f \end{matrix} \middle| y \right] {}_1F_1 \left[\begin{matrix} 1 - f + c & - \\ 2 - f & -y \end{matrix} \right] \\
 &= {}_2F_3 \left[\begin{matrix} 1/2 + f/2 - c, 1/2 - f/2 + c \\ f/2 + 1/2, 3/2 - f/2, 1/2 \end{matrix} \middle| y^2/4 \right] \\
 &\quad + \frac{(f - 1)(f - 2c)y}{f(2 - f)} {}_2F_3 \left[\begin{matrix} 1 - c + f/2, 1 + c - f/2 \\ 2 - f/2, 1 + f/2, 3/2 \end{matrix} \middle| y^2/4 \right].
 \end{aligned}$$

The present paper is devoted to the investigation of general multiple series identities which extend and generalise the theorems of Bailey [1, page 239 (4.6)], Pathan [5, page 1115 (3.2)] and Preece [7, page 378 (11)]. These theorems given in Section 2 will be seen to be extremely useful, in that most properties of hypergeometric series carry over naturally and simply for these identities and provide connections with various classes of well-known hypergeometric functions and even new representations for special cases of these functions. Some applications of these theorems are given in Section 3. Clearly, the same procedure could have been utilised to extend many more results on hypergeometric functions. But, instead, we deduce some special cases in Section 4.

2. Multivariable analogues of the Preece theorem

Motivated by the works of Bailey, Pathan and Preece, we shall establish the following theorems for multiple series, which are more generalised than the multiple

Gaussian hypergeometric functions $F^{(3)}$, $F^{(4)}$ and $F_p^{(4)}$.

THEOREMS. Let $S(i, j, k, p)$ be the generalised coefficients of arbitrary complex numbers, where x, y, z are complex variables and c, f are arbitrary independent complex parameters (where $2f \neq 0, -1, \pm 2, \pm 3, \dots$) and any values of numerator and denominator parameters and variables x, y, z leading to results which do not make sense are tacitly excluded, then

$$\sum_{i,j,k,p=0}^{\infty} S(i+k, j, p) \frac{(-1)^k (c-2f+1)_i (c)_k x^j y^{i+k} z^p}{(2-2f)_i (2f)_k i! j! k! p!}$$

$$= \sum_{t=0}^1 \frac{y^t}{t!} \Omega(t) \sum_{i,j,p=0}^{\infty} S(2i+t, j, p) \Lambda(t, i) \frac{(y^2/4)^i x^j z^p}{j! p!} \tag{2.1}$$

$$= \sum_{t,u=0}^1 \frac{y^t x^u}{t! u!} \Omega(t) \sum_{i,j,p=0}^{\infty} S(2i+t, 2j+u, p) \Lambda(t, i)$$

$$\times \frac{(y^2/4)^i (x^2/4)^j z^p}{((1+u)/2)_j (1+u/2)_j p!} \tag{2.2}$$

$$= \sum_{t,u,w=0}^1 \frac{y^t x^u z^w}{t! u! w!} \Omega(t) \sum_{i,j,p=0}^{\infty} S(2i+t, 2j+u, 2p+w) \Lambda(t, i)$$

$$\times \frac{(y^2/4)^i (x^2/4)^j (z^2/4)^p}{((1+u)/2)_j (1+u/2)_j ((1+w)/2)_p (1+w/2)_p}, \tag{2.3}$$

where, for convenience,

$$\Omega(t) = \frac{\Gamma(2f)\Gamma\left(\frac{1+2f-t}{2}\right)\Gamma\left(\frac{1+2f-2c+t}{2}\right)\Gamma(2f-c-t)(c-2f+1)_t}{\Gamma(2f-c)\Gamma\left(\frac{1+2f+t}{2}\right)\Gamma\left(\frac{1+2f-2c-t}{2}\right)\Gamma(2f-t)(2-2f)_t}, \tag{2.4}$$

$$\Lambda(t, i) = \frac{\left(\frac{1+2f-2c+t}{2}\right)_i \left(\frac{1-2f+2c+t}{2}\right)_i}{\left(\frac{3-2f+t}{2}\right)_i \left(\frac{1+2f+t}{2}\right)_i \left(\frac{1+t}{2}\right)_i \left(\frac{2+t}{2}\right)_i},$$

provided that each multiple series involved converges absolutely.

PROOF OF THEOREMS (2.1)–(2.3). Let L denote the left-hand side of (2.1). Then using the series identity [8, page 56, Lemma 10(1)] defined as

$$\sum_{i,j,k,p=0}^{\infty} A(i, j, k, p) = \sum_{i,j,p=0}^{\infty} \sum_{k=0}^i A(i-k, j, k, p),$$

we may write

$$L = \sum_{i,j,p=0}^{\infty} S(i, j, p) \frac{(c-2f+1)_i y^i x^j z^p}{(2-2f)_i i! j! p!} {}_3F_2 \left[\begin{matrix} 2f-1-i, -i, c \\ 2f, 2f-c-i \end{matrix} \middle| 1 \right]. \tag{2.5}$$

Using Dixon’s theorem [8, page 92, Theorem 33] in (2.5), we may then write

$$L = \sum_{i,j,p=0}^{\infty} S(i, j, p) \frac{(c - 2f + 1)_i y^i x^j z^p}{(2 - 2f)_i i! j! p!} \times \frac{\Gamma(2f)\Gamma\left(\frac{1+2f-i}{2}\right)\Gamma\left(\frac{1+2f-2c+i}{2}\right)\Gamma(2f - c - i)}{\Gamma\left(\frac{1+2f-2c-i}{2}\right)\Gamma\left(\frac{1+2f+i}{2}\right)\Gamma(2f - i)\Gamma(2f - c)}. \tag{2.6}$$

Now applying the following identities [13, page 214 (8), page 217 (12)]:

$$\begin{aligned} \sum_{i=0}^{\infty} A(i) &= \sum_{t=0}^1 \sum_{i=0}^{\infty} A(2i + t), \\ \sum_{i,j=0}^{\infty} B(i, j) &= \sum_{t,u=0}^1 \sum_{i,j=0}^{\infty} B(2i + t, 2j + u), \\ \sum_{i,j,p=0}^{\infty} C(i, j, p) &= \sum_{t,u,w=0}^1 \sum_{i,j,p=0}^{\infty} C(2i + t, 2j + u, 2p + w) \end{aligned}$$

in (2.6), then using the series rearrangement technique, we get the right-hand sides of (2.1)–(2.3) respectively. □

3. Applications of Theorems (2.1)–(2.3)

3.1. The Preece Theorem associated with Srivastava’s function $F^{(3)}$. In Theorems (2.1) and (2.2), setting

$$S(i, j, k) = \frac{[(a_A)]_{i+j+k} [(d_D)]_{i+k} [(g_G)]_j}{[(b_B)]_{i+j+k} [(e_E)]_{i+k} [(h_H)]_j}$$

and $z = 0$ and using the series rearrangement technique, we get

$$\begin{aligned} &F^{(3)} \left[\begin{matrix} (a_A) :: -; (d_D); - : (g_G); c - 2f + 1; c; \\ (b_B) :: -; (e_E); - : (h_H); 2 - 2f; 2f; \end{matrix} \quad x, y, -y \right] \\ &= \sum_{t=0}^1 \frac{[(a_A)]_t [(d_D)]_t y^t}{[(b_B)]_t [(e_E)]_t t!} \Omega(t) \\ &\times F_{B:2E+3;H}^{A:2D+2;G} \left[\begin{matrix} [(a_A) + t : 2, 1] : [\Delta(2; (d_D + t) : 1)], [\frac{1+t}{2} + f - c : 1], \\ [(b_B) + t : 2, 1] : [\Delta(2; (e_E + t) : 1)], [\frac{3+t}{2} - f : 1], \\ [\frac{1+t}{2} - f + c : 1]; [(g_G) : 1]; \\ [\frac{1+t}{2} + f - c : 1], [\frac{1+t}{2} : 1]; [(h_H) : 1]; \end{matrix} \quad y^2/4^{1+E-D}, x \right] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{t,u=0}^1 \frac{[(a_A)]_{t+u} [(d_D)]_t [(g_G)]_u y^t x^u}{[(b_B)]_{t+u} [(e_E)]_t [(h_H)]_u t! u!} \Omega(t) \\
 &\times F_{2B:2E+3;2H+1}^{2A:2D+2;2G} \left[\begin{array}{l} \Delta[2; (a_A) + t + u]; \frac{1+t}{2} + f - c, \frac{1+t}{2} - f + c, \\ \Delta[2; (b_B) + t + u]; \frac{3}{2} - f + \frac{t}{2}, \frac{1}{2} + f + \frac{t}{2}, \Delta^*(2; 1 + t), \\ \Delta[2; (d_D) + t]; \Delta[2; (g_G) + u]; \frac{4^{A+D} y^2}{4^{1+B+E}}, \frac{4^{A+G} x^2}{4^{1+B+H}} \end{array} \right], \\
 &\Delta[2; (e_E) + t]; \Delta^*(2; 1 + u), \Delta[2; (h_H) + u];
 \end{aligned}$$

provided that the denominator parameters are neither zero nor negative integers and for convenience, the symbol $\Delta(m; b)$ abbreviates the array of m parameters given by

$$\frac{b}{m}, \frac{(b + 1)}{m}, \frac{(b + 2)}{m}, \dots, \frac{(b + m - 1)}{m},$$

where $m = 1, 2, 3, \dots$

The asterisk in $\Delta^*(N; j + 1)$ represents the fact that the (denominator) parameter N/N is always omitted for $0 \leq j \leq (N - 1)$, so that the set $\Delta^*(N; j + 1)$ obviously contains only $N - 1$ parameters [13, page 214].

The notation $\Delta[N; (b_B)]$ denotes the array of BN parameters [13, page 47(8), pages 193–194] given by $\Delta(N; b_1), \dots, \Delta(N; b_B)$ and $\Omega(t)$ is given by (2.4).

3.2. The Preece theorem associated with Srivastava’s function $F^{(4)}$. In Theorem (2.3), setting

$$S(i, j, k, p) = \frac{[(a_A)]_{i+j+k+p} [(m_M)]_{j+p} [(d_D)]_{i+k} [(g_G)]_j [(q_Q)]_p}{[(b_B)]_{i+j+k+p} [(n_N)]_{j+p} [(e_E)]_{i+k} [(h_H)]_j [(r_R)]_p},$$

we get

$$\begin{aligned}
 &F^{(4)} \left[\begin{array}{l} (a_A) :: c - 2f + 1; (d_D); (g_G); (m_M) : c; (d_D); (q_Q); (m_M); \\ (b_B) :: 2 - 2f; (e_E); (h_H); (n_N) : 2f; (e_E); (r_R); (n_N); \end{array} y, x, -y, z \right] \\
 &= \sum_{t,u,w=0}^1 \frac{[(a_A)]_{t+u+w} [(m_M)]_{u+w} [(d_D)]_t [(g_G)]_u [(q_Q)]_w y^t x^u z^w}{[(b_B)]_{t+u+w} [(n_N)]_{u+w} [(e_E)]_t [(h_H)]_u [(r_R)]_w t! u! w!} \Omega(t) \\
 &\times F^{(3)} \left[\begin{array}{l} \Delta[2; (a_A) + t + u + w] :: -; \Delta[2; (m_M) + u + w]; - : \\ \Delta[2; (b_B) + t + u + w] :: -; \Delta[2; (n_N) + u + w]; - : \\ \frac{1+t}{2} + f - c, \frac{1+t}{2} - f + c, \Delta[2; (d_D) + t]; \\ \frac{3+t}{2} - f, \frac{1+t}{2} + f, \Delta^*(2; 1 + t), \Delta[2; (e_E) + t]; \\ \Delta[2; (g_G) + u]; \Delta[2; (q_Q) + w]; \\ \Delta^*(2; 1 + u), \Delta[2; (h_H) + u]; \Delta^*(2; 1 + w), \Delta[2; (r_R) + w]; \\ \frac{4^{A+D} y^2}{4^{1+B+E}}, \frac{4^{A+G+M} x^2}{4^{1+B+H+N}}, \frac{4^{A+M+Q} z^2}{4^{1+B+N+R}} \end{array} \right]. \tag{3.1}
 \end{aligned}$$

$$\Delta(2; h + u); \Delta^*(2; 1 + w), \Delta(2; r + w); y^2, x^2, z^2 \Big].$$

(II) Setting $A = N = Q = G = 1, a_1 = a, n_1 = n, g_1 = g, q_1 = s$ and $B = D = E = M = H = R = 0$ in (3.1), we get

$$\begin{aligned} &K_{13}[a, a, a, a; g, s, c - 2f + 1, c; n, n, 2 - 2f, 2f; x, z, y, -y] \\ &= \sum_{t,u,w=0}^1 \frac{(a)_{t+u+w}(g)_u(s)_w y^t x^u z^w}{(n)_{u+w} t! u! w!} \Omega(t) \\ &\times F^{(3)} \left[\begin{array}{c} \Delta(2; a + t + u + w) :: -; -; \text{---}; \Delta(2; g + u) ; \\ \text{---} :: -; -; \Delta(2; n + u + w); \Delta^*(2; 1 + u) ; \\ \frac{1}{2} + f - c + \frac{t}{2}, \frac{1}{2} - f + c + \frac{t}{2} ; \Delta(2; s + w) ; \\ \frac{3}{2} - f + \frac{t}{2}, \frac{1}{2} + f + \frac{t}{2}, \Delta^*(2; 1 + t); \Delta^*(2; 1 + w); \end{array} ; x^2, y^2, z^2 \right]. \end{aligned}$$

(III) Setting $A = M = N = Q = R = 1, a_1 = a, m_1 = m, n_1 = n, q_1 = q, r_1 = r$ and $B = D = E = G = H = 0$ in (3.2), we get

$$\begin{aligned} &F_A^{(4)}[a, c, q, m, c - 2f + 1; 2f, r, n, 2 - 2f; -y, z, x, y] \\ &= \sum_{t,u,w=0}^1 \frac{(a)_{t+u+w}(m)_u(q)_w y^t x^u z^w}{(n)_u (r)_w t! u! w!} \Omega(t) \\ &\times F^{(3)} \left[\begin{array}{c} \Delta(2; a + t + u + w) :: -; -; -; \Delta(2; m + u) ; \\ \text{---} :: -; -; -; \Delta^*(2; 1 + u), \Delta(2; n + u) ; \\ \Delta(2; q + w) ; \\ \Delta^*(2; 1 + w), \Delta(2; r + w) ; \\ \frac{1+t}{2} + f - c, \frac{1+t}{2} - f + c ; \\ \frac{3+t}{2} - f, \frac{1+t}{2} + f, \Delta^*(2; 1 + t); \end{array} ; x^2, z^2, y^2 \right]. \end{aligned}$$

(IV) In (2.1), setting $S(i, j, k) = (a)_{2j+i+k}/(b)_j$ and $z = 0$, we get

$$\begin{aligned} &X_8[a, c - 2f + 1, c; b, 2 - 2f, 2f; x, y, -y] \\ &= \sum_{t=0}^1 \frac{y^t (a)_t}{t!} \Omega(t) \\ &\times F_{0;3;1}^{2;2;0} \left[\begin{array}{c} \Delta(2; a + t) : \frac{1+t}{2} + f - c, \frac{1+t}{2} - f + c ; -; \\ \text{---} : \frac{3+t}{2} - f, \frac{1+t}{2} + f, \Delta^*(2; 1 + t); b ; \end{array} ; y^2, 4x \right]. \end{aligned}$$

(V) In (2.1), setting $S(j, p, i, k) = (a)_{2j+2p+i+k}/(b)_j (d)_j$, we get

$${}^{(2)}H_4^{(4)}[a, c - 2f + 1, c; b, d, 2 - 2f, 2f; x, z, y, -y]$$

$$\begin{aligned}
 &= \sum_{t=0}^1 \frac{y^t(a)_t}{t!} \Omega(t) \\
 &\times F^{(3)} \left[\begin{array}{c} \Delta(2; a+t) :: -; -; -; -; -; \\ \hline :: -; -; -; b; d; \\ \frac{1+t}{2} + f - c, \frac{1+t}{2} - f + c \quad ; 4x, 4z, y^2 \\ \frac{3+t}{2} - f, \frac{1+t}{2} + f, \Delta^*(2; 1+t); \end{array} \right].
 \end{aligned}$$

Here K_{10} , K_{13} , X_8 , ${}^{(2)}H_4^{(4)}$ and $F_A^{(4)}$ are the multiple Gaussian hypergeometric functions of Exton [2, page 79 (3.3.10, 3.3.13)], [3, page 113 (1.8)], [2, page 97 (3.5.2)] and Lauricella [4, page 150], respectively.

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