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RESEARCH ARTICLE

Kodaira–Spencer isomorphisms and degeneracy maps on Iwahori-level Hilbert modular varieties: the saving trace

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Abstract
We consider integral models of Hilbert modular varieties with Iwahori level structure at primes over p, first proving a Kodaira–Spencer isomorphism that gives a concise description of their dualizing sheaves. We then analyze fibres of the degeneracy maps to Hilbert modular varieties of level prime to p and deduce the vanishing of higher direct images of structure and dualizing sheaves, generalizing prior work with Kassaei and Sasaki (for p unramified in the totally real field F). We apply the vanishing results to prove flatness of the finite morphisms in the resulting Stein factorizations, and combine them with the Kodaira–Spencer isomorphism to simplify and generalize the construction of Hecke operators at primes over p on Hilbert modular forms (integrally and mod p).

Contents

1. Introduction

The motivation for this paper is two-fold: the construction of Galois representations and of Hecke operators at primes over p in the setting of mod p Hilbert modular forms for a totally real field F. By mod p Hilbert modular forms, we mean sections of automorphic line bundles of arbitrary weight over mod p Hilbert modular varieties of level prime to p. We will explain this below more precisely, discuss what was done previously, and describe how the results here complete the picture for Hecke operators and feed into an argument that does so for Galois representations.

First, let us recall the nature of the difficulty in constructing Hecke operators at p in characteristic p. One issue already arises in the setting of classical modular forms. For simplicity, assume $N > 4$ and let $X_1(N)$ denote the modular curve $\Gamma_1(N)\backslash\mathfrak{H}^*$, so that $X_1(N)$ may be viewed as parametrizing pairs (E, P), where E is a generalized elliptic curve over $\mathbb{C}$ and P is a point of E of order N (see, for example, [DR73]). Similarly, for p not dividing N and $\Gamma_1(N;p) = \Gamma_1(N) \cap \Gamma_0(p)$, the modular curve $X_1(N;p) = \Gamma_1(N;p)\backslash\mathfrak{H}^*$ parametrizes data of the form (E, P, C), where (E, P) is as above and C is a subgroup of E of order p, and there are two natural degeneracy maps $\pi_1, \pi_2 : X_1(N;p) \to X_1(N)$, with $\pi_1(E, P, C) = (E, P)$ and $\pi_2(E, P, C) = (E/C, P \bmod C)$. The space of modular forms of weight k with respect to $\Gamma_1(N)$ may be identified with $H^0(X_1(N), \omega^{\otimes k})$, where the line bundle $\omega = 0^*\Omega^1_{E/X_1(N)}$ is the pull-back of $\Omega^1_{E/X_1(N)}$ along the zero section $0 : X_1(N) \to E$. The Hecke operator T_p can then be defined as the composite

$$\begin{aligned} H^0(X_1(N), \omega^{\otimes k}) &\longrightarrow H^0(X_1(N;p), \pi_2^*\omega^{\otimes k}) \\ &\longrightarrow H^0(X_1(N;p), \pi_1^*\omega^{\otimes k}) \longrightarrow H^0(X_1(N), \omega^{\otimes k}) \end{aligned}$$

divided by p, where the first map is pull-back, the second is induced by the universal isogeny over $X_1(N;p)$, and the third is the trace. To define T_p integrally or in characteristic p, one can work instead with integral models of the curves and show (as in [Con07, §4.5], for example) that the resulting composite morphism

$$\pi_{1,*}\left(\pi_2^*\omega^{\otimes k}\right) \longrightarrow \pi_{1,*}\left(\pi_1^*\omega^{\otimes k}\right) \longrightarrow \omega^{\otimes k} \tag{1.1}$$

of sheaves is divisible by p (assuming $k \geq 1$).

This paper offers an alternative to the standard approach just described to the construction of Hecke operators at p. In particular, it allows for a more general and direct definition of the morphism which produces (1.1) after multiplication by p. Our perspective thus breaks the mindset reflected in [Cal20, §3.4], which described all prior constructions by saying that *the 'correct' definition of T involves first defining a map coming from a correspondence and then showing that it is 'divisible' by the correct power of p.*

To construct integral and mod p Hecke operators at primes over p over more general Shimura varieties, one encounters the further difficulty that the degeneracy maps no longer necessarily extend to finite flat morphisms on the usual integral models, so the definition of a trace morphism requires a more sophisticated application of Grothendieck–Serre duality. Emerton, Reduzzi and Xiao take this approach in [ERX17a] to defining Hecke operators at primes over p for Hilbert modular forms, integrally and mod p, but for a restricted set of weights. However, Fakhruddin and Pilloni set up a general framework in

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[FP23] and prove results for Hilbert modular forms that are optimal in terms of the weights[1] considered, but they assume p is unramified in F.

This paper contains both a conceptual innovation and a technical improvement on previous work, yielding an optimal result. The innovation takes the form of a Kodaira–Spencer isomorphism describing the dualizing (or canonical) sheaf on integral models of Iwahori-level Hilbert modular varieties. In addition to being strikingly natural and simple to state, it can also be viewed as encoding integrality properties of Hecke operators at p. The technical advance is a cohomological vanishing theorem for the dualizing sheaf, showing that its higher direct images relative to certain degeneracy maps are trivial. This generalizes such a result in [DKS23], where it is proved under the assumption that p is unramified in F, and we apply it here to define Hecke operators integrally and in characteristic p, obtaining them directly from morphisms of coherent sheaves rather than (rescaled) morphisms of complexes in a derived category, as in [ERX17a] and [FP23].

The cohomological vanishing is also useful in proving the existence of Galois representations associated to mod p Hilbert modular eigenforms of arbitrary weight. The construction of such Galois representations is proved independently by Emerton, Reduzzi and Xiao in [ERX17b], and by Goldring and Koskivirta in [GK19], under parity conditions on the weight. These conditions are inherited from the obvious parity obstruction to algebraicity for automorphic forms on $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$; however, they are unnecessary, and unnatural, in the consideration of automorphic forms in finite characteristic. The construction of Galois representations for mod p Hilbert modular forms of arbitrary weight is carried out in [DS23] under the assumption that p is unramified in F, with the cohomological vanishing result in [DKS23] playing a crucial role in the argument. The generalization proved in this paper can similarly be used, in conjunction with the methods of [DS23] and [ERX17b], to prove the existence of Galois representations associated to mod p Hilbert modular eigenforms in full generality.

The cohomological vanishing results also have applications to the study of integral models for Iwahori-level Hilbert modular varieties. Indeed, similar arguments to those for the dualizing sheaves also show that the higher direct images of their structure sheaves vanish, and together with Grothendieck–Serre duality, this implies the flatness of the finite morphisms in the Stein factorizations of the degeneracy maps. As a result, we obtain Cohen–Macaulay models for the Iwahori-level varieties which are finite flat over the smooth models for the varieties of prime-to-p level.

We now describe our results in more detail.

We fix a prime p and a totally real field F, and let $\mathcal{O}_F$ denote the ring of integers of F. Fix also embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \to \mathbb{C}$, and let $\mathcal{O}$ denote the ring of integers of a sufficiently large finite extension K of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$.

Let U be a sufficiently small open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. A construction of Pappas and Rapoport [PR05] yields a smooth model Y over $\mathcal{O}$ for the Hilbert modular variety with complex points

$$\mathrm{GL}_2(F)_+\backslash(\mathfrak{H}^\Sigma \times \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/U),$$

where $\mathfrak{H}$ is the complex upper-half plane and Σ is the set of embeddings $F \to \overline{\mathbb{Q}}$. The scheme Y is equipped with line bundles[2] ω and δ arising from its interpretation as a coarse moduli space parametrizing abelian schemes with additional data including an $\mathcal{O}_F$-action. Since Y is smooth over $\mathcal{O}$, its relative dualizing sheaf $\mathcal{K}_{Y/\mathcal{O}}$ is identified with $\wedge^{[F:\mathbb{Q}]}\Omega^1_{Y/\mathcal{O}}$, and in [RX17] (see also [Dia23, §3.3]), Reduzzi and Xiao establish an integral version of the Kodaira–Spencer isomorphism, taking the form

$$\mathcal{K}_{Y/\mathcal{O}} \cong \delta^{-1}\omega^{\otimes 2}.$$

[1]The results in [FP23] are only formulated integrally, necessitating a parity condition on the weight, but the hypothesis is not essential to the methods there.

[2]We remark that δ is trivializable, but only non-canonically, and we systematically incorporate it into constructions to render them Hecke-equivariant.

Let $\mathfrak{p}$ be a prime of $\mathcal{O}_F$ containing p. Denote its residue degree by $f_\mathfrak{p}$ and let

$$U_0(\mathfrak{p}) = \left\{ \left. \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U \,\right|\, c_\mathfrak{p} \in \mathfrak{p}\mathcal{O}_{F,\mathfrak{p}} \right\}.$$

Augmenting the moduli problem for Y with suitable isogeny data then produces a model $Y_0(\mathfrak{p})$ for the Hilbert modular variety of level $U_0(\mathfrak{p})$, equipped with a pair of degeneracy maps $\pi_1, \pi_2 : Y_0(\mathfrak{p}) \longrightarrow Y$. The morphisms π_1 and π_2 are projective, but not finite or flat unless $F_\mathfrak{p} = \mathbb{Q}_p$.

The scheme $Y_0(\mathfrak{p})$ is syntomic over $\mathcal{O}$, and hence has an invertible dualizing sheaf $\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}$. Our Iwahori-level version of the Kodaira–Spencer isomorphism is the following (Theorem 3.5; see there for the precise meaning of Hecke-equivariant).

Theorem A. *There is a Hecke-equivariant isomorphism*

$$\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}} \cong \pi_2^*(\delta^{-1}\omega) \otimes \pi_1^*\omega.$$

The idea of the proof is that, for sufficiently small U, the Hecke correspondence

$$(\pi_1, \pi_2) : Y_0(\mathfrak{p}) \longrightarrow Y \times Y$$

is a closed immersion whose conormal bundle can be described using deformation-theoretic considerations similar to those applied to the diagonal embedding $Y \longrightarrow Y \times Y$ in the proof of the Kodaira–Spencer isomorphism for level prime to p. We mention also that Theorem A can be expressed in terms of upper-shriek functors as an isomorphism

$$\pi_2^!\omega \cong \pi_1^*\omega,$$

and that $\pi_1^*\delta$ and $\pi_2^*\delta$ are canonically isomorphic, so that π_1 and π_2 can be interchanged in these statements.

We now describe our cohomological vanishing results. For simplicity, we focus on the vanishing of $R^i\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}$ for $i > 0$. Since $\pi_{1,K} : Y_0(\mathfrak{p})_K \longrightarrow Y_K$ is finite, the problem reduces to proving the vanishing of $R^i\overline{\pi}_{1,*}\mathcal{K}_{\overline{Y}_0(\mathfrak{p})/\overline{\mathbb{F}}_p}$, where $\overline{\pi}_1 : \overline{Y}_0(\mathfrak{p}) \longrightarrow \overline{Y}$ is the reduction[3] of π_1. Then $\overline{Y}_0(\mathfrak{p})$ is a local complete intersection which may be written as a union of smooth subschemes $\overline{Y}_0(\mathfrak{p})_J$ indexed by subsets J of $\Sigma_\mathfrak{p}$, where $\Sigma_\mathfrak{p}$ is the set of embeddings $F_\mathfrak{p} \longrightarrow \overline{\mathbb{Q}}_p$. The technical heart of this paper is a complete description of the fibres of the restriction of $\overline{\pi}_1$ to $\overline{Y}_0(\mathfrak{p})_J$, proving the following (Corollary 4.14; see there for the definition of m and δ, and Theorem 4.13 for an even more precise version).

Theorem B. *Every nonempty fibre of $\overline{Y}_0(\mathfrak{p})_J \longrightarrow \overline{Y}$ is isomorphic to $(\mathbb{P}^1)^m \times S^\delta$, where S =* Spec $(\overline{\mathbb{F}}_p[T]/T^p)$ *and m and δ are determined by J.*

This generalizes Theorem D of [DKS23], where it is proved under the assumption that p is unramified in F. Work in this direction, for arbitrary behavior of p, was also carried out[4] in [ERX17a, §4]. The approach taken here to the general case relies on a brute force analysis of the local deformation theory of the fibres, undertaken in §§4.3–4.5. With this in hand, the proof is similar to the one in [DKS23], as is the deduction of cohomological vanishing via the 'dicing' argument introduced there. Furthermore, we observe here that similar arguments give the vanishing of $R^i\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})}$ for $i > 0$, and combining these results with Grothendieck–Serre duality gives the following (see Corollaries 5.4 and 5.7):

Theorem C. *The sheaves $R^i\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}$ and $R^i\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})}$ vanish for $i > 0$, and are locally free of rank $1 + p^{f_\mathfrak{p}}$ if $i = 0$. Furthermore, there is a Hecke-equivariant isomorphism*

$$\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}, \mathcal{K}_{Y/\mathcal{O}}).$$

[3] We work here over $\overline{\mathbb{F}}_p$, slightly deviating from the notation in the paper.
[4] There is, however, a serious gap in the argument in [ERX17a]; see Remark 4.12 below.

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As in [DKS23], we in fact treat the degeneracy map $\varphi : Y_1(\mathfrak{P}) \to Y$ where $\mathfrak{P}$ is the radical of $p\mathcal{O}_F$ and $Y_1(\mathfrak{P})$ is a model for the Hilbert modular variety of level $U_1(\mathfrak{P})$, this being what is needed for the construction of Galois representations. The case where $\mathfrak{P}$ contains the radical, for example $\mathfrak{P} = \mathfrak{p}$, is a mild generalization, and this implies the result for π_1. It follows also that the models for the Hilbert modular varieties of level $U_1(\mathfrak{p})$ and $U_0(\mathfrak{p})$ defined by

$$\mathbf{Spec}(\varphi_*\mathcal{O}_{Y_1(\mathfrak{p})}) \quad \text{and} \quad \mathbf{Spec}(\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})})$$

are finite and flat over Y, and hence Cohen–Macaulay. We remark that if p is unramified in F, such a model for level $U_1(p)$ (and indeed $U_1(p^r)$) was constructed by Kottwitz and Wake in [KW17]; it is natural to ask how these models may be related. Since our (finite flat over Y) models are defined via Stein factorization and $Y_0(\mathfrak{p})$ is normal, we obtain the following, perhaps surprising, further consequence (Corollary 5.12), pointed out to us by G. Pappas:

Corollary D. *The normalization of Y in $Y_0(\mathfrak{p})_K$ is flat over Y.*

Note also that combining Theorems A and C with the Kodaira–Spencer isomorphism over Y and the isomorphism $\pi_1^*\delta \cong \pi_2^*\delta$ gives an isomorphism

$$\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})} \cong \mathcal{H}om_{\mathcal{O}_Y}(\pi_{1,*}\pi_2^*\omega,\, \omega),$$

and hence a canonical morphism

$$\pi_{1,*}\pi_2^*\omega \longrightarrow \omega$$

which we call the *saving trace*. Over Y_K it coincides with the composite

$$(\pi_{1,*}\pi_2^*\omega)_K \longrightarrow (\pi_{1,*}\pi_1^*\omega)_K \longrightarrow \omega_K$$

divided by $p^{f_\mathfrak{p}}$, where the first morphism is induced by the universal isogeny and the second is the trace relative to $\pi_{1,K}$. (See (5.21); note that for $F = \mathbb{Q}$, this recovers the divisibility of (1.1) by p in the case $k = 1$.)

The saving trace can be used to define the Hecke operator $T_\mathfrak{p}$ on Hilbert modular forms with coefficients in an arbitrary $\mathcal{O}$-algebra R; for simplicity, we restrict our attention in this introduction to $R = \overline{\mathbb{F}}_p$. First, recall that for any $\mathbf{k} = (k_\theta)_{\theta\in\Sigma} \in \mathbb{Z}^\Sigma$ (and sufficiently small U), there is an associated automorphic bundle $\overline{\mathcal{A}}_\mathbf{k}$ over $\overline{Y}$ (denoted $\mathcal{A}_{\mathbf{k},\mathbf{0},\overline{\mathbb{F}}_p}$ in §2.3; in particular, $\overline{\mathcal{A}}_\mathbf{1} = \overline{\omega} = \omega_{\overline{\mathbb{F}}_p}$). If $F \neq \mathbb{Q}$, so that the Koecher Principle holds, then $M_\mathbf{k}(U; \overline{\mathbb{F}}_p) = H^0(\overline{Y}, \overline{\mathcal{A}}_\mathbf{k})$ is the space of Hilbert modular forms over $\overline{\mathbb{F}}_p$ of weight $\mathbf{k}$ and level U; if $F = \mathbb{Q}$, one needs to compactify $\overline{Y}$ and extend $\overline{\mathcal{A}}_k = \overline{\omega}^{\otimes k}$ to define $M_k(U; \overline{\mathbb{F}}_p)$. Taking the direct limit over all (sufficiently small) U containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$, we obtain a $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$-module $M_\mathbf{k}(\overline{\mathbb{F}}_p)$. We remark that even forms of paritious weight $\mathbf{k}$ in characteristic p do not necessarily lift to characteristic zero; indeed, this is already the case for $F = \mathbb{Q}$ and $k = 1$.

If $k_\theta \geq 1$ for all $\theta \in \Sigma_\mathfrak{p}$ (a mild hypothesis in view of the main result of [DK23] and [DDW24, Prop. 1.13]), then the universal isogeny induces a morphism $\overline{\pi}_2^*\overline{\mathcal{A}}_{\mathbf{k}-\mathbf{1}} \to \overline{\pi}_1^*\overline{\mathcal{A}}_{\mathbf{k}-\mathbf{1}}$ over $\overline{Y}_0(\mathfrak{p})$. Twisting its direct image by the saving trace $\overline{\pi}_{1,*}\overline{\pi}_2^*\overline{\omega} \cong (\pi_{1,*}\pi_2^*\omega)_{\overline{\mathbb{F}}_p} \to \overline{\omega}$ over $Y_{\overline{\mathbb{F}}_p}$ then yields a morphism

$$\overline{\pi}_{1,*}\overline{\pi}_2^*\overline{\mathcal{A}}_\mathbf{k} \longrightarrow \overline{\mathcal{A}}_\mathbf{k},$$

and hence an endomorphism $T_\mathfrak{p}$ of $M_\mathbf{k}(U; \overline{\mathbb{F}}_p)$ (see (5.25)). Taking the limit over U thus defines a $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$-equivariant endomorphism $T_\mathfrak{p}$ of $M_\mathbf{k}(\overline{\mathbb{F}}_p)$. See Theorem 5.13 for a statement applicable to more general weights, coefficients and cohomology degrees.

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2. Hilbert modular varieties

2.1. Notation

Fix a prime p and a totally real field F of degree $d = [F : \mathbb{Q}]$. We let $\mathcal{O}_F$ denote the ring of integers of F, $\mathfrak{d}$ its different, S_p the set of primes over p. For each $\mathfrak{p} \in S_p$, we write $F_\mathfrak{p}$ for the completion of F at $\mathfrak{p}$, and we let $F_{\mathfrak{p},0}$ denote its maximal unramified subextension, $\mathbb{F}_\mathfrak{p}$ the residue field $\mathcal{O}_F/\mathfrak{p}$, $f_\mathfrak{p}$ the residue degree $[F_{\mathfrak{p},0} : \mathbb{Q}_p] = [\mathbb{F}_\mathfrak{p} : \mathbb{F}_p]$, and $e_\mathfrak{p}$ the ramification index $[F_\mathfrak{p} : F_{\mathfrak{p},0}]$. We also fix a choice of totally positive $\varpi_\mathfrak{p} \in F$ so that $v_\mathfrak{p}(\varpi_\mathfrak{p}) = 1$ and $v_{\mathfrak{p}'}(\varpi_\mathfrak{p}) = 0$ for all other $\mathfrak{p}' \in S_p$. In particular, $\varpi_\mathfrak{p}$ is a uniformizer in $F_\mathfrak{p}$, and for each $\mathfrak{p} \in S_p$, and let $E_\mathfrak{p}(u) \in W(\mathbb{F}_\mathfrak{p})[u]$ denote its minimal polynomial over $F_{\mathfrak{p},0}$, whose ring of integers we identify with $W(\mathbb{F}_\mathfrak{p})$.

We adopt much of the notation and conventions of [Dia23] for notions and constructions associated to embeddings of F. In particular, for each $\mathfrak{p} \in S_p$, we let $\Sigma_\mathfrak{p}$ denote the set of embeddings $F_\mathfrak{p} \to \overline{\mathbb{Q}}_p$, and we identify $\Sigma := \coprod_{\mathfrak{p}\in S_p} \Sigma_\mathfrak{p}$ with the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$ via the canonical bijection. We also let $\Sigma_{\mathfrak{p},0}$ denote the set of embeddings $F_{\mathfrak{p},0} \to \overline{\mathbb{Q}}_p$, which we identify with the set of embeddings $W(\mathbb{F}_\mathfrak{p}) \to W(\overline{\mathbb{F}}_p)$, or equivalently $\mathbb{F}_\mathfrak{p} \to \overline{\mathbb{F}}_p$, and we let $\Sigma_0 = \coprod_{\mathfrak{p}\in S_p} \Sigma_{\mathfrak{p},0}$.

For each $\mathfrak{p} \in S_p$, we fix a choice of embedding $\tau_{\mathfrak{p},0} \in \Sigma_{\mathfrak{p},0}$, and for $i \in \mathbb{Z}/f_\mathfrak{p}\mathbb{Z}$, we let $\tau_{\mathfrak{p},i} = \phi^i \circ \tau_{\mathfrak{p},0}$ where ϕ is the Frobenius automorphism of $\overline{\mathbb{F}}_p$. We also fix an ordering $\theta_{\mathfrak{p},i,1}, \theta_{\mathfrak{p},i,2}, \dots, \theta_{\mathfrak{p},i,e_\mathfrak{p}}$ of the embeddings $\theta \in \Sigma_\mathfrak{p}$ restricting to $\tau_{\mathfrak{p},i}$, so that

$$\Sigma = \{\, \theta_{\mathfrak{p},i,j} \,|\, \mathfrak{p} \in S_p, i \in \mathbb{Z}/f_\mathfrak{p}\mathbb{Z}, 1 \le j \le e_\mathfrak{p} \,\}.$$

Finally, we let σ denote the permutation of Σ defined by $\sigma(\theta_{\mathfrak{p},i,j}) = \theta_{\mathfrak{p},i,j+1}$ if $j \ne e_\mathfrak{p}$ and $\sigma(\theta_{\mathfrak{p},i,e_\mathfrak{p}}) = \theta_{\mathfrak{p},i+1,1}$.

Choose a finite extension K of $\mathbb{Q}_p$ sufficiently large to contain the images of all $\theta \in \Sigma$; let $\mathcal{O}$ denote its ring of integers, ϖ a uniformizer, and k its residue field.

For $\tau \in \Sigma_{\mathfrak{p},0}$, we define $E_\tau(u) \in \mathcal{O}[u]$ to be the image of $E_\mathfrak{p}$ under the homomorphism induced by τ, so that

$$\mathcal{O}_F \otimes \mathcal{O} = \bigoplus_{\mathfrak{p}\in S_p} \bigoplus_{\tau\in\Sigma_{\mathfrak{p},0}} \mathcal{O}_{F,\mathfrak{p}} \otimes_{W(\mathbb{F}_\mathfrak{p}),\tau} \mathcal{O} \cong \bigoplus_{\tau\in\Sigma_0} \mathcal{O}[u]/(E_\tau).$$

For any $\mathcal{O}_F \otimes \mathcal{O}$-module M, we obtain a corresponding decomposition $M = \bigoplus_{\tau\in\Sigma_0} M_\tau$, and we also write $M_{\mathfrak{p},i}$ for M_τ if $\tau = \tau_{\mathfrak{p},i}$. Similarly, if S is a scheme over $\mathcal{O}$ and $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_F \otimes \mathcal{O}_S$-modules on S, then we write $\mathcal{M}_\tau = \mathcal{M}_{\mathfrak{p},i}$ for the corresponding summand of $\mathcal{M}$.

For each $\theta = \theta_{\mathfrak{p},i,j} \in \Sigma$, we factor $E_\tau = s_\theta t_\theta$ where $\tau = \tau_{\mathfrak{p},i}$,

$$\begin{aligned} s_\theta &= s_{\tau,j} = (u - \theta_{\mathfrak{p},i,1}(\varpi_\mathfrak{p})) \cdots (u - \theta_{\mathfrak{p},i,j}(\varpi_\mathfrak{p})) \\ \text{and} \quad t_\theta &= t_{\tau,j} = (u - \theta_{\mathfrak{p},i,j+1}(\varpi_\mathfrak{p})) \cdots (u - \theta_{\mathfrak{p},i,e_\mathfrak{p}}(\varpi_\mathfrak{p})); \end{aligned} \tag{2.1}$$

note that the ideals (s_θ) and (t_θ) in $\mathcal{O}[u]/(E_\tau)$ are each other's annihilators, and that the corresponding ideals in $\mathcal{O}_{F,\mathfrak{p}} \otimes_{W(\mathbb{F}_\mathfrak{p}),\tau} \mathcal{O}$ are independent of the choice of the uniformizer $\varpi_\mathfrak{p}$.

2.2. The Pappas-Rapoport model

We now recall the definition, due to Pappas and Rapoport, for smooth integral models of Hilbert modular varieties of level prime to p,

We let $\mathbb{A}_{F,\mathbf{f}} = F \otimes \widehat{\mathbb{Z}}$ denote the finite adeles of F, and we let $\mathbb{A}_{F,\mathbf{f}}^{(p)} = \mathbb{Q} \otimes \widehat{\mathcal{O}}_F^{(p)} = F \otimes \widehat{\mathbb{Z}}^{(p)}$, where $\widehat{\mathcal{O}}_F^{(p)}$ (resp. $\widehat{\mathbb{Z}}^{(p)}$) is the prime-to-$p$ completion of $\mathcal{O}_F$ (resp. $\mathbb{Z}$), so $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell\ne p} \mathbb{Z}_\ell$ and $\widehat{\mathcal{O}}_F^{(p)} = \mathcal{O}_F \otimes \widehat{\mathbb{Z}}^{(p)} = \prod_{v\nmid p} \mathcal{O}_{F,v}$.

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Let U be an open compact subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F) \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ of the form U_pU^p, where $U_p = \mathrm{GL}_2(\mathcal{O}_{F,p})$. Consider the functor which associates, to a locally Noetherian $\mathcal{O}$-scheme S, the set of isomorphism classes of data $(A, \iota, \lambda, \eta, \mathcal{F}^\bullet)$, where

- $s : A \to S$ is an abelian scheme of relative dimension d;
- $\iota : \mathcal{O}_F \to \mathrm{End}_S(A)$ is a homomorphism;
- λ is an $\mathcal{O}_F$-linear quasi-polarization of A such that for each connected component S_i of S, λ induces an isomorphism $\mathfrak{c}_i\mathfrak{d} \otimes_{\mathcal{O}_F} A_{S_i} \to A^\vee_{S_i}$ for some fractional ideal $\mathfrak{c}_i$ of F prime to p;
- η is a level U^p structure on A; that is, a $\pi_1(S_i, \overline{s}_i)$-invariant U^p-orbit of $\widehat{\mathcal{O}}_F^{(p)}$-linear isomorphisms

$$\eta_i : (\widehat{\mathcal{O}}_F^{(p)})^2 \to \mathfrak{d} \otimes_{\mathcal{O}_F} T^{(p)}(A_{\overline{s}_i})$$

  for a choice of geometric point $\overline{s}_i$ on each connected component S_i of S, where $T^{(p)}$ denotes the product over $\ell \neq p$ of the ℓ-adic Tate modules, and $g \in U^p$ acts on η_i by pre-composing with right multiplication by g^{-1};
- $\mathcal{F}^\bullet$ is a collection of Pappas–Rapoport filtrations; that is, for each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$, an increasing filtration of $\mathcal{O}_{F,\mathfrak{p}} \otimes_{W(\mathbb{F}_\mathfrak{p}),\tau} \mathcal{O}_S$-modules

$$0 = \mathcal{F}_\tau^{(0)} \subset \mathcal{F}_\tau^{(1)} \subset \cdots \subset \mathcal{F}_\tau^{(e_\mathfrak{p}-1)} \subset \mathcal{F}_\tau^{(e_\mathfrak{p})} = (s_*\Omega^1_{A/S})_\tau$$

  such that for $j = 1, \ldots, e_\mathfrak{p}$, the quotient $\mathcal{F}_\tau^{(j)}/\mathcal{F}_\tau^{(j-1)}$ is a line bundle on S on which $\mathcal{O}_F$ acts via $\theta_{\mathfrak{p},i,j}$.

If U^p is sufficiently small, then the functor is representable by an infinite disjoint union $\widetilde{Y}_U$ of smooth, quasi-projective schemes of relative dimension d over $\mathcal{O}$. Furthermore, we have an action of $\nu \in \mathcal{O}^\times_{F,(p),+}$ on $\widetilde{Y}_U$ defined by composing the quasi-polarization with $\iota(\nu)$, and the action factors through a free action of $\mathcal{O}^\times_{F,(p),+}/(U \cap \mathcal{O}_F^\times)^2$ by which the quotient is representable by a smooth quasi-projective scheme of relative dimension d over $\mathcal{O}$, which we denote by Y_U. We also have a natural right action of $g \in \mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ on the inverse system of schemes $\widetilde{Y}_U$ defined by pre-composing the level structure η with right-multiplication by g^{-1}, and the action descends to one on the inverse system of schemes Y_U. Furthermore, we have a compatible system of isomorphisms of the $Y_U(\mathbb{C})$ (for any choice of $\mathcal{O} \to \mathbb{C}$) with the Hilbert modular varieties

$$\mathrm{GL}_2(F)_+ \backslash (\mathfrak{H}^\Sigma \times \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/U)$$

under which the action of $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ corresponds to right multiplication.

2.3. Automorphic bundles

Let $\underline{A} = (A, \iota, \lambda, \eta, \mathcal{F}^\bullet)$ denote the universal object over $\widetilde{Y}_U$. Recall that $\mathcal{H}^1_{\mathrm{dR}}(A/\widetilde{Y}_U)$ is a (Zariski-)locally free sheaf of rank two $\mathcal{O}_F \otimes \mathcal{O}_{\widetilde{Y}_U}$-modules, and hence decomposes as $\oplus_{\tau\in\Sigma}\mathcal{H}^1_{\mathrm{dR}}(A/\widetilde{Y}_U)_\tau$, where each $\mathcal{H}^1_{\mathrm{dR}}(A/\widetilde{Y}_U)_\tau$ is locally free of rank two over $\mathcal{O}_{\widetilde{Y}_U}[u]/(E_\tau)$.

For $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j}$, we let $\mathcal{L}_\theta$ denote the line bundle $\mathcal{F}_\tau^{(j)}/\mathcal{F}_\tau^{(j-1)}$ on $\widetilde{Y}_U$. We let $\mathcal{G}_\tau^{(j)}$ denote the pre-image of $\mathcal{F}_\tau^{(j-1)}$ in $\mathcal{H}^1_{\mathrm{dR}}(A/\widetilde{Y}_U)_\tau$ under $u - \theta(\varpi_\mathfrak{p})$, so that $\mathcal{P}_\theta := \mathcal{G}_\tau^{(j)}/\mathcal{F}_\tau^{(j-1)}$ is a rank two vector bundle on $\widetilde{Y}_U$, containing $\mathcal{L}_\theta$ as a sub-bundle. We let $\mathcal{M}_\theta = \mathcal{P}_\theta/\mathcal{L}_\theta = \mathcal{G}_\tau^{(j)}/\mathcal{F}_\tau^{(j)}$ and $\mathcal{N}_\theta = \mathcal{L}_\theta \otimes_{\mathcal{O}_{\widetilde{Y}_U}} \mathcal{M}_\theta = \wedge^2_{\mathcal{O}_{\widetilde{Y}_U}} \mathcal{P}_\theta$. For any $\mathbf{k}, \mathbf{m} \in \mathbb{Z}^\Sigma$, we let $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}$ denote the line bundle

$$\bigotimes_{\theta\in\Sigma}\left(\mathcal{L}_\theta^{\otimes k_\theta} \otimes_{\mathcal{O}_{\widetilde{Y}_U}} \mathcal{N}_\theta^{\otimes m_\theta}\right)$$

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on $\widetilde{Y}_U$. We also let $\widetilde{\omega} = \widetilde{\mathcal{A}}_{\mathbf{1},\mathbf{0}}$ and $\widetilde{\delta} = \widetilde{\mathcal{A}}_{\mathbf{0},\mathbf{1}}$. Note that $\widetilde{\omega}$ may be identified with $\wedge^d_{\mathcal{O}_{\widetilde{Y}_U}}(s_*\Omega^1_{A/\widetilde{Y}_U})$; we remark that, less obviously, $\widetilde{\delta}$ may be identified with $\wedge^{2d}_{\mathcal{O}_{\widetilde{Y}_U}}(\mathcal{H}^1_{\mathrm{dR}}(A/\widetilde{Y}_U))$ (and more naturally with $\mathrm{Disc}_{F/\mathbb{Q}} \otimes \wedge^{2d}_{\mathcal{O}_{\widetilde{Y}_U}}(\mathcal{H}^1_{\mathrm{dR}}(A/\widetilde{Y}_U))$).

There is a natural action of $\mathcal{O}^\times_{F,(p),+}$ on the vector bundles $\mathcal{F}^{(j)}_\tau$ and $\mathcal{G}^{(j)}_\tau$ over the one on $\widetilde{Y}_U$, inducing actions on $\mathcal{L}_\theta$, $\mathcal{P}_\theta$, $\mathcal{M}_\theta$ and $\mathcal{N}_\theta$; however, the restriction to $(U \cap \mathcal{O}^\times_F)^2$ is nontrivial, so it fails to define descent data on these sheaves to Y_U (see [Dia23, §3.2]). More precisely, if $\nu = \mu^2$ for some $\mu \in U \cap \mathcal{O}^\times_F$, then ν acts on $\mathcal{L}_\theta$, $\mathcal{P}_\theta$ and $\mathcal{M}_\theta$ (resp. $\mathcal{N}_\theta$) by $\theta(\mu)$ (resp. $\theta(\nu)$). Thus, if $\mathbf{k} + 2\mathbf{m}$ is parallel, in the sense that $k_\theta + 2m_\theta$ is independent of θ, then we obtain descent data on $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}$ (provided $\mathrm{Nm}_{F/\mathbb{Q}}(U \cap \mathcal{O}^\times_F) = \{1\}$), and we let $\mathcal{A}_{\mathbf{k},\mathbf{m}}$ denote the resulting line bundle on Y_U; note in particular this applies to $\widetilde{\omega}$ and $\widetilde{\delta}$, yielding the line bundles on Y_U which we denote ω and δ. More generally, if R is an $\mathcal{O}$-algebra in which the image of $\prod_\theta \theta(\mu)^{k_\theta+2m_\theta}$ is 1 for all $\mu \in U \cap \mathcal{O}^\times_F$, then $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m},R} := \widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}} \otimes_{\mathcal{O}} R$ descends to a line bundle on $Y_{U,R} := Y_U \times_{\mathcal{O}} R$ which we denote by $\mathcal{A}_{\mathbf{k},\mathbf{m},R}$; note in particular that if $p^N R = 0$ for some N, then this applies to all $\mathbf{k}, \mathbf{m}$ whenever U is sufficiently small (depending on N).

We also have natural left action of $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ on the various vector bundles over its right action on the inverse system $\widetilde{Y}_U$. More precisely, suppose that $g \in \mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ is such that $g^{-1}Ug \subset U'$, where U and U' are sufficiently small open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. Let $\underline{A}' = (A', \iota', \lambda', \eta', \mathcal{F}'^\bullet)$ denote the universal object over $\widetilde{Y}_{U'}$, and similarly let $\mathcal{L}'_\theta$, etc., denote the associated vector bundles. As described in [Dia23, §3.2], the morphism $\widetilde{\rho}_g : \widetilde{Y}_U \longrightarrow \widetilde{Y}_{U'}$ is associated with a (prime-to-p) quasi-isogeny $A \longrightarrow \widetilde{\rho}_g^* A'$ inducing isomorphisms $\widetilde{\rho}_g^* \mathcal{F}'^{(j)}_\tau \xrightarrow{\sim} \mathcal{F}^{(j)}_\tau$, which in turn give rise to isomorphisms $\widetilde{\rho}_g^* \mathcal{L}'_\theta \xrightarrow{\sim} \mathcal{L}_\theta$, etc., satisfying the usual compatibilities. Furthermore, if $\mathbf{k} + 2\mathbf{m}$ is parallel, then the resulting isomorphisms $\widetilde{\rho}_g^* \widetilde{\mathcal{A}}'_{\mathbf{k},\mathbf{m}} \xrightarrow{\sim} \widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}$ descend to isomorphisms $\rho_g^* \mathcal{A}'_{\mathbf{k},\mathbf{m}} \xrightarrow{\sim} \mathcal{A}_{\mathbf{k},\mathbf{m}}$, where $\rho_g : Y_U \longrightarrow Y_{U'}$ is the morphism obtained by descent from $\widetilde{\rho}_g$, and more generally, we obtain isomorphisms $\rho_g^* \mathcal{A}'_{\mathbf{k},\mathbf{m},R} \xrightarrow{\sim} \mathcal{A}_{\mathbf{k},\mathbf{m},R}$ whenever the image in R of $\prod_\theta \theta(\mu)^{k_\theta+2m_\theta}$ is 1 for all $\mu \in U' \cap \mathcal{O}^\times_F$.

If $R = \mathbb{C}$ and $\mathbf{k} + 2\mathbf{m}$ is parallel, then we recover the usual automorphic line bundles on $Y_U(\mathbb{C})$ whose global sections are Hilbert modular forms of weight $(\mathbf{k}, \mathbf{m})$ and level U, along with the usual[5] Hecke action of $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ on their direct limit over U.

Finally, recall from [Dia23, §3.2] that the quasi-polarization and Poincaré duality induce a perfect alternating pairing on $\mathcal{P}_\theta$ (depending on the choice of $\varpi_{\mathfrak{p}}$), and hence a trivialization of $\mathcal{N}_\theta$, but their products do not descend to trivializations of the bundles $\mathcal{A}_{\mathbf{0},\mathbf{m},R}$ over Y_U (if $\mathbf{m} \neq \mathbf{0}$). These are, however, torsion bundles, which are furthermore non-canonically trivializable for sufficiently small U. The Hecke action on their global sections is given by [Dia23, Prop. 3.2.2].

2.4. Iwahori–level structure

For $\mathfrak{p} \in S_p$, we let $I_0(\mathfrak{p})$ denote the Iwahori subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{p}}) \,\middle|\, c \in \mathfrak{p}\mathcal{O}_{F,\mathfrak{p}} \right\}.$$

In this section, we recall the definition of suitable integral models of Hilbert modular varieties with such level structure at a set of primes over p.

Fix an ideal $\mathfrak{P}$ of $\mathcal{O}_F$ containing the radical of $p\mathcal{O}_F$, so that $\mathfrak{P} = \prod_{\mathfrak{p}\in P} \mathfrak{p}$ for some subset P of S_p. We are mainly interested in the cases $P = \{\mathfrak{p}\}$ and $P = S_p$, but the additional generality introduces no difficulties, and it may be instructive (and amusing) to note how some of our results specialize to well-known ones in the case $P = \emptyset$. Let U be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ as above, so that

[5] Up to a factor of $||\det||$, depending on normalizations.

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$U = U_pU^p$ where $U_p = \mathrm{GL}_2(\mathcal{O}_{F,p})$ and U^p is a sufficiently small open subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F^{(p)})$, and we let

$$U_0(\mathfrak{P}) = \{\, g \in U \mid g_{\mathfrak{p}} \in I_0(\mathfrak{p}) \text{ for all } \mathfrak{p}|\mathfrak{P}\,\}.$$

d We let $\varpi_{\mathfrak{P}} = \prod \varpi_{\mathfrak{p}}$, $f_{\mathfrak{P}} = \sum f_{\mathfrak{p}}$, $d_{\mathfrak{P}} = \sum e_{\mathfrak{p}} f_{\mathfrak{p}}$, $\Sigma_{\mathfrak{P}} = \coprod \Sigma_{\mathfrak{p}}$ and $\Sigma_{\mathfrak{P},0} = \coprod \Sigma_{\mathfrak{p},0}$, with the product, sums and unions taken over the set of $\mathfrak{p}$ dividing $\mathfrak{P}$.

For a locally Noetherian $\mathcal{O}$-scheme S, we consider the functor which associates to S the set of isomorphism classes of triples $(\underline{A}_1, \underline{A}_2, \psi)$, where $\underline{A}_i = (A_i, \iota_i, \lambda_i, \eta_i, \mathcal{F}_i^\bullet)$ define elements of $Y_U(S)$ for $i = 1, 2$ and $\psi : A_1 \longrightarrow A_2$ is an isogeny of degree $p^{f_{\mathfrak{P}}}$ such that

- $\ker(\psi) \subset A_1[\mathfrak{P}]$;
- ψ is $\mathcal{O}_F$-linear (i.e., $\psi \circ \iota_1(\alpha) = \iota_2(\alpha) \circ \psi$ for all $\alpha \in \mathcal{O}_F$);
- $\lambda_1 \circ \iota_1(\varpi_{\mathfrak{P}}) = \psi^\vee \circ \lambda_2 \circ \psi$;
- $\psi \circ \eta_1 = \eta_2$ (as U^p-orbits on each connected component of S);
- $\psi^*\mathcal{F}_2^\bullet \subset \mathcal{F}_1^\bullet$ (i.e., $\psi^*\mathcal{F}_{\tau,2}^{(j)} \subset \mathcal{F}_{\tau,1}^{(j)}$ for all $\mathfrak{p} \in S_p$, $\tau \in \Sigma_{0,\mathfrak{p}}$ and $j = 1, \ldots, e_{\mathfrak{p}}$).

For such an isogeny ψ, consider also the isogeny $\xi : A_2 \longrightarrow \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A_1$ such that $\xi \circ \psi : A_1 \longrightarrow \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A_1$ is the canonical isogeny with kernel $A_1[\mathfrak{P}]$. The compatibility with the quasi-polarizations λ_1 and λ_2 then implies the commutativity of the resulting diagram of p-integral quasi-isogenies (over each connected component of the base S):

$$\begin{tikzcd} A_2 \arrow[r, "\xi"] \arrow[d, "\lambda_2"'] & \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A_1 \arrow[d, "\varpi_{\mathfrak{P}} \otimes \lambda_2"] \\ (\mathfrak{d} \otimes_{\mathcal{O}_F} A_2)^\vee \arrow[r, "(1\otimes\psi)^\vee"] & (\mathfrak{d} \otimes_{\mathcal{O}_F} A_1)^\vee. \end{tikzcd}$$

This in turn implies the commutativity of the diagram

$$\begin{tikzcd} \mathfrak{P} \otimes_{\mathcal{O}_F} \mathcal{H}_{\tau,1} \arrow[r, "\xi_\tau^*"] \arrow[d, "\varpi_{\mathfrak{P}}^{-1} \otimes \mu_{\tau,1}"'] & \mathcal{H}_{\tau,2} \arrow[d, "\mu_{\tau,2}"] \\ \mathcal{H}_{\tau,1}^\vee \arrow[r, "(\psi_\tau^*)^\vee"] & \mathcal{H}_{\tau,2}^\vee \end{tikzcd}$$

for each $\tau \in \Sigma_0$, where $\mathcal{H}_{\tau,i}$ is the τ-component of the locally free $\mathcal{O}_F \otimes \mathcal{O}_S$-module $\mathcal{H}^1_{\mathrm{dR}}(A_i/S)$ for $i = 1, 2$, the superscript $^\vee$ denotes its $\mathcal{O}_S[u]/(E_\tau)$-dual, and the $\mu_{\tau,i}$ are the τ-components of the isomorphisms obtained from the polarizations and Poincaré duality (see [Dia23, (4)]). The condition that $\psi^*\mathcal{F}_{\tau,2}^{(j)} \subset \mathcal{F}_{\tau,1}^{(j)}$ for all j therefore implies that $\xi_\tau^*(\mathfrak{P} \otimes_{\mathcal{O}_F} (\mathcal{F}_{\tau,1}^{(j)})^\perp) \subset (\mathcal{F}_{\tau,2}^{(j)})^\perp$, so it follows from [Dia23, Lemma 3.1.1] that

$$\xi_\tau^*(\mathfrak{P} \otimes_{\mathcal{O}_F} \mathcal{F}_{\tau,1}^{(j)}) \subset (\mathcal{F}_{\tau,2}^{(j)}). \tag{2.2}$$

The functor defined above is representable by a scheme $\widetilde{Y}_{U_0(\mathfrak{P})}$, projective over $\widetilde{Y}_U$ relative to either of the forgetful morphisms.[6] In the next section, we prove that the schemes $\widetilde{Y}_{U_0(\mathfrak{P})}$ are syntomic of relative dimension d over $\mathcal{O}$ (see also [ERX17a, Prop. 3.3]). Note that we again have a free action of $\mathcal{O}_{F,(p),+}^\times/(U \cap \mathcal{O}_F^\times)^2$ for which the quotient is representable by a quasi-projective scheme of relative

[6] The compatibility in (2.2) is included in other references as a further requirement on the isogeny ψ in the moduli problem defining the model. As we have shown that it is automatically satisfied, it follows that our definition agrees with the one used in [DK23, §7], for example, except that there it is assumed that $\mathfrak{P} = \mathfrak{p}$, $U = U(\mathfrak{n})$ for some $\mathfrak{n}$ and the polarization ideals are fixed (and normalized differently). Our $\widetilde{Y}_{U_0(\mathfrak{p})}$ is therefore isomorphic to an infinite disjoint union of the schemes denoted Y^{PR} in *loc. cit.* in the case $U = U(\mathfrak{n})$, and its quotient by a finite étale cover if $U(\mathfrak{n}) \subset U$.

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dimension d over $\mathcal{O}$, which we denote by $Y_{U_0(\mathfrak{P})}$, and a natural right action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ on the inverse system of the $\widetilde{Y}_{U_0(\mathfrak{P})}$ (over U^p) descending to one on the inverse system of $Y_{U_0(\mathfrak{P})}$. Furthermore, the schemes $Y_{U_0(\mathfrak{P})}$ are independent of the choices of $\varpi_{\mathfrak{p}}$, and we again have isomorphisms of their sets of complex points with Hilbert modular varieties

$$Y_{U_0(\mathfrak{P})}(\mathbb{C}) \cong \mathrm{GL}_2(F)_+\backslash(\mathfrak{H}^\Sigma \times \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/U_0(\mathfrak{P}))$$

under which the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ corresponds to right multiplication.

Consider also the forgetful morphisms $\widetilde{\pi}_i : \widetilde{Y}_{U_0(\mathfrak{P})} \to \widetilde{Y}_U$ sending $(\underline{A}_1, \underline{A}_2, \psi)$ to $\underline{A}_i$ for $i = 1, 2$, and their product

$$\widetilde{h} := (\widetilde{\pi}_1, \widetilde{\pi}_2) : \widetilde{Y}_{U_0(\mathfrak{P})} \to \widetilde{Y}_U \times_{\mathcal{O}} \widetilde{Y}_U,$$

descending to morphisms $\pi_i : Y_{U_0(\mathfrak{P})} \to Y_U$ and $h : Y_{U_0(\mathfrak{P})} \to Y_U \times_{\mathcal{O}} Y_U$, which are again independent of the choices of $\varpi_{\mathfrak{p}}$.

Proposition 2.1. *For sufficiently small U, the morphisms $\widetilde{h}$ and h are closed immersions.*

Proof. Since $\widetilde{h}$ is projective, it suffices to prove that it is injective on geometric points and their tangent spaces. Furthermore, the assertion for h follows from the one for $\widetilde{h}$.

Suppose then that $(\underline{A}_1, \underline{A}_2, \psi)$ and $(\underline{A}'_1, \underline{A}'_2, \psi')$ correspond to geometric points of $\widetilde{Y}_{U_0(\mathfrak{P})}$ with the same image under $\widetilde{h}$. Thus, there are isomorphisms $f_i : A_i \to A'_i$ for $i = 1, 2$ which are compatible with all auxiliary data. Let $\xi : A'_2 \to A_1$ be the unique isogeny such that $\xi \circ \psi' \circ f_1 = p$, and consider the $\mathcal{O}_F$-linear endomorphism $\alpha := \xi \circ f_2 \circ \psi$ of A_1 (where we identify $\mathcal{O}_F$ with a subalgebra of $\mathrm{End}\,(A_1)$ via ι_1). We wish to prove that $\alpha = p$, as this implies that $f_2 \circ \psi = \psi' \circ f_1$, giving the desired injectivity on geometric points.

First, note that the compatibility of ψ' and f_1 with quasi-polarizations implies that $\xi^\vee \circ \lambda_1 \circ \varpi_{\mathfrak{P}} \circ \xi = p^2\lambda'_2$. Combining this with the compatibility of ψ and f_2 with quasi-polarizations, it follows that $\alpha^\vee \circ \lambda_1 \circ \alpha = p^2\lambda_1$. In particular, $F(\alpha)$ is stable under the λ_1-Rosati involution of $\mathrm{End}^0(A_1)$, which sends α to $p^2\alpha^{-1}$. By the classification of endomorphism algebras of abelian varieties, it follows that either $\alpha \in F$, in which case $\alpha = \pm p$, or $F(\alpha)$ is a quadratic CM-extension of F, in which case α is in its ring of integers and $\alpha\overline{\alpha} = p^2$.

Next, note that the compatibility of ψ, ψ', f_1 and f_2 with level structures implies that if $U \subset U(\mathfrak{n})$, then $\alpha \circ \overline{\eta}_1 = p\overline{\eta}_1$, where $\overline{\eta}_1 : (\mathcal{O}_F/\mathfrak{n})^2 \cong A_1[\mathfrak{n}]$ is the isomorphism induced by η_1. It follows that $\alpha - p$ annihilates $A_1[\mathfrak{n}]$, and therefore so does $\overline{\alpha} - p$ (writing $\overline{\alpha} = \alpha$ if $\alpha \in F$). Letting β denote the element $\alpha + \overline{\alpha} - 2p \in \mathcal{O}_F$, it follows that β annihilates $A_1[\mathfrak{n}]$, and hence $\beta \in \mathfrak{n}$. Furthermore, since α is a root of

$$X^2 - (\beta + 2p)X + p^2$$

and either $\alpha = \pm p$ or $F(\alpha)$ is a CM-extension of F, it follows that $|\theta(\beta)| \le 4p$ for all embeddings $\theta : F \hookrightarrow \mathbb{R}$. If $\mathfrak{n}$ is such that $N_{F/\mathbb{Q}}(\mathfrak{n}) > (4p)^{[F:\mathbb{Q}]}$, this implies that $\beta = 0$, and hence, $\alpha = p$.

We have now proved that $\widetilde{h}$ is injective on geometric points. The injectivity on tangent spaces is immediate from the Grothendieck–Messing Theorem, a version of which we recall below for convenience and for later use. $\square$

Let S be a scheme and $i : S \hookrightarrow T$ a nilpotent divided power thickening. If $t : B \to T$ is an abelian scheme of dimension d, then the restriction over S of the crystal $R^1t_{\mathrm{crys},*}\mathcal{O}_{B,\mathrm{crys}}$ is canonically identified with $R^1\overline{t}_{\mathrm{crys},*}\mathcal{O}_{\overline{B},\mathrm{crys}}$, where $\overline{t} : \overline{B} \to S$ is the base-change of $t : B \to T$. The image of $t_*\Omega^1_{B/T}$ under the resulting isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(B/T) \xrightarrow{\sim} (R^1t_{\mathrm{crys},*}\mathcal{O}_{B,\mathrm{crys}})_T \xrightarrow{\sim} (R^1\overline{t}_{\mathrm{crys},*}\mathcal{O}_{\overline{B},\mathrm{crys}})_T \tag{2.3}$$

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is thus an $\mathcal{O}_T$-subbundle $\mathcal{V}_B$ of $(R^1\bar{t}_{\mathrm{crys},*}\mathcal{O}_{\overline{B},\mathrm{crys}})_T$ whose restriction to S corresponds to $\bar{t}_*\Omega^1_{\overline{B}/S}$ under the canonical isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(\overline{B}/S) \xrightarrow{\sim} (R^1\bar{t}_{\mathrm{crys},*}\mathcal{O}_{\overline{B},\mathrm{crys}})_S \longrightarrow i^*(R^1\bar{t}_{\mathrm{crys},*}\mathcal{O}_{\overline{B},\mathrm{crys}})_T.$$

The Grothendieck–Messing Theorem (as in [Gro74, Ch. V, §4]) states that the functor sending B to $(\overline{B}, \mathcal{V}_B)$ defines an equivalence between the categories of abelian schemes over T and that of pairs $(A, \mathcal{V})$, where $s : A \to S$ is an abelian scheme and $\mathcal{V}$ is an $\mathcal{O}_T$-subbundle of $(R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_T$ such that $i^*\mathcal{V}$ corresponds to $s_*\Omega^1_{A/S}$ under the canonical isomorphism of $i^*(R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_T$ with $\mathcal{H}^1_{\mathrm{dR}}(A/S)$. A morphism of pairs $(A, \mathcal{V}_1) \to (A', \mathcal{V}')$ being a pair of morphisms $(A \to A', \mathcal{V}' \to \mathcal{V})$ satisfying the evident compatibility, it follows that if $t : B \to T$ and $t' : B' \to T$ are abelian schemes and $\psi : \overline{B} \to \overline{B}'$ is a morphism of their base-changes to S, then ψ extends (necessarily uniquely) to a morphism $B \to B'$ if and only if the morphism

$$\psi^*_{\mathrm{crys}} : (R^1\bar{t}'_{\mathrm{crys},*}\mathcal{O}_{\overline{B}',\mathrm{crys}})_T \longrightarrow (R^1\bar{t}_{\mathrm{crys},*}\mathcal{O}_{\overline{B},\mathrm{crys}})_T$$

sends $\mathcal{V}'$ to $\mathcal{V}$. In particular, the functor sending B to $\overline{B}$ is faithful, which is all that is needed in the proof of Proposition 2.1.

For later reference, we note if the data $\underline{A} = (A, \iota, \lambda, \eta, \mathcal{F}^\bullet)$ corresponds to an element of $\widetilde{Y}_U(S)$, then to give a lift to an element of $\widetilde{Y}_U(T)$ is equivalent to giving a lift $\mathcal{E}^\bullet$ of the Pappas–Rapoport filtrations to $(R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_T$, by which we mean a collection of $\mathcal{O}_{F,\mathfrak{p}} \otimes_{W(\mathbb{F}_\mathfrak{p}),\tau} \mathcal{O}_T$-submodules

$$0 = \mathcal{E}^{(0)}_\tau \subset \mathcal{E}^{(1)}_\tau \subset \cdots \subset \mathcal{E}^{(e_\mathfrak{p}-1)}_\tau \subset \mathcal{E}^{(e_\mathfrak{p})}_\tau$$

of $(R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_{T,\tau}$ for each $\tau = \tau_{\mathfrak{p},i}$ such that

- $\mathcal{E}^{(j)}_\tau / \mathcal{E}^{(j-1)}_\tau$ is a line bundle on T on which $\mathcal{O}_F$ acts via $\theta_{\mathfrak{p},i,j}$.
- $\iota^*\mathcal{E}^{(j)}_\tau$ corresponds to $\mathcal{F}^{(j)}_\tau$ under the canonical isomorphism

$$\iota^*(R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_T \cong (R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_S \cong \mathcal{H}^1_{\mathrm{dR}}(A/S)$$

for $j = 1, \ldots, e_\mathfrak{p}$. The bijection is defined by sending the data of a lift $\widetilde{\underline{A}} = (\widetilde{A}, \widetilde{\iota}, \widetilde{\lambda}, \widetilde{\eta}, \widetilde{\mathcal{F}}^\bullet)$ to the lift of filtrations corresponding to $\widetilde{\mathcal{F}}^\bullet$ under the canonical isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(\widetilde{A}/T) \cong (R^1\widetilde{s}_{\mathrm{crys},*}\mathcal{O}_{\widetilde{A},\mathrm{crys}})_T \cong (R^1 s_{\mathrm{crys},*}\mathcal{O}_{A,\mathrm{crys}})_T.$$

The injectivity of the map is a straightforward consequence of the Grothendieck–Messing Theorem; for the surjectivity, one needs also to know that the quasi-polarization λ extends to the lift of A associated to $(A, \mathcal{V})$ with $\mathcal{V} = \oplus_\tau \mathcal{E}^{(e_\mathfrak{p})}_\tau$, which is ensured by [Vol05, Prop. 2.10]. Since the theorem provides an equivalence of categories, it follows also that if a triple $(\underline{A}_1, \underline{A}_2, \psi)$ corresponds to an element of $\widetilde{Y}_{U_0(\mathfrak{P})}(S)$, then to give a lift to an element of $\widetilde{Y}_{U_0(\mathfrak{P})}(T)$ is equivalent to giving lifts $\mathcal{E}^\bullet_i$ of the Pappas–Rapoport filtrations $\mathcal{F}^\bullet_i$ to $(R^1 s_{i,\mathrm{crys},*}\mathcal{O}_{A_i,\mathrm{crys}})_T$ for $i = 1, 2$, such that $\mathcal{E}^\bullet_1$ and $\mathcal{E}^\bullet_2$ are compatible with

$$\psi^*_{\mathrm{crys},T} : (R^1 s_{2,\mathrm{crys},*}\mathcal{O}_{A_2,\mathrm{crys}})_T \longrightarrow (R^1 s_{1,\mathrm{crys},*}\mathcal{O}_{A_1,\mathrm{crys}})_T.$$

2.5. *Local structure: an example*

In the next section, we will recall the analysis of the local structure of $\widetilde{Y}_{U_0(\mathfrak{P})}$. First, however, at the suggestion of the referee, we consider the case where F is a quadratic extension ramified at p. This will already illustrate the key ideas and techniques; the general case is then mainly a matter of transforming them into an inductive argument.

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To further fix ideas and make the analysis more concrete, let $F = \mathbb{Q}(\sqrt{p})$ and $\mathfrak{P} = \mathfrak{p} = \varpi_{\mathfrak{p}}\mathcal{O}_F$, where $\varpi_{\mathfrak{p}} = \sqrt{p}$. We also let $\mathcal{O} = \mathbb{Z}_p[\varpi]$ and $\Sigma = \{\theta_1, \theta_2\}$, where $\varpi^2 = p$, $\theta_1(\varpi_{\mathfrak{p}}) = \varpi$ and $\theta_2(\varpi_{\mathfrak{p}}) = -\varpi$, so that $\mathcal{O}_F \otimes \mathcal{O} \cong \mathcal{O}[u]/(u^2 - p)$, in which $s_1 = u - \varpi$ and $t_1 = u + \varpi$ in the notation of §2.1.

Suppose now that $y \in \widetilde{Y}_{U_0(\mathfrak{p})}(\mathbb{F}_p)$, and let S denote the local ring $\mathcal{O}_{\widetilde{Y}_{U_0(\mathfrak{p})},y}$ and $(\underline{A}, \underline{A}', \psi)$ denote the corresponding triple over S. Thus, the $S[u]/(u^2 - p)$-module $H^0(A, \Omega^1_{A/S})$ is free of rank two over S, equipped with a filtration of S-modules

$$0 = F^{(0)} \subset F^{(1)} \subset F^{(2)} = H^0(A, \Omega^1_{A/S})$$

such that $(u - \varpi)F^{(1)} = 0$, $(u + \varpi)F^{(2)} \subset F^{(1)}$, and $L_1 := F^{(1)}$ and $L_2 := F^{(2)}/F^{(1)}$ are each free of rank one. Viewing $H^0(A, \Omega^1_{A/S})$ as a submodule of the free rank two $S[u]/(u^2 - p)$-module $H^1_{\mathrm{dR}}(A/S)$, and letting $G^{(1)} = (u + \varpi)H^1_{\mathrm{dR}}(A/S)$ (or equivalently, the kernel of $u - \varpi$ on $H^1_{\mathrm{dR}}(A/S)$) and $G^{(2)} = (u + \varpi)^{-1}F^{(1)}$ (i.e., the preimage of $F^{(1)}$ in $H^1_{\mathrm{dR}}(A/S)$ under $u + \varpi$), we obtain inclusions of free S-modules

$$\begin{tikzcd}[column sep=small] & & F^{(2)} \arrow[dr, hook] & & \\ 0 \subset & F^{(1)} \arrow[ur, hook] \arrow[dr, hook] & & G^{(2)} \subset H^1_{\mathrm{dR}}(A/S) \\ & & G^{(1)} \arrow[ur, hook] & & \end{tikzcd}$$

such that each successive quotient is free of rank one over S.

Furthermore, the free rank two S-modules $P_1 := G^{(1)}$ and $P_2 := G^{(2)}/F^{(1)}$ are equipped with perfect alternating pairings $\langle \cdot, \cdot \rangle_i$ whose construction we briefly recall (see [Dia23, §3.1] for more details). Firstly, Poincaré duality and the polarization on A yield an isomorphism

$$H^1_{\mathrm{dR}}(A/S) \xrightarrow{\sim} \mathrm{Hom}_S(\mathfrak{d}^{-1} \otimes_{\mathcal{O}_F} H^1_{\mathrm{dR}}(A/S), S) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{O}_F \otimes S}(H^1_{\mathrm{dR}}(A/S), \mathcal{O}_F \otimes S),$$

and hence a perfect alternating pairing $\langle \cdot, \cdot \rangle_{\mathrm{dR}}$ on $H^1_{\mathrm{dR}}(A/S)$ over $\mathcal{O}_F \otimes S = S[u]/(u^2 - p)$. Furthermore, one finds that this induces an isomorphism

$$G^{(1)} \xrightarrow{\sim} \mathrm{Hom}_S(H^1_{\mathrm{dR}}(A/S)/(u - \varpi)H^1_{\mathrm{dR}}(A/S), (u + \varpi)(\mathcal{O}_F \otimes S)),$$

and our perfect pairing on $P_1 = G^{(1)}$ (over S) is then obtained from the isomorphisms

$$H^1_{\mathrm{dR}}(A/S)/(u - \varpi)H^1_{\mathrm{dR}}(A/S) \xrightarrow{\sim} G^{(1)} \quad \text{and} \quad S \xrightarrow{\sim} (u + \varpi)(\mathcal{O}_F \otimes S)$$

defined by multiplication by $u + \varpi$. However, one finds that $F^{(1)}$ and $G^{(2)}$ are orthogonal complements, and the perfect pairing on $P_2 = G^{(2)}/F^{(1)}$ is then obtained from the resulting isomorphism

$$G^{(2)}/F^{(1)} \xrightarrow{\sim} \mathrm{Hom}_S(G^{(2)}/F^{(1)}, (u - \varpi)(\mathcal{O}_F \otimes S)).$$

Similarly, we have the filtration

$$0 = F'^{(0)} \subset F'^{(1)} \subset F'^{(2)} = H^0(A', \Omega^1_{A'/S}),$$

which we use to define S-modules $L'_i \subset P'_i$ for $i = 1, 2$ such that L'_i and P'_i/L'_i, free of rank one and P'_i is equipped with a perfect alternating pairing. Furthermore, the $S[u]/(u^2 - \varpi)$-linear map

$\psi_{\mathrm{dR}}^*: H^1_{\mathrm{dR}}(A'/S) \to H^1_{\mathrm{dR}}(A/S)$ induces morphisms $\psi_i^*: P_i' \to P_i$ restricting to $L_i' \to L_i$, and the equation $\psi \circ \lambda' \circ \psi^\vee = \varpi_{\mathfrak{p}}\lambda$ (where λ and λ' are the polarizations on A and A') implies that

$$\langle \psi_{\mathrm{dR}}^*(x), \psi_{\mathrm{dR}}^*(z)\rangle_{\mathrm{dR}} = u\langle x, z\rangle'_{\mathrm{dR}},$$

and hence, $\langle \psi_i^*(x), \psi_i^*(z)\rangle_i = \theta_i(\varpi_{\mathfrak{p}})\langle x, z\rangle_i' = \pm\varpi\langle x, z\rangle_i'$ for $i = 1, 2$ and $x, z \in P_i'$.

Consider the behavior of the maps ψ_i^* at the closed point (i.e., $\psi_{i,0}^*: P_{i,0}' \to P_{i,0}$), where we use $\cdot_0$ to denote $\cdot \otimes_S \mathbb{F}_p$. We find that $\psi_{i,0}^*$ has rank one (see the argument in the general case), and since $\psi_{i,0}^*(L_{i,0}') \subset L_{i,0}$, it follows that $\psi_{i,0}^*(L_{i,0}') = 0$ or $\psi_{i,0}^*(P_{i,0}') = L_{i,0}$. To fix ideas even further, let us suppose that both these hold for $i = 1$, and that only the first holds for $i = 2$. We will prove that in this case, the completion of S at its maximal ideal $\mathfrak{n}$ is isomorphic to the $\mathcal{O}$-algebra

$$\widehat{R} := \mathcal{O}[[X_1, X_1', X_2]]/(X_1X_1' + \varpi).$$

In order to do so, we will construct a homomorphism $\widehat{R} \to \widehat{S}$ using a parametrization of the S-lines $L_i \subset P_i$ and $L_i' \subset P_i'$ with respect to suitable choices of bases for the ambient S-planes. We will then use the Grothendieck–Messing Theorem to prove inductively that the resulting homomorphism $R_n \to S_n$ is an isomorphism for all $n \geq 1$, where $S_1 = S/(\varpi, \mathfrak{n})$, $S_n = S/\mathfrak{n}^n$ for $n \geq 2$, and R_n is defined similarly.

First, note that our assumptions on the $\psi_{i,0}^*$ imply that $F_0'^{(2)} = uH^1_{\mathrm{dR}}(A_0'/\mathbb{F}_p)$. Indeed, we have

$$F_0'^{(1)} \subset \ker(\psi_{\mathrm{dR},0}^*) \quad \text{and} \quad \psi_{\mathrm{dR},0}^*(F_0'^{(2)}) \subset F_0^{(1)} = \psi_{\mathrm{dR},0}^*(G_0'^{(1)}),$$

so that $F_0'^{(2)} \subset G_0'^{(1)}$, and comparing dimensions gives equality. However,

$$F_0^{(1)} \subset \psi_{\mathrm{dR},0}^*(G_0'^{(2)}) = \psi_{\mathrm{dR},0}^*(u^{-1}F_0'^{(1)}) \subset G_0^{(1)},$$

and comparing dimensions implies $\psi_{\mathrm{dR},0}^*(G_0'^{(2)}) = G_0^{(1)}$, but $\psi_{\mathrm{dR},0}^*(G_0'^{(2)}) \not\subset F_0^{(2)}$, so $F_0^{(2)} \neq G_0^{(1)}$. It follows that $F_0^{(2)}$ is a cyclic $\mathbb{F}_p[u]/(u^2)$-module and $F_0^{(1)} = uF_0^{(2)}$. We may therefore choose bases (x, z) for $H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p)$ and (x', z') for $H^1_{\mathrm{dR}}(A_0'/\mathbb{F}_p)$ as $\mathbb{F}_p[u]/(u^2)$-modules so that

$$F_0^{(1)} = \langle ux\rangle,\ F_0^{(2)} = \langle x\rangle,\ F_0'^{(1)} = \langle ux'\rangle \ \text{ and } \ F_0'^{(2)} = \langle ux', uz'\rangle$$

(where $\langle\cdot\rangle$ denotes generation as $\mathbb{F}_p[u]/(u^2)$-modules). Elementary manipulations show that we can furthermore choose these bases so that the matrix of $\psi_{\mathrm{dR},0}^*$ is $\begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}$ and $\langle x, z\rangle_{\mathrm{dR},0} = \langle x', z'\rangle'_{\mathrm{dR},0} = 1$.

We then have bases for $P_{1,0}$ and $P_{1,0}'$ defined by

$$e_{1,0} = ux,\ f_{1,0} = uz, \quad \text{and} \quad e_{1,0}' = ux',\ f_{1,0}' = uz',$$

so that $L_{1,0} = \mathbb{F}_p e_{1,0}$, $L_{1,0}' = \mathbb{F}_p e_{1,0}'$, $\langle e_{1,0}, f_{1,0}\rangle_{1,0} = \langle e_{1,0}', f_{1,0}'\rangle_{1,0}' = 1$, and the matrix of $\psi_{1,0}^*$ takes the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Consider the isomorphisms $P_{1,0} \otimes_{\mathbb{F}_p} S_1 \cong P_{1,1}$ and $P_{1,0}' \otimes_{\mathbb{F}_p} S_1 \cong P_{1,1}'$ obtained from the canonical isomorphisms

$$H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} S_1 \xrightarrow{\alpha} H^1_{\mathrm{dR}}(A_1/S_1) \quad \text{and} \quad H^1_{\mathrm{dR}}(A_0'/\mathbb{F}_p) \otimes_{\mathbb{F}_p} S_1 \xrightarrow{\alpha'} H^1_{\mathrm{dR}}(A_1'/S_1)$$

(systematically using $\cdot_n$ for $\otimes_S S_n$). Their functoriality properties further ensure that the matrix of $\psi_{1,1}^*$ has the same form as above with respect to the corresponding bases for $P_{1,1}$ and $P_{1,1}'$, which we denote $(e_{1,1}, f_{1,1})$ and $(e_{1,1}', f_{1,1}')$, and that $\langle e_{1,1}, f_{1,1}\rangle_{1,1} = \langle e_{1,1}', f_{1,1}'\rangle_{1,1}' = 1$. However $e_{1,1}$ and $e_{1,1}'$ are no longer bases for $L_{1,1}$ and $L_{1,1}'$; instead we have $L_{1,1} = S_1(e_{1,1} - s_{1,1}f_{1,1})$ and $L_{1,1}' = S_1(e_{1,1}' - s_{1,1}'f_{1,1})$ for some (unique) $s_{1,1}, s_{1,1}' \in \mathfrak{n}S_1$. An elementary matrix calculation then shows that we may lift the chosen bases (e_1, f_1) for P_1 and (e_1', f_1') for P_1' over S so that $\langle e_1, f_1\rangle_1 = \langle e_1', f_1'\rangle_1' = 1$ and the resulting

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matrix of ψ_1^* is $\left(\begin{smallmatrix} 0 & 1 \\ -\varpi & 0 \end{smallmatrix}\right)$. We then have $L_1 = S(e_1 - s_1 f_1)$ and $L_1' = S(e_1' - s_1' f_1)$ for some uniquely determined $s_1, s_1' \in \mathfrak{n}$, and the fact that $\psi_1^*(L_1') \subset L_1$ means that

$$\begin{pmatrix} 0 & 1 \\ -\varpi & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -s_1' \end{pmatrix} = \begin{pmatrix} -s_1' \\ -\varpi \end{pmatrix} = -s_1'\begin{pmatrix} 1 \\ -s_1 \end{pmatrix},$$

and therefore $s_1 s_1' = -\varpi$.

Similarly, using the bases for $P_{2,0}$ and $P_{2,0}'$ defined by

$$e_{2,0} = x + F_0^{(1)},\ f_{2,0} = uz + F_0^{(1)}, \quad \text{and} \quad e_{2,0}' = uz' + F_0'^{(1)},\ f_{2,0}' = -x' + F_0'^{(1)}$$

gives $L_{2,0} = \mathbb{F}_p e_{2,0}$, $L_{2,0}' = \mathbb{F}_p e_{2,0}'$, $\langle e_{2,0}, f_{2,0}\rangle_{2,0} = \langle e_{2,0}', f_{2,0}'\rangle_{2,0}' = 1$, and the matrix of $\psi_{2,0}^*$ takes the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note however that the canonical isomorphism α does not (necessarily) send $L_{1,0} = F_0^{(1)}$ to $L_{1,1} = F_1^{(1)}$, and therefore does not yield an isomorphism between $P_{2,0} \otimes_{\mathbb{F}_p} S_1$ and $P_{2,1}$. However letting $S_{1/2} = S/I$, where $I = (s_1, s_1', \mathfrak{n}^2) \subset (\varpi, \mathfrak{n}^2)$, we find that α and α' still induce isomorphisms $P_{2,0} \otimes_{\mathbb{F}_p} S_{1/2} \cong P_{2,1/2}$ and $P_{2,0}' \otimes_{\mathbb{F}_p} S_{1/2} \cong P_{2,1/2}'$. We may then lift the resulting bases for $P_{2,1/2}$ and $P_{2,1/2}'$ to ones, say (e_2, f_2) for P_2 and (e_2', f_2') for P_2', such that $\langle e_2, f_2\rangle_2 = \langle e_2', f_2'\rangle_2' = 1$ and the matrix for ψ_2^* has the form $\left(\begin{smallmatrix} -\varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Defining $s_2, s_2' \in \mathfrak{n}$ by $L_2 = S(e_2 - s_2 f_2)$ and $L_2' = (e_2' - s_2' f_2')$, the fact that $\psi_2^*(L_2') \subset L_2$ now translates into the equation $s_2' = -\varpi s_2$.

We now define the homomorphism $\rho : \widehat{R} \to \widehat{S}$ by $X_1 \mapsto s_1$, $X_1' \mapsto s_1'$ and $X_2 \mapsto s_2$, and we will sketch the proof that it is an isomorphism.

We first prove that ρ is surjective, or equivalently, that the $\mathbb{F}_p$-vector space $\mathfrak{n}/(\varpi, \mathfrak{n}^2)$ is spanned by $s_{1,1}$, $s_{1,1}'$ and $s_{2,1}$, or equivalently, if $\delta : S_1 \to T := \mathbb{F}_p[\epsilon]/(\epsilon^2)$ is an $\mathbb{F}_p$-algebra homomorphism such that $\delta(s_{1,1}) = \delta(s_{1,1}') = \delta(s_{2,1}) = 0$, then δ factors through $\mathbb{F}_p$. This in turn is equivalent to the assertion that for such a δ, the triple $(\underline{\widetilde{A}}, \underline{\widetilde{A}}', \widetilde{\psi}) := \delta^*(\underline{A}_1, \underline{A}_1', \psi_1)$, is isomorphic to the base-change (via $\mathbb{F}_p \hookrightarrow T$) of $(\underline{A}_0, \underline{A}_0', \psi_0)$. By the Grothendieck–Messing Theorem,[7] this in turn is equivalent to the assertion that for $i = 1, 2$, the T-modules $F_1^{(i)} \otimes_{S_1} T$ and $F_1'^{(i)} \otimes_{S_1} T$ correspond (respectively) to $F_0^{(i)} \otimes_{\mathbb{F}_p} T$ and $F_0'^{(i)} \otimes_{\mathbb{F}_p} T$ under the canonical isomorphisms

$$H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} T \cong H^1_{\mathrm{dR}}(\widetilde{A}/T) \quad \text{and} \quad H^1_{\mathrm{dR}}(A_0'/\mathbb{F}_p) \otimes_{\mathbb{F}_p} T \cong H^1_{\mathrm{dR}}(\widetilde{A}'/T)$$

(induced by α and α'). Since $F_1^{(1)} = L_{1,1} = S_1(e_{1,1} - s_{1,1} f_{1,1})$ and $\delta(s_{1,1}) = 0$, we have that $F_1^{(1)} \otimes_{S_1} T = T(e_{1,1} \otimes 1)$, and by construction, $e_{1,1}$ corresponds to $e_{1,0} \otimes 1$ under α. Similarly, we see that $F_1'^{(1)} \otimes_{S_1} T$ corresponds to $F_0'^{(1)} \otimes_{\mathbb{F}_p} T$. Note also that δ factors through $S_{1/2}$, so the same argument shows that

$$(F_1^{(2)}/F_1^{(1)}) \otimes_{S_1} T = L_{2,1} \otimes_{S_1} T = L_{2,1/2} \otimes_{S_{1/2}} T$$

corresponds to $L_{2,0} \otimes_{\mathbb{F}_p} T = (F_0^{(2)}/F_0^{(1)}) \otimes_{\mathbb{F}_p} T$ under the isomorphism induced by α, and hence that $F_1^{(2)} \otimes_{S_1} T$ corresponds to $F_0^{(1)} \otimes_{\mathbb{F}_p} T$. Finally, the fact that $L_{2,1/2}' \otimes_S T = T(e_{2,0}' \otimes 1)$ follows automatically from the description of ψ_2^*, so we similarly obtain the desired conclusion for $F_1'^{(2)} \otimes_{S_1} T$.

In order to prove that $\rho_1 : R_1 \to S_1$ is an isomorphism, we could again consider the map on tangent spaces, but in order to be more indicative of the inductive step treating ρ_n, we will interpret the argument as the construction of a surjective homomorphism $S_1 \to R_1$. Note that if $\widetilde{y} \in \widetilde{Y}_{U_0(\mathfrak{p})}(R_1)$ is a lift of y, then the induced map $S = \mathcal{O}_{\widetilde{Y}_{U_0(\mathfrak{p})}, y} \to R_1$ factors through S_1. Furthermore, by the Grothendieck–Messing

[7] As described at the end of §2.4, but in the simplest case, namely for the thickening Spec $(\mathbb{F}_p) \hookrightarrow$ Spec (T), and using the canonical isomorphisms $H^1_{\mathrm{crys}}(A_0/T) \cong H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} T$ and $H^1_{\mathrm{crys}}(A_0'/T) \cong H^1_{\mathrm{dR}}(A_0'/\mathbb{F}_p) \otimes_{\mathbb{F}_p} T$ to reinterpret (2.3).

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Theorem (with $i : \mathrm{Spec}\,(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}\,(R_1)$), to give such a lift is equivalent to giving lifts

$$\widetilde{F}^{(1)} \subset \widetilde{F}^{(2)} \subset H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} R_1 \quad \text{and} \quad \widetilde{F}'^{(1)} \subset \widetilde{F}'^{(2)} \subset H^1_{\mathrm{dR}}(A'_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} R_1$$

of the Pappas–Rapoport filtrations compatible with $\psi^*_{\mathrm{dR},0} \otimes 1$. More precisely, we require that the $\widetilde{F}^{(i)}$ and $\widetilde{F}'^{(i)}$ (for $i = 1, 2$) be free R_1-modules[8] such that

- $\widetilde{F}^{(i)} \otimes_{R_1} \mathbb{F}_p = F_0^{(i)}$ and $\widetilde{F}'^{(i)} \otimes_{R_1} \mathbb{F}_p = F_0'^{(i)}$;
- $u\widetilde{F}^{(i)} \subset \widetilde{F}^{(i-1)}$ and $u\widetilde{F}'^{(i)} \subset \widetilde{F}'^{(i-1)}$ (where $\widetilde{F}^{(0)} = \widetilde{F}'^{(0)} = 0$);
- $(\psi^*_{\mathrm{dR},0} \otimes 1)(\widetilde{F}'^{(i)}) \subset \widetilde{F}^{(i)}$.

We will write down such lifts in terms of the bases $(\widetilde{x}, \widetilde{z})$ and $(\widetilde{x}', \widetilde{z}')$, where (x, z) and (x', z') are the bases already chosen for $H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p)$ and $H^1_{\mathrm{dR}}(A'_0/\mathbb{F}_p)$ and $\widetilde{x} = x \otimes 1$, etc. Letting $\widetilde{P}_1 = uH^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} R_1$ and $\widetilde{P}'_1 = uH^1_{\mathrm{dR}}(A'_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} R_1$, note that the matrix of the R_1-linear map $\widetilde{\psi}_1 : \widetilde{P}'_1 \to \widetilde{P}_1$ induced by $\psi^*_{\mathrm{dR},0}$ with respect to the bases $(u\widetilde{x}, u\widetilde{z})$ and $(u\widetilde{x}', u\widetilde{z}')$ is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that

$$\widetilde{F}^{(1)} = R_1 u(\widetilde{x} - \overline{X}_1 \widetilde{z}) \quad \text{and} \quad \widetilde{F}'^{(1)} = R_1 u(\widetilde{x}' - \overline{X}'_1 \widetilde{z}')$$

are free R_1-modules such that $\widetilde{\psi}^*_1(\widetilde{F}'^{(1)}) \subset \widetilde{F}^{(1)}$, where $\overline{X}_1$ denotes the image of X_1 in R_1, etc. Letting $\widetilde{P}_2$ denote the free rank two R_1-module $(u^{-1}\widetilde{F}^{(1)})/\widetilde{F}^{(1)}$ and similarly defining $\widetilde{P}'_2$, one finds that the matrix of the R_1-linear map $\widetilde{\psi}_2 : \widetilde{P}'_2 \to \widetilde{P}_2$ induced by $\psi^*_{\mathrm{dR},0}$ is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with respect to the bases

$$(\widetilde{x} - \overline{X}_1 \widetilde{z} + \widetilde{F}^{(1)}, u\widetilde{z} + \widetilde{F}^{(1)}) \quad \text{and} \quad (\overline{X}_1 \widetilde{x}' + u\widetilde{z}' + \widetilde{F}'^{(1)}, -\widetilde{x}' + \overline{X}'_1 \widetilde{z}' + \widetilde{F}'^{(1)}).$$

(Note that $\overline{X}_1 \widetilde{x}' + u\widetilde{z}' \in u^{-1}\widetilde{F}'^{(1)}$ since $u(\overline{X}_1 \widetilde{x}' + u\widetilde{z}') = u\overline{X}_1(\widetilde{x}' - \overline{X}'_1 \widetilde{z}')$.) We thus find that

$$\widetilde{F}^{(2)} = \langle \widetilde{x} - (\overline{X}_1 + u\overline{X}_2)\widetilde{z} \rangle, \quad \text{and} \quad \widetilde{F}'^{(2)} = \langle u(\widetilde{x}' - \overline{X}'_1 \widetilde{z}'), \overline{X}_1 \widetilde{x}' + u\widetilde{z}' \rangle$$

yields lifts of the Pappas–Rapoport filtrations with the desired properties (where $\langle \cdot \rangle$ here denotes generation as $R_1[u]/(u^2)$-modules). As already explained, this yields a homomorphism $S_1 \to R_1$. Furthermore our definition of the parameters $s_{1,1}$, $s'_{1,1}$ and $s_{2,1}$ ensures that $s_{1,1} \mapsto \overline{X}_1$, $s'_{1,1} \mapsto \overline{X}'_1$ and $s_{2,1} \mapsto \overline{X}_2 \bmod (\overline{X}_1, \overline{X}'_1)$, so the map is surjective. Since R_1 and S_1 are Artinian, and $\rho_1 : R_1 \to S_1$ is also surjective, it follows that the lengths of R_1 and S_1 are the same, and therefore that ρ_1 is an isomorphism.

Suppose now that $n \geq 1$ and that $\rho_n : R_n \to S_n$ is an isomorphism. In order to prove that ρ_{n+1} is an isomorphism, it suffices (arguing as above) to show[9] that the composite $S \twoheadrightarrow S_n \xrightarrow{\rho_n^{-1}} R_n$ lifts to a surjective homomorphism $S \to R_{n+1}$. Using the thickening $\mathrm{Spec}\,(S_n) \hookrightarrow \mathrm{Spec}\,(R_{n+1})$, with $H^1_{\mathrm{crys}}(A_n/R_{n+1})$ instead of $H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} R_1$, the construction of lifts of the Pappas–Rapoport filtrations on $H^1_{\mathrm{dR}}(A_n/S_n)$ is similar to the one above, once one has abstracted the argument establishing the existence of suitable bases for $\widetilde{P}_i$ and $\widetilde{P}'_i$ (see the proof in the general case). While the isomorphism $H^1_{\mathrm{crys}}(A_0/R_1) \cong H^1_{\mathrm{dR}}(A_0/\mathbb{F}_p) \otimes_{\mathbb{F}_p} R_1$ was implicitly used in the proof of the surjectivity of the resulting homomorphism $S_1 \to R_1$, no such isomorphism is available relative to the thickening $\mathrm{Spec}\,(S_n) \hookrightarrow \mathrm{Spec}\,(R_{n+1})$, nor is it needed, since the surjectivity of the resulting homomorphism $S_{n+1} \to R_{n+1}$ is immediate from that of its composite with $R_{n+1} \to R_n$.

[8]The first bullet renders this equivalent to the successive quotients being free over R_1, and the second implies that they are in fact $R_1[u]/(u^2)$-modules.

[9]In this case, we have that $R_2 = R_1$ and $S_2 = S_1$, so we could assume $n \geq 2$ if we wanted.

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2.6. Local structure: in general

We now proceed to analyze the local structure of $\widetilde{Y}_0(\mathfrak{P})$ for arbitrary F and $\mathfrak{P}$, proving[10] in particular that the schemes are reduced and syntomic (i.e., flat local complete intersections, over $\mathcal{O}$).

Let y be a closed point of $\widetilde{Y}_{U_0(\mathfrak{P})}$ in characteristic p. We let S denote the local ring $\mathcal{O}_{\widetilde{Y}_{U_0(\mathfrak{P})},y}$, $\mathfrak{n}$ its maximal ideal, S_0 the residue field, $S_1 = S/(\mathfrak{n}^2, \varpi)$ and $S_n = S/\mathfrak{n}^n$ for $n > 1$. We let $(\underline{A}, \underline{A}', \psi)$ denote the corresponding triple over S, and similarly write $(\underline{A}_n, \underline{A}'_n, \psi_n)$ for the triple over S_n for $n \geq 0$. For each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$, we view the Pappas–Rapoport filtrations as being defined by $S[u]/(E_\tau)$-submodules $F_\tau^{(j)} \subset H^0(A, \Omega^1_{A/S})_\tau$ and their quotients $F_{\tau,n}^{(j)} \subset H^0(A_n, \Omega^1_{A_n/S_n})_\tau$ over $S_n[u]/(E_\tau)$ for $n \geq 0$, and similarly with A replaced by A'. For each $\theta = \theta_{\mathfrak{p},i,j} \in \Sigma$, we let $L_\theta = F_\tau^{(j)}/F_\tau^{(j-1)}$ and $P_\theta = G_\tau^{(j)}/F_\tau^{(j-1)}$, where $G_\tau^{(j)}$ denotes the preimage of $F_\tau^{(j-1)}$ under multiplication by $u - \theta(\varpi_\mathfrak{p})$ on $H^1_{\mathrm{dR}}(A/S)_\tau$, so that L_θ is a rank one summand of the free rank two S-module P_θ. We similarly define L'_θ and P'_θ, and systematically use $\cdot_n$ to denote $\cdot \otimes_S S_n$.

By definition, the isogeny $\psi : A \to A'$ induces morphisms $\psi_\theta^* : P'_\theta \to P_\theta$ for $\theta \in \Sigma$ such that $\psi_\theta^*(L'_\theta) \subset L_\theta$. Note that ψ_θ^* is an isomorphism for $\theta \notin \Sigma_\mathfrak{P}$. Recall also from [Dia23, §3.1] that Poincaré duality and the quasi-polarizations give rise to perfect alternating pairings on P_θ and P'_θ, which we denote by $\langle \cdot, \cdot \rangle_\theta$ and $\langle \cdot, \cdot \rangle'_\theta$. Furthermore, the compatibility of ψ with the quasi-polarizations implies the commutativity of the resulting diagram

$$\begin{tikzcd} \wedge_S^2 P'_\theta \arrow[r, "\wedge^2\psi_\theta^*"] \arrow[d, "\wr"] & \wedge_S^2 P_\theta \arrow[d, "\wr"] \\ S \arrow[r, "\theta(\varpi_\mathfrak{P})"] & S. \end{tikzcd}$$

Note in particular if $\theta \in \Sigma_\mathfrak{P}$, then the S_0-linear map $\psi_{\theta,0}^* : P'_{\theta,0} \to P_{\theta,0}$ is not invertible (since $\wedge_{S_0}^2 \psi_{\theta,0}^* = 0$); we will show that it has rank one. To that end, let $W = W(S_0)$ and consider the free rank two $\mathcal{O}_F \otimes W$-module $D := H^1_{\mathrm{crys}}(A_0/W)$. Thus, D decomposes as $\oplus_\tau D_\tau$ where each D_τ is free of rank two over $W[u]/(E_\tau)$, and we let $\widetilde{G}_\tau^{(j)}$ denote the preimage of $G_{\tau,0}^{(j)} = u^{-1}F_{\tau,0}^{(j-1)}$ in D_τ under the canonical projection to $D_\tau/pD_\tau \cong H^1_{\mathrm{dR}}(A_0/S_0)_\tau$. Note that if $\tau = \tau_{\mathfrak{p},i}$, then we have a sequence of inclusions

$$u^{e_\mathfrak{p}-1}D_\tau = \widetilde{G}_\tau^{(1)} \subset \widetilde{G}_\tau^{(2)} \subset \cdots \subset \widetilde{G}_\tau^{(e_\mathfrak{p}-1)} \subset \widetilde{G}_\tau^{(e_\mathfrak{p})}$$

such that $\dim_{S_0}(\widetilde{G}_\tau^{(j)}/pD_\tau) = \dim_{S_0}(G_{\tau,0}^{(j)}) = j+1$ and the projection to $H^1_{\mathrm{dR}}(A_0/S_0)_\tau$ identifies $\widetilde{G}_\tau^{(j)}/u\widetilde{G}_\tau^{(j)}$ with $P_{\theta,0}$ for $j = 1, \ldots, e_\mathfrak{p}$ and $\theta = \theta_{\mathfrak{p},i,j}$. Similarly, decomposing $D' := H^1_{\mathrm{crys}}(A'_0/W)$ as $\oplus_\tau D'_\tau$, and defining $\widetilde{G}'^{(j)}_\tau$ as the preimage of $G'^{(j)}_{\tau,0}$ in D'_τ, the isogeny $\psi_0 : A_0 \to A'_0$ induces an injective $\mathcal{O}_F \otimes W$-linear homomorphism $D' \to D$ whose cokernel is killed by u and has dimension $f_\mathfrak{P}$ over S_0. Furthermore, the restrictions $\widetilde{\psi}_\tau^* : D'_\tau \to D_\tau$ are isomorphisms for $\tau \notin \Sigma_{\mathfrak{P},0}$, and restrict to homomorphisms $\widetilde{G}'^{(j)}_\tau \to \widetilde{G}_\tau^{(j)}$ whose cokernel projects onto that of $\psi_{\theta,0}^*$ for each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$, $\theta = \theta_{\mathfrak{p},i,j}$. Therefore, it suffices to prove that the cokernel of each $\widetilde{G}'^{(j)}_\tau \to \widetilde{G}_\tau^{(j)}$, necessarily nontrivial if $\tau \in \Sigma_{\mathfrak{P},0}$, has length (at most) one over W. For $j = 1$, this follows from the identifications $u^{e_\mathfrak{p}-1}D_\tau = \widetilde{G}_\tau^{(1)}$ and $u^{e_\mathfrak{p}-1}D'_\tau = \widetilde{G}'^{(1)}_\tau$, which implies that the sum over τ of the lengths of the cokernels is $f_\mathfrak{P}$. For $j > 1$, it then follows by induction from the injectivity of $\widetilde{\psi}_\tau^*$ and the fact that $\widetilde{G}_\tau^{(j)}/\widetilde{G}_\tau^{(j-1)}$ and $\widetilde{G}'^{(j)}_\tau/\widetilde{G}'^{(j-1)}_\tau$ each have length one.

[10]The result is essentially Proposition 3.3 of [ERX17a], but we will give a complete proof for several reasons: 1) our different definition of the moduli problem, 2) our greater degree of generality, and 3) the omission of the proof in *loc. cit.* that the maps to Grassmannians induce isomorphisms on tangent spaces, where our perspective provides the basis for the construction of an Iwahori-level Kodaira–Spencer isomorphism.

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We then define

$$\Sigma_y = \{\,\theta \in \Sigma_{\mathfrak{P}} \mid \mathrm{im}(\psi^*_{\theta,0}) = L_{\theta,0}\,\} \quad \text{and} \quad \Sigma'_y = \{\,\theta \in \Sigma_{\mathfrak{P}} \mid \ker(\psi^*_{\theta,0}) = L'_{\theta,0}\,\}.$$

Note that $\Sigma_{\mathfrak{P}} = \Sigma_y \cup \Sigma'_y$, and define

$$R = \mathcal{O}[X_\theta, X'_\theta]_{\theta\in\Sigma}/(g_\theta)_{\theta\in\Sigma},$$

$$\text{where} \quad g_\theta = \begin{cases} X_\theta - \theta(\varpi_{\mathfrak{P}})X'_\theta, & \text{if } \theta \in \Sigma \setminus \Sigma'_y, \\ X'_\theta - \theta(\varpi_{\mathfrak{P}})X_\theta, & \text{if } \theta \in \Sigma'_y \setminus \Sigma_y, \\ X_\theta X'_\theta + \theta(\varpi_{\mathfrak{P}}), & \text{if } \theta \in \Sigma_y \cap \Sigma'_y. \end{cases}$$

We will show that a suitable parametrization of the lines L_θ and L'_θ by the variables X_θ and X'_θ defines an $\mathcal{O}$-algebra homomorphism $R \longrightarrow S$ inducing an isomorphism $W \otimes_{W(k)} \widehat{R}_{\mathfrak{m}} \xrightarrow{\sim} \widehat{S}_{\mathfrak{n}}$, where $\mathfrak{m}$ is the maximal ideal of R generated by ϖ and the variables X_θ and X'_θ for $\theta \in \Sigma$. (Recall that k is the residue field of $\mathcal{O}$, so that $R/\mathfrak{m} = k$, and that $W(k)$ denotes the ring of Witt vectors of k.)

Let α denote the composite of the canonical isomorphisms

$$H^1_{\mathrm{dR}}(A_0/S_0) \otimes_{S_0} S_1 \xrightarrow{\sim} H^1_{\mathrm{crys}}(A_0/S_1) \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_1/S_1)$$

(where we write $H^1_{\mathrm{crys}}(A_0/S_1)$ for $(R^1 s_{0,\mathrm{crys},*}\mathcal{O}_{A_0,\mathrm{crys}})(S_1))$, and similarly define

$$\alpha' : H^1_{\mathrm{dR}}(A'_0/S_0) \otimes_{S_0} S_1 \xrightarrow{\sim} H^1_{\mathrm{dR}}(A'_1/S_1).$$

Thus, α and α' are $\mathcal{O}_F \otimes S_1$-linear and compatible with ψ^*_0 and ψ^*_1. It follows that for each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$, α and α' decompose as direct sums of $S_1[u]/u^{e_{\mathfrak{p}}}$-linear isomorphisms α_τ and α'_τ such that the diagrams

$$\begin{tikzcd} H^1_{\mathrm{dR}}(A'_0/S_0)_\tau \otimes_{S_0} S_1 \arrow[r, "\psi^*_{\tau,0}\otimes 1"] \arrow[d, "\alpha'_\tau"'] & H^1_{\mathrm{dR}}(A_0/S_0)_\tau \otimes_{S_0} S_1 \arrow[d, "\alpha_\tau"] \\ H^1_{\mathrm{dR}}(A'_1/S_1)_\tau \arrow[r, "\psi^*_{\tau,1}"] & H^1_{\mathrm{dR}}(A_1/S_1)_\tau \end{tikzcd}$$

commute. We then consider the chain of ideals

$$(\mathfrak{n}^2, \varpi) = I^{(0)}_\tau \subset I^{(1)}_\tau \subset \cdots \subset I^{(e_{\mathfrak{p}}-1)}_\tau \subset I^{(e_{\mathfrak{p}})}_\tau \subset \mathfrak{n}$$

where $I^{(j)}_\tau$ is defined by the vanishing of the maps

$$F^{(\ell)}_{\tau,0} \otimes_{S_0} S_1 \longrightarrow H^1_{\mathrm{dR}}(A_1/S_1)_\tau/F^{(\ell)}_{\tau,1} \quad \text{and} \quad F'^{(\ell)}_{\tau,0} \otimes_{S_0} S_1 \longrightarrow H^1_{\mathrm{dR}}(A'_1/S_1)_\tau/F'^{(\ell)}_{\tau,1}$$

induced by α_τ and α'_τ for $\ell = 1, \ldots, j$. For $\theta = \theta_{\mathfrak{p},i,j}$, we let $S_\theta = S/I^{(j-1)}_\tau$, so that α_τ and α'_τ restrict in particular to $S_\theta[u]$-linear isomorphisms

$$F^{(j-1)}_{\tau,0} \otimes_{S_0} S_\theta \xrightarrow{\sim} F^{(j-1)}_\tau \otimes_S S_\theta \quad \text{and} \quad F'^{(j-1)}_{\tau,0} \otimes_{S_0} S_\theta \xrightarrow{\sim} F'^{(j-1)}_\tau \otimes_S S_\theta,$$

and hence to $S_\theta[u]$-linear isomorphisms

$$G^{(j)}_{\tau,0} \otimes_{S_0} S_\theta \xrightarrow{\sim} G^{(j)}_\tau \otimes_S S_\theta \quad \text{and} \quad G'^{(j)}_{\tau,0} \otimes_{S_0} S_\theta \xrightarrow{\sim} G'^{(j)}_\tau \otimes_S S_\theta.$$

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We thus obtain S_θ-linear isomorphisms α_θ and α'_θ such that the diagram

$$\begin{tikzcd} P'_{\theta,0}\otimes_{S_0} S_\theta \arrow[r, "\psi^*_{\theta,0}\otimes 1"] \arrow[d, "\alpha'_\theta"'] & P_{\theta,0}\otimes_{S_0} S_\theta \arrow[d, "\alpha_\theta"'] \\ P'_\theta \otimes_S S_\theta \arrow[r, "\psi^*_\theta\otimes 1"] & P_\theta\otimes_S S_\theta \end{tikzcd}$$

commutes. Furthermore, α_θ is compatible with the pairings $\langle\cdot,\cdot\rangle_{\theta,0}$ and $\langle\cdot,\cdot\rangle_\theta$, and similarly, α'_θ is compatible with $\langle\cdot,\cdot\rangle'_{\theta,0}$ and $\langle\cdot,\cdot\rangle'_\theta$.

We claim that bases $\mathcal{B}_\theta = (e_\theta, f_\theta)$ for P_θ and $\mathcal{B}'_\theta = (e'_\theta, f'_\theta)$ for P'_θ may be chosen so that

- $e_\theta \otimes_S S_\theta = \alpha_\theta(L_{\theta,0}\otimes_{S_0} S_\theta)$ and $e'_\theta\otimes_S S_\theta = \alpha'_\theta(L'_{\theta,0}\otimes_{S_0} S_\theta)$;
- $\langle e_\theta, f_\theta\rangle_\theta = \langle e'_\theta, f'_\theta\rangle'_\theta = 1$;
- the matrix of ψ^*_θ with respect to $\mathcal{B}_\theta$ and $\mathcal{B}'_\theta$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & \theta(\varpi_{\mathfrak{P}})\end{pmatrix}, \quad \begin{pmatrix} \theta(\varpi_{\mathfrak{P}}) & 0 \\ 0 & 1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ -\theta(\varpi_{\mathfrak{P}}) & 0\end{pmatrix} \tag{2.4}$$

  according to whether[11] $\theta \in \Sigma\setminus\Sigma'_y$, $\Sigma'_y\setminus\Sigma_y$ or $\Sigma_y\cap\Sigma'_y$.

Indeed, first choose bases $\mathcal{B}_{\theta,0} = (e_{\theta,0}, f_{\theta,0})$ for $P_{\theta,0}$ and $\mathcal{B}'_{\theta,0} = (e'_{\theta,0}, f'_{\theta,0})$ for $P'_{\theta,0}$ so that $L_{\theta,0} = S_0 e_{\theta,0}$, $L'_{\theta,0} = S_0 e'_{\theta,0}$, $\langle e_{\theta,0}, f_{\theta,0}\rangle_{\theta,0} = \langle e'_{\theta,0}, f'_{\theta,0}\rangle'_{\theta,0} = 1$ and $\psi^*_{\theta,0}$ has the required form (mod $\mathfrak{n}$), and lift the bases $(\alpha_\theta(e_{\theta,0}\otimes 1), \alpha_\theta(f_{\theta,0}\otimes 1))$ and $(\alpha'_\theta(e'_{\theta,0}\otimes 1), \alpha'_\theta(f'_{\theta,0}\otimes 1))$ arbitrarily to bases $\mathcal{B}_\theta$ and $\mathcal{B}'_\theta$ over S satisfying the condition on the pairings. The matrix T of ψ^*_θ with respect to $\mathcal{B}_\theta$ and $\mathcal{B}'_\theta$ then has the required form mod $I_\tau^{(j-1)}$ and satisfies $\det(T) = \theta(\varpi_{\mathfrak{P}})$. We may then replace $\mathcal{B}_\theta$ and $\mathcal{B}'_\theta$ by $\mathcal{B}_\theta U$ and $\mathcal{B}'_\theta U'$ for some $U, U' \in \ker(\mathrm{SL}_2(S)\to\mathrm{SL}_2(S_\theta))$ so that the resulting matrix $U^{-1}TU'$ has the required form.

We now have $L_\theta = S(e_\theta - s_\theta f_\theta)$ and $L'_\theta = S(e'_\theta - s'_\theta f'_\theta)$ for unique $s_\theta, s'_\theta \in \mathfrak{n}$, and the fact that $\psi^*_\theta(L'_\theta)\subset L_\theta$ means that

- $s_\theta = \theta(\varpi_{\mathfrak{P}})s'_\theta$ if $\theta\in\Sigma\setminus\Sigma'_y$;
- $s'_\theta = \theta(\varpi_{\mathfrak{P}})s_\theta$ if $\theta\in\Sigma'_y\setminus\Sigma_y$;
- $s_\theta s'_\theta = -\theta(\varpi_{\mathfrak{P}})$ if $\theta\in\Sigma_y\cap\Sigma'_y$.

Thus, the $\mathcal{O}$-algebra homomorphism $\mathcal{O}[X_\theta, X'_\theta]_{\theta\in\Sigma}\longrightarrow S$ defined by $X_\theta\mapsto s_\theta$, $X'_\theta\mapsto s'_\theta$ factors through R. We will prove that the resulting morphism $\rho: W\otimes_{W(k)}\widehat{R}_{\mathfrak{m}}\to\widehat{S}_{\mathfrak{n}}$ is an isomorphism.

We let $R_1 = W\otimes_{W(k)} R/(\mathfrak{m}^2,\varpi)$ and $R_n = W\otimes_{W(k)} R/\mathfrak{m}^n$, so we obtain morphisms $\rho_n: R_n\to S_n$. It suffices to prove that each of these is an isomorphism, which we achieve by induction on n.

We start by showing that ρ_1 is surjective (i.e., that the homomorphism induced by ρ on tangent spaces is injective). First, note that $I_\tau^{(j)} = (I_\tau^{(j-1)}, s_\theta, s'_\theta)$ for all $\tau = \tau_{\mathfrak{p},i}$, $\theta = \theta_{\mathfrak{p},i,j}$. Indeed, by construction, the ideal $I_\tau^{(j)}/I_\tau^{(j-1)}$ of S_θ is defined by the vanishing of the maps

$$L_{\theta,0}\otimes_{S_0} S_\theta\longrightarrow (P_\theta/L_\theta)\otimes_S S_\theta \quad\text{and}\quad L'_{\theta,0}\otimes_{S_0} S_\theta\longrightarrow (P'_\theta/L'_\theta)\otimes_S S_\theta$$

induced by α_θ and α'_θ, and these are precisely s_θ and s'_θ (mod $I_\tau^{(j-1)}$) relative to the bases $\alpha_\theta^{-1}(e_\theta\otimes 1)$, $(f_\theta + L_\theta)\otimes 1$, $\alpha'^{-1}_\theta(e'_\theta\otimes 1)$ and $(f'_\theta + L'_\theta)\otimes 1$. It follows that the image of ρ_1 contains each $I_\tau^{(e_\mathfrak{p})}$, so letting T denote the quotient of S by the ideal generated by $I_\tau^{(e_\mathfrak{p})}$ for all $\tau\in\Sigma_0$, it suffices to prove the projection $T\to S_0$ is an isomorphism (i.e., that the triple $(\underline{A}, \underline{A}', \psi)\otimes_S T$ is isomorphic to $(\underline{A}_0, \underline{A}'_0, \psi_0)\otimes_{S_0} T$). By the definition of T, we have that $F_{\tau,0}^{(j)}\otimes_{S_0} T$ corresponds to $F_{\tau,1}^{(j)}\otimes_{S_1} T$ under $\alpha_\tau\otimes_{S_1} T$ for all τ and j,

[11] Note that if $\theta\notin\Sigma_{\mathfrak{P}}$, then $\theta(\varpi_{\mathfrak{P}})\in\mathcal{O}^\times$ and ψ^*_θ is an isomorphism. This is the simplest case, but we toss it in with the case $\theta\in\Sigma_y\setminus\Sigma'_y$ for convenience; note that we could just as well have included it with the case $\theta\in\Sigma'_y\setminus\Sigma_y$.

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and similarly for $F'^{(j)}_{\tau,0}$, $F'^{(j)}_{\tau,1}$ and α'_τ, so the conclusion is immediate from the equivalence established at the end of §2.4.

To deduce that ρ_1 is an isomorphism, it suffices to prove that $\lg(R_1) \le \lg(S_1)$, which will follow from the existence of a surjection $S \to R_1$. We construct such a surjection by defining suitable lifts of the Pappas–Rapoport filtrations $F_0^\bullet$ and $F_0'^\bullet$ to $H^1_{\mathrm{crys}}(A_0/R_1)$ and $H^1_{\mathrm{crys}}(A'_0/R_1)$. For $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j}$, we let $R_\theta = R_1/J_\tau^{(j-1)}$ where the ideals $J_\tau^{(j)}$ are defined inductively by $J_\tau^{(0)} = (0)$ and $J_\tau^{(j)} = (J_\tau^{(j-1)}, X_\theta, X'_\theta)$. We will inductively define chains of $R_1[u]/u^{e_\mathfrak{p}}$-submodules

$$
\begin{aligned}
0 = \widetilde{F}_\tau^{(0)} \subset \widetilde{F}_\tau^{(1)} \subset \cdots \subset \widetilde{F}_\tau^{(e_\mathfrak{p})} \subset H^1_{\mathrm{crys}}(A_0/R_1)_\tau \\
\text{and } 0 = \widetilde{F}_\tau'^{(0)} \subset \widetilde{F}_\tau'^{(1)} \subset \cdots \subset \widetilde{F}_\tau'^{(e_\mathfrak{p})} \subset H^1_{\mathrm{crys}}(A'_0/R_1)_\tau
\end{aligned}
$$

such that the following hold for $j = 1, \dots, e_\mathfrak{p}$ and $\theta = \theta_{\mathfrak{p},i,j}$:

- $\widetilde{F}_\tau^{(j)}$ and $\widetilde{F}_\tau'^{(j)}$ are free (of rank j) over R_1;
- $u\widetilde{F}_\tau^{(j)} \subset \widetilde{F}_\tau^{(j-1)}$, $u\widetilde{F}_\tau'^{(j)} \subset \widetilde{F}_\tau'^{(j-1)}$ and $\psi^*_{0,\mathrm{crys}}(\widetilde{F}_\tau'^{(j)}) \subset \widetilde{F}_\tau^{(j)}$;
- $\widetilde{F}_\tau^{(j-1)} \otimes_{R_1} R_\theta$ corresponds to $F^{(j-1)}_{\tau,0} \otimes_{S_0} R_\theta$ and $\widetilde{F}_\tau'^{(j-1)} \otimes_{R_1} R_\theta$ corresponds to $F'^{(j-1)}_{\tau,0} \otimes_{S_0} R_\theta$ under the canonical isomorphisms $H^1_{\mathrm{dR}}(A_0/S_0) \otimes_{S_0} R_\theta \cong H^1_{\mathrm{crys}}(A_0/R_1) \otimes_{R_1} R_\theta$ and $H^1_{\mathrm{dR}}(A'_0/S_0) \otimes_{S_0} R_\theta \cong H^1_{\mathrm{crys}}(A'_0/R_1) \otimes_{R_1} R_\theta$, which therefore induce injective R_θ-linear homomorphisms
$$
\begin{aligned}
\beta_\theta : P_{\theta,0} \otimes_{S_0} R_\theta \longrightarrow (H^1_{\mathrm{crys}}(A_0/R_1)_\tau/\widetilde{F}_\tau^{(j-1)}) \otimes_{R_1} R_\theta \\
\text{and } \beta'_\theta : P'_{\theta,0} \otimes_{S_0} R_\theta \longrightarrow (H^1_{\mathrm{crys}}(A'_0/R_1)_\tau/\widetilde{F}_\tau'^{(j-1)}) \otimes_{R_1} R_\theta;
\end{aligned}
$$
- $(\widetilde{F}_\tau^{(j)}/\widetilde{F}_\tau^{(j-1)}) \otimes_{R_1} R_\theta$ is generated over R_θ by $\beta_\theta(e_{\theta,0} \otimes 1 - f_{\theta,0} \otimes X_\theta)$ and $(\widetilde{F}_\tau'^{(j)}/\widetilde{F}_\tau'^{(j-1)}) \otimes_{R_1} R_\theta$ is generated over R_θ by $\beta'_\theta(e'_{\theta,0} \otimes 1 - f'_{\theta,0} \otimes X'_\theta)$.

Suppose then that $1 \le j \le e_\mathfrak{p}$ and that

$$
0 = \widetilde{F}_\tau^{(0)} \subset \widetilde{F}_\tau^{(1)} \subset \cdots \subset \widetilde{F}_\tau^{(j-1)} \quad \text{and} \quad 0 = \widetilde{F}_\tau'^{(0)} \subset \widetilde{F}_\tau'^{(1)} \subset \cdots \subset \widetilde{F}_\tau'^{(j-1)}
$$

have been constructed as above. We let $\widetilde{G}_\tau^{(j)} = u^{-1}\widetilde{F}_\tau^{(j-1)}$, that is, the preimage of $\widetilde{F}_\tau^{(j-1)}$ in $H^1_{\mathrm{crys}}(A_0/R_1)_\tau$ under u, and $\widetilde{P}_\theta = \widetilde{G}_\tau^{(j)}/\widetilde{F}_\tau^{(j-1)}$. Since $H^1_{\mathrm{crys}}(A_0/R_1)_\tau$ is free of rank two over $R_1[u]/(u^{e_\mathfrak{p}})$, we see that $\widetilde{F}_\tau^{(j-1)} \subset uH^1_{\mathrm{crys}}(A_0/R_1)_\tau$, so that $\widetilde{G}_\tau^{(j)}$ and $\widetilde{P}_\theta$ are free (of ranks $j+1$ and 2, respectively) over R_1. Similarly, we let $\widetilde{G}_\tau'^{(j)} = u^{-1}\widetilde{F}_\tau'^{(j-1)}$ and $\widetilde{P}'_\theta = \widetilde{G}_\tau'^{(j)}/\widetilde{F}_\tau'^{(j-1)}$. The conditions on $\widetilde{F}_\tau^{(j-1)}$ and $\widetilde{F}_\tau'^{(j-1)}$ imply that the third bullet holds, that $\psi^*_{0,\mathrm{crys}}$ induces an R_1-linear homomorphism $\widetilde{\psi}^*_\theta : \widetilde{P}'_\theta \to \widetilde{P}_\theta$, and that β_θ and β'_θ define isomorphisms $P_{\theta,0} \otimes_{S_0} R_\theta \xrightarrow{\sim} \widetilde{P}_\theta \otimes_{R_1} R_\theta$ and $P'_{\theta,0} \otimes_{S_0} R_\theta \xrightarrow{\sim} \widetilde{P}'_\theta \otimes_{R_1} R_\theta$ compatible with $\psi^*_{\theta,0}$ and $\widetilde{\psi}^*_\theta$. Furthermore, the construction of $\widetilde{F}_\tau^{(j)}$ and $\widetilde{F}_\tau'^{(j)}$ satisfying the remaining conditions is equivalent to that of invertible R_1-submodules $\widetilde{L}_\theta \subset \widetilde{P}_\theta$ and $\widetilde{L}'_\theta \subset \widetilde{P}'_\theta$ such that

- $\widetilde{\psi}^*_\theta(\widetilde{L}'_\theta) \subset \widetilde{L}_\theta$;
- $\widetilde{L}_\theta \otimes_{R_1} R_\theta = \beta_\theta(e_{\theta,0} \otimes 1 - f_{\theta,0} \otimes X_\theta)R_\theta$;
- $\widetilde{L}'_\theta \otimes_{R_1} R_\theta = \beta'_\theta(e'_{\theta,0} \otimes 1 - f'_{\theta,0} \otimes X'_\theta)R_\theta$.

Exactly as for de Rham cohomology (see [Dia23, §3.1]), we obtain a perfect pairing

$$
H^1_{\mathrm{crys}}(A_0/R_1)_\tau \xrightarrow{\sim} \mathrm{Hom}_{R_1[u]/u^{e_\mathfrak{p}}}(H^1_{\mathrm{crys}}(A_0/R_1)_\tau, R_1[u]/(u^{e_\mathfrak{p}}))
$$

from the homomorphisms induced by the quasi-polarization on A_0 and Poincaré duality on crystalline cohomology. Furthermore, the pairing is compatible with the corresponding one on $H^1_{\mathrm{dR}}(B/R_1)_\tau$, where

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B and the resulting quasi-polarization are associated to an arbitrarily chosen extension of

$$0 \subset \widetilde{F}_\tau^{(1)} \subset \cdots \subset \widetilde{F}_\tau^{(j-1)}$$

to a lift $\widetilde{F}^\bullet$ of the Pappas–Rapoport filtrations to $H^1_{\mathrm{crys}}(A_0/R_1)$. We may therefore apply [Dia23, Lemma 3.1.1] to conclude that $(\widetilde{F}_\tau^{(j-1)})^\perp = u^{j+1-e_\mathfrak{p}}\widetilde{F}_\tau^{(j-1)} = u^{j-e_\mathfrak{p}}\widetilde{G}_\tau^{(j)}$, and hence obtain a perfect alternating R_1-valued pairing on $\widetilde{P}_\theta$, compatible via β_θ with $\langle\cdot,\cdot\rangle_{\theta,0}$. We similarly obtain a perfect alternating R_1-valued pairing on $\widetilde{P}'_\theta$ compatible via β'_θ with $\langle\cdot,\cdot\rangle'_{\theta,0}$. Furthermore, the compatibility of ψ_0 with the quasi-polarizations again implies that $\det\widetilde{\psi}^*_\theta = \theta(\varpi_\mathfrak{P})$ $(=0)$ with respect to the pairings.

The same argument as in the construction of the bases $\mathcal{B}_\theta$ and $\mathcal{B}'_\theta$ then yields lifts of

$$\beta_\theta(e_{\theta,0}\otimes 1, f_{\theta,0}\otimes 1) \quad \text{and} \quad \beta'_\theta(e'_{\theta,0}\otimes 1, f'_{\theta,0}\otimes 1)$$

to bases $(\widetilde{e}_\theta, \widetilde{f}_\theta)$ for $\widetilde{P}_\theta$ and $(\widetilde{e}'_\theta, \widetilde{f}'_\theta)$ for $\widetilde{P}'_\theta$ with respect to which the matrix of $\widetilde{\psi}^*_\theta$ is as in (2.4). Defining $\widetilde{L}_\theta = (\widetilde{e}_\theta - X_\theta\widetilde{f}_\theta)R_1$ and $\widetilde{L}'_\theta = (\widetilde{e}'_\theta - X'_\theta\widetilde{f}'_\theta)R_1$ then completes the construction of $\widetilde{F}_\tau^{(j)}$ and $\widetilde{F}_\tau'^{(j)}$ satisfying the desired properties.

We may now apply the Grothendieck–Messing Theorem (as described at the end of §2.4) to obtain a triple $(\underline{\widetilde{A}}, \underline{\widetilde{A}}', \widetilde{\psi})$ over R_1 lifting $(\underline{A}_0, \underline{A}'_0, \psi_0)$. Furthermore, the properties of the filtrations $\widetilde{F}^\bullet$ and $\widetilde{F}'^\bullet$ ensure, by induction on j for each τ, that the resulting morphism $S \to R_1 \to R_\theta$ factors through S_θ and induces isomorphisms

$$\begin{array}{ccccccc} P_\theta \otimes_S R_\theta & \xrightarrow{\sim} & \widetilde{P}_\theta \otimes_{R_1} R_\theta & & P'_\theta \otimes_S R_\theta & \xrightarrow{\sim} & \widetilde{P}'_\theta \otimes_{R_1} R_\theta \\ \cup & & \cup & \text{and} & \cup & & \cup \\ L_\theta \otimes_S R_\theta & \xrightarrow{\sim} & \widetilde{L}_\theta \otimes_{R_1} R_\theta & & L'_\theta \otimes_S R_\theta & \xrightarrow{\sim} & \widetilde{L}'_\theta \otimes_{R_1} R_\theta \end{array}$$

sending $(e_\theta\otimes 1, f_\theta\otimes 1)$ to $(\widetilde{e}_\theta\otimes 1, \widetilde{f}_\theta\otimes 1)$ and $(e'_\theta\otimes 1, f'_\theta\otimes 1)$ to $(\widetilde{e}'_\theta\otimes 1, \widetilde{f}'_\theta\otimes 1)$, so that $s_\theta \mapsto X_\theta$ and $s'_\theta \mapsto X'_\theta \pmod{J_\tau^{(j-1)}}$. We therefore conclude that the morphism $S \to R_1$ is surjective, and hence that $\rho_1 : R_1 \to S_1$ is an isomorphism.

Suppose then that $n \geq 1$ and that $\rho_n : R_n \to S_n$ is an isomorphism. The surjectivity of ρ_{n+1} is already immediate from that of ρ_n, so it suffices to prove that $\lg(R_{n+1}) \leq \lg(S_{n+1})$, which will follow from the existence of a lift of ρ_n^{-1} to morphism $S \to R_{n+1}$, which necessarily factors through a surjective morphism from S_{n+1}.

As in the case $n = 0$, we proceed by defining suitable lifts of the Pappas–Rapoport filtrations $F_n^\bullet$ and $F_n'^\bullet$ to the crystalline cohomology of A_n and A'_n evaluated on the thickening $R_{n+1} \to R_n \xrightarrow{\sim} S_n$ (with trivial divided power structure), but the argument is now simpler since the isomorphism on tangent spaces has already been established. More precisely, we inductively define chains of $R_{n+1}[u]/(E_\tau)$-submodules

$$\begin{aligned} 0 = \widetilde{F}_\tau^{(0)} \subset \widetilde{F}_\tau^{(1)} \subset \cdots \subset \widetilde{F}_\tau^{(e_\mathfrak{p})} \subset H^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau \\ \text{and } 0 = \widetilde{F}_\tau'^{(0)} \subset \widetilde{F}_\tau'^{(1)} \subset \cdots \subset \widetilde{F}_\tau'^{(e_\mathfrak{p})} \subset H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau \end{aligned}$$

such that the following hold for $j = 1, \ldots, e_\mathfrak{p}$ and $\theta = \theta_{\mathfrak{p},i,j}$:

- $\widetilde{F}_\tau^{(j)}$ and $\widetilde{F}_\tau'^{(j)}$ are free (of rank j) over R_{n+1};
- $(u - \theta(\varpi_\mathfrak{P}))\widetilde{F}_\tau^{(j)} \subset \widetilde{F}_\tau^{(j-1)}$, $(u - \theta(\varpi_\mathfrak{P}))\widetilde{F}_\tau'^{(j)} \subset \widetilde{F}_\tau'^{(j-1)}$ and $\psi^*_{n,\mathrm{crys}}(\widetilde{F}_\tau'^{(j)}) \subset \widetilde{F}_\tau^{(j)}$;
- $\widetilde{F}_\tau^{(j)} \otimes_{R_{n+1}} S_n$ corresponds to $F_{\tau,n}^{(j)}$ and $\widetilde{F}_\tau'^{(j)} \otimes_{R_{n+1}} S_n$ to $F_{\tau,n}'^{(j)}$ under the canonical isomorphisms $H^1_{\mathrm{dR}}(A_n/S_n) \cong H^1_{\mathrm{crys}}(A_n/R_{n+1}) \otimes_{R_{n+1}} S_n$ and $H^1_{\mathrm{dR}}(A'_n/S_n) \cong H^1_{\mathrm{crys}}(A'_n/R_{n+1}) \otimes_{R_{n+1}} S_n$.

Suppose then that $1 \leq j \leq e_\mathfrak{p}$ and that

$$0 = \widetilde{F}_\tau^{(0)} \subset \widetilde{F}_\tau^{(1)} \subset \cdots \subset \widetilde{F}_\tau^{(j-1)} \quad \text{and} \quad 0 = \widetilde{F}_\tau'^{(0)} \subset \widetilde{F}_\tau'^{(1)} \subset \cdots \subset \widetilde{F}_\tau'^{(j-1)}$$

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have been constructed as above. Let $\widetilde{G}_\tau^{(j)} = (u-\theta(\varpi_\mathfrak{p}))^{-1}\widetilde{F}_\tau^{(j-1)}$, $\widetilde{P}_\theta = \widetilde{G}_\tau^{(j)}/\widetilde{F}_\tau^{(j-1)}$ and similarly define $\widetilde{P}'_\theta$, so that $\widetilde{P}_\theta$ and $\widetilde{P}'_\theta$ are free of rank two over R_{n+1}. Furthermore, we have canonical isomorphisms

$$\beta_\theta : P_{\theta,n} \xrightarrow{\sim} \widetilde{P}_\theta \otimes_{R_{n+1}} S_n \quad \text{and} \quad \beta'_\theta : P'_{\theta,n} \xrightarrow{\sim} \widetilde{P}'_\theta \otimes_{R_{n+1}} S_n,$$

and $\psi^*_{n,\mathrm{crys}}$ induces an R_{n+1}-linear homomorphism $\widetilde{\psi}^*_\theta : \widetilde{P}'_\theta \to \widetilde{P}_\theta$ compatible with $\psi^*_{\theta,n}$. The construction of the required $\widetilde{F}_\tau^{(j)}$ and $\widetilde{F}_\tau'^{(j)}$ is then equivalent to that of invertible R_{n+1}-submodules $\widetilde{L}_\theta \subset \widetilde{P}_\theta$ and $\widetilde{L}'_\theta \subset \widetilde{P}'_\theta$ such that

- $\widetilde{\psi}^*_\theta(\widetilde{L}'_\theta) \subset \widetilde{L}_\theta$;
- $\widetilde{L}_\theta \otimes_{R_{n+1}} S_n = \beta_\theta(e_{\theta,n} - s_\theta f_{\theta,n})S_n$;
- $\widetilde{L}'_\theta \otimes_{R_{n+1}} S_n = \beta'_\theta(e'_{\theta,n} - s'_\theta f'_{\theta,n})S_n$.

The same argument as for $n = 0$ (with t_j^{-1} generalizing $u^{j-e_\mathfrak{p}}$) yields perfect alternating R_{n+1}-valued pairings on $\widetilde{P}_\theta$ on $\widetilde{P}'_\theta$ compatible via β_θ and β'_θ with $\langle\cdot,\cdot\rangle_{\theta,n}$ and $\langle\cdot,\cdot\rangle'_{\theta,n}$, and with respect to which $\det\widetilde{\psi}^*_\theta = \theta(\varpi_\mathfrak{P})$. We may thus lift $\beta_\theta(e_{\theta,n}, f_{\theta,n})$ and $\beta'_\theta(e'_{\theta,n}, f'_{\theta,n})$ to bases $(\widetilde{e}_\theta, \widetilde{f}_\theta)$ for $\widetilde{P}_\theta$ and $(\widetilde{e}'_\theta, \widetilde{f}'_\theta)$ for $\widetilde{P}'_\theta$ with respect to which the matrix of $\widetilde{\psi}^*_\theta$ is as in (2.4). Defining $\widetilde{L}_\theta = (\widetilde{e}_\theta - X_\theta \widetilde{f}_\theta)R_{n+1}$ and $\widetilde{L}'_\theta = (\widetilde{e}'_\theta - X'_\theta \widetilde{f}'_\theta)R_{n+1}$ thus yields the desired filtrations $\widetilde{F}^\bullet$ and $\widetilde{F}'^\bullet$, and hence a lift of (A_n, A'_n, ψ_n) to a triple over R_{n+1}.

This completes the proof that ρ is isomorphism, and hence that $\mathcal{O}^\wedge_{\widetilde{Y}_{U_0(\mathfrak{P})},y}$ is isomorphic to

$$W \otimes_{W(k)} \mathcal{O}[[X_\theta, X'_\theta]]_{\theta\in\Sigma}/(g_\theta)_{\theta\in\Sigma}.$$

As this is a reduced complete intersection, flat of relative dimension d over $\mathcal{O}$, and $\widetilde{Y}_{U_0(\mathfrak{P}),K}$ is smooth of dimension d over K, it follows that $\widetilde{Y}_{U_0(\mathfrak{P})}$ is reduced and syntomic of dimension d over $\mathcal{O}$, and that the same holds for $Y_{U_0(\mathfrak{P})}$ since $\widetilde{Y}_{U_0(\mathfrak{P})} \to Y_{U_0(\mathfrak{P})}$ is an étale cover.

3. The Kodaira–Spencer isomorphism

3.1. The layers of a thickening

Recall that the Kodaira–Spencer isomorphism describes the dualizing sheaf of the smooth scheme Y_U over $\mathcal{O}$ in terms of automorphic line bundles; more precisely we have a canonical[12] isomorphism

$$\delta^{-1} \otimes \omega^{\otimes 2} \xrightarrow{\sim} \mathcal{K}_{Y_U/\mathcal{O}},$$

where ω and δ are the line bundles on Y_U defined in §2.3 and $\mathcal{K}_{Y_U/\mathcal{O}} = \wedge^d_{\mathcal{O}_{Y_U}} \Omega^1_{Y_U/\mathcal{O}}$ (see [Dia23, §3.3]). We will prove an analogous result for $Y_{U_0(\mathfrak{P})}$, which by the results of the preceding section is Gorenstein over $\mathcal{O}$, and hence has an invertible dualizing sheaf $\mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}}$.

First, recall that the Kodaira–Spencer isomorphism for Y_U was established in [RX17] by constructing a suitable filtration on the sheaf of relative differentials and relating its graded pieces to automorphic line bundles. In [Dia23, §3.3], we gave a version of the construction which can be viewed as peeling off layers of the first infinitesimal thickening of the diagonal in $\widetilde{Y}_U \times_\mathcal{O} \widetilde{Y}_U$. We use the same approach here to describe the layers of the first infinitesimal thickening of $\widetilde{Y}_{U_0(\mathfrak{P})}$ in $\widetilde{Y}_U \times_\mathcal{O} \widetilde{Y}_U$ along

$$\widetilde{h} := (\widetilde{\pi}_1, \widetilde{\pi}_2) : \widetilde{Y}_{U_0(\mathfrak{P})} \to \widetilde{Y}_U \times_\mathcal{O} \widetilde{Y}_U,$$

[12] A precise sense in which it is canonical is its compatibility with the natural actions of $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$. The isomorphism, however, depends up to sign on a choice of ordering of Σ, which we fix in retrospect and perpetuity.

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which we recall is a closed immersion for sufficiently small U by Proposition 2.1. Indeed the Kodaira–Spencer filtration for Y_U may be viewed as the particular case where $\mathfrak{P} = \mathcal{O}_F$, for which the proof in [Dia23, §3.3] requires only minor modifications to establish the desired generalization.

We assume throughout this section that U is sufficiently small that Proposition 2.1 holds. To simplify notation and render it more consistent with that of [Dia23, §3.3], we will write S for $\widetilde{Y}_U$, X for $S \times_{\mathcal{O}} S$, Y for $\widetilde{Y}_{U_0(\mathfrak{P})}$, $\mathcal{J}$ for the sheaf of ideals of $\mathcal{O}_X$ defining the image of $\widetilde{h} : Y \hookrightarrow X$, $\iota : Y \hookrightarrow Z = \mathbf{Spec}(\mathcal{O}_X/\mathcal{J}^2)$ for the first infinitesimal thickening of Y in X, and $\mathcal{I} = \mathcal{J}/\mathcal{J}^2$ for the sheaf of ideals of $\mathcal{O}_Z$ defining the image of ι. Since X is smooth over $\mathcal{O}$ of dimension $2d$ and Y is syntomic over $\mathcal{O}$ of dimension d (all dimensions being relative), it follows that $\widetilde{h}$ is a regular immersion (see for example [StaX, §0638]) and its conormal bundle

$$\widetilde{\mathcal{C}} := \widetilde{h}^*\mathcal{J} = \iota^*\mathcal{I}$$

is locally free of rank d on $Y = \widetilde{Y}_{U_0(\mathfrak{P})}$.

Theorem 3.1. *There exists a decomposition $\widetilde{\mathcal{C}} = \bigoplus_{\tau\in\Sigma_0} \widetilde{\mathcal{C}}_\tau$, together with an increasing filtration*

$$0 = \mathrm{Fil}^0(\widetilde{\mathcal{C}}_\tau) \subset \mathrm{Fil}^1(\widetilde{\mathcal{C}}_\tau) \subset \cdots \subset \mathrm{Fil}^{e_\mathfrak{p}-1}(\widetilde{\mathcal{C}}_\tau) \subset \mathrm{Fil}^{e_\mathfrak{p}}(\widetilde{\mathcal{C}}_\tau) = \widetilde{\mathcal{C}}_\tau$$

for each $\tau = \tau_{\mathfrak{p},i}$, such that $\mathrm{gr}^j(\widetilde{\mathcal{C}}_\tau) \cong \widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \otimes_{\mathcal{O}_Y} \widetilde{\pi}_2^*\mathcal{L}_\theta$ for each $j = 1, \ldots, e_\mathfrak{p}$, where $\theta = \theta_{\mathfrak{p},i,j}$.

Proof. Let A denote the universal abelian scheme over S, let $\psi : A_1 \to A_2$ denote the universal isogeny over Y, and let $B_i = q_i^*A$ for $i = 1, 2$, where q_i denotes the restriction to Z of the i^{th} projection $S \times_{\mathcal{O}} S \to S$. Consider the $\mathcal{O}_F \otimes \mathcal{O}_Z$-linear morphism

$$\begin{aligned} q_2^*\mathcal{H}_{\mathrm{dR}}(A/S) \cong \mathcal{H}^1_{\mathrm{dR}}(B_2/Z) &\cong (R^1 s_{2,\mathrm{crys},*}\mathcal{O}_{A_2,\mathrm{crys}})_Z \\ &\xrightarrow{-\psi^*_{\mathrm{crys}}} (R^1 s_{1,\mathrm{crys},*}\mathcal{O}_{A_1,\mathrm{crys}})_Z \cong \mathcal{H}^1_{\mathrm{dR}}(B_1/Z) \cong q_1^*\mathcal{H}^1_{\mathrm{dR}}(A/S) \end{aligned} \tag{3.1}$$

extending[13] $\iota^*\mathcal{H}^1_{\mathrm{dR}}(B_2/Z) \cong \mathcal{H}^1_{\mathrm{dR}}(A_2/Y) \xrightarrow{-\psi^*} \mathcal{H}^1_{\mathrm{dR}}(A_1/Y) \cong \iota^*\mathcal{H}^1_{\mathrm{dR}}(B_1/Z)$, where the isomorphisms flanking $-\psi^*_{\mathrm{crys}}$ are the ones in (2.3), for $\iota : Y \hookrightarrow Z$ with $B = B_i$. Fixing for the moment $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$ and $\theta = \theta_{\mathfrak{p},i,1}$, it follows from the definition of $\mathcal{P}_\theta = \mathcal{G}_\tau^{(1)}$ that (3.1) restricts to an $\mathcal{O}_Z$-linear morphism $q_2^*\mathcal{P}_\theta \to q_1^*\mathcal{P}_\theta$. Furthermore, the composite

$$q_2^*\mathcal{L}_\theta \hookrightarrow q_2^*\mathcal{P}_\theta \to q_1^*\mathcal{P}_\theta \twoheadrightarrow q_1^*\mathcal{M}_\theta$$

has trivial pull-back to Y, so it factors through a morphism

$$\iota_*\widetilde{\pi}_2^*\mathcal{L}_\theta = q_2^*\mathcal{L}_\theta \otimes_{\mathcal{O}_Z} (\mathcal{O}_Z/\mathcal{I}) \longrightarrow q_1^*\mathcal{M}_\theta \otimes_{\mathcal{O}_Z} \mathcal{I} = \iota_*\widetilde{\pi}_1^*\mathcal{M}_\theta \otimes_{\mathcal{O}_Z} \mathcal{I},$$

and hence induces a morphism

$$\Xi_\theta : \iota_*(\widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \otimes_{\mathcal{O}_Y} \widetilde{\pi}_2^*\mathcal{L}_\theta) \longrightarrow \mathcal{I}.$$

We then define the sheaf of ideals $\mathcal{I}_\tau^{(1)}$ on Z to be the image of Ξ_θ, and we let $Z_\tau^{(1)}$ denote the subscheme of Z defined by $\mathcal{I}_\tau^{(1)}$, and $q_{\tau,i}^{(1)}$ (for $i = 1, 2$) the restrictions of the projection maps to $Z_\tau^{(1)} \to S$. By construction, the pull-back of Ξ_θ to $Z_\tau^{(1)}$ is trivial, and hence so is that of the

[13] The choice of sign is made for consistency with the conventions of the classical Kodaira–Spencer isomorphism, that is, the case $\mathfrak{P} = \mathcal{O}_F$, where $\widetilde{h}$ is the diagonal embedding, ψ is the identity, and we would ordinarily consider the resulting morphism $q_1^*\mathcal{H}^1_{\mathrm{dR}}(A/S) \to q_2^*\mathcal{H}^1_{\mathrm{dR}}(A/S)$. In general, the morphism induced by ψ goes in the opposite direction, so we introduce a minus sign here to obviate factors of $(-1)^d$ in later compatibility formulas.

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morphism $q_2^*\mathcal{L}_\theta \to q_1^*\mathcal{M}_\theta$, which implies that the pull-back of (3.1) maps $q_{\tau,2}^{(1)*}\mathcal{L}_\theta = q_{\tau,2}^{(1)*}\mathcal{F}_\tau^{(1)}$ to $q_{\tau,1}^{(1)*}\mathcal{L}_\theta = q_{\tau,1}^{(1)*}\mathcal{F}_\tau^{(1)}$. It therefore follows from the definition of $\mathcal{G}_\tau^{(2)}$ and the $\mathcal{O}_F \otimes \mathcal{O}_Z$-linearity of (3.1) that its pull-back to $Z_\tau^{(1)}$ restricts to an $\mathcal{O}_{Z_\tau^{(1)}}$-linear morphism $q_{\tau,2}^{(1)*}\mathcal{G}_\tau^{(2)} \to q_{\tau,1}^{(1)*}\mathcal{G}_\tau^{(2)}$ (if $e_\mathfrak{p} > 1$), and hence to a morphism

$$q_{\tau,2}^{(1)*}\mathcal{P}_{\theta'} \longrightarrow q_{\tau,1}^{(1)*}\mathcal{P}_{\theta'},$$

where $\theta' = \theta_{\mathfrak{p},i,2}$. The same argument as above now yields a morphism

$$\Xi_{\theta'} : \iota_*(\widetilde{\pi}_1^*\mathcal{M}_{\theta'}^{-1} \otimes_{\mathcal{O}_Y} \widetilde{\pi}_2^*\mathcal{L}_{\theta'}) \longrightarrow \mathcal{I}/\mathcal{I}_\tau^{(1)}$$

whose image is $\mathcal{I}_\tau^{(2)}/\mathcal{I}_\tau^{(1)}$ for some sheaf of ideals $\mathcal{I}_\tau^{(2)} \supset \mathcal{I}_\tau^{(1)}$ on Z.

Iterating the above construction thus yields, for each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$, a chain of sheaves of ideals

$$0 = \mathcal{I}_\tau^{(0)} \subset \mathcal{I}_\tau^{(1)} \subset \cdots \subset \mathcal{I}_\tau^{(e_\mathfrak{p})}$$

on Z such that for $j = 1, \ldots, e_\mathfrak{p}$ and $\theta = \theta_{\mathfrak{p},i,j}$, the morphism (3.1) induces

- $\mathcal{O}_F \otimes \mathcal{O}_{Z_\tau^{(j)}}$-linear morphisms $q_{\tau,2}^{(j)*}\mathcal{F}_\tau^{(j)} \longrightarrow q_{\tau,1}^{(j)*}\mathcal{F}_\tau^{(j)}$
- and surjections $\Xi_\theta : \iota_*(\widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \otimes_{\mathcal{O}_Y} \mathcal{L}_\theta) \twoheadrightarrow \mathcal{I}_\tau^{(j)}/\mathcal{I}_\tau^{(j-1)}$,

where $Z_\tau^{(j)}$ is the closed subscheme of Z defined by $\mathcal{I}_\tau^{(j)}$, and $q_{\tau,1}^{(j)}, q_{\tau,2}^{(j)}$ are the projections $Z_\tau^{(j)} \to S$.

Furthermore, we claim that the map

$$\bigoplus_{\mathfrak{p}\in S_p} \bigoplus_{\tau\in\Sigma_{\mathfrak{p},0}} \mathcal{I}_\tau^{(e_\mathfrak{p})} \to \mathcal{I}$$

is surjective. Indeed, let Z' denote the closed subscheme of Z defined by the image, so Z' is the scheme-theoretic intersection of the $Z_\tau^{(e_\mathfrak{p})}$. For $i = 1, 2$, let q_i' denote the projection map $Z' \to S$ and $s_i' : B_i' \to Z'$ the pull-back of $s : A \to S$. By construction, (3.1) pulls back to a morphism $q_2'^*\mathcal{H}^1_{\mathrm{dR}}(A/S) \to q_1'^*\mathcal{H}^1_{\mathrm{dR}}(A/S)$ under which $q_2'^*\mathcal{F}_\tau^{(j)}$ maps to $q_1'^*\mathcal{F}_\tau^{(j)}$ for all τ and j. In particular, $s'_{2,*}\Omega^1_{B_2'/T} = q_2'^*(s_*\Omega^1_{A/S})$ maps to $q_1'^*(s_*\Omega^1_{A/S}) = s'_{1,*}\Omega^1_{B_1'/T}$, which the Grothendieck–Messing Theorem therefore implies is induced by an isogeny $\widetilde{\psi} : B_1' \to B_2'$ of abelian schemes lifting ψ, which is necessarily compatible with $\mathcal{O}_F$-actions, quasi-polarizations and level structures on $\underline{B}_i' := q_i'^*\underline{A}$. Since $\widetilde{\psi}$ also respects the Pappas–Rapoport filtrations $q_i'^*\mathcal{F}^\bullet$, it follows that the triple $(\underline{B}_1', \underline{B}_2', \widetilde{\psi})$ corresponds to a morphism $r : Z' \to Y$ such that $\widetilde{h} \circ r$ is the closed immersion $Z' \hookrightarrow X$. Since $\widetilde{h}$ is also a closed immersion, it follows that r is an isomorphism, yielding the desired surjectivity.

Now defining $\widetilde{\mathcal{C}}_\tau = \iota^*\mathcal{I}_\tau^{(e_\mathfrak{p})}$ and $\mathrm{Fil}^j(\widetilde{\mathcal{C}}_\tau) = \iota^*\mathcal{I}_\tau^{(j)}$ for each $\tau = \tau_{\mathfrak{p},i}$ and $j = 1, \ldots, e_\mathfrak{p}$, we obtain surjective morphisms

$$\bigoplus_{\tau\in\Sigma_0} \widetilde{\mathcal{C}}_\tau \twoheadrightarrow \widetilde{\mathcal{C}}, \qquad \text{and} \qquad \widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \otimes_{\mathcal{O}_Y} \widetilde{\pi}_2^*\mathcal{L}_\theta \twoheadrightarrow \mathrm{gr}^j(\widetilde{\mathcal{C}}_\tau) \quad \text{for each } \theta = \theta_{\mathfrak{p},i,j}.$$

Since the $\widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \otimes_{\mathcal{O}_Y} \widetilde{\pi}_2^*\mathcal{L}_\theta$ are line bundles and $\widetilde{\mathcal{C}}$ is a vector bundle of rank d, it follows that all the maps are isomorphisms. ◻

Rewriting the line bundles in the statement of the theorem as

$$\widetilde{\mathcal{D}}_\theta = \widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \otimes \widetilde{\pi}_2^*\mathcal{L}_\theta \cong \widetilde{\pi}_1^*\mathcal{N}_\theta^{-1} \otimes \widetilde{\pi}_1^*\mathcal{L}_\theta \otimes \widetilde{\pi}_2^*\mathcal{L}_\theta$$

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(all tensor products being over $\mathcal{O}_Y$), we consequently obtain an isomorphism

$$\bigwedge^d \widetilde{\mathcal{C}} \cong \bigotimes_{\theta\in\Sigma} \widetilde{\mathcal{D}}_\theta \cong \widetilde{\pi}_1^*\widetilde{\delta}^{-1} \otimes \widetilde{\pi}_1^*\widetilde{\omega} \otimes \widetilde{\pi}_2^*\widetilde{\omega}.$$

Recall that the action of $\mathcal{O}_{F,(p),+}^\times$ on Y is defined by its action on the quasi-polarizations of both abelian schemes in the triple $(\underline{A}_1, \underline{A}_2, \psi)$. In particular, the action factors through $\mathcal{O}_{F,(p),+}^\times/(U \cap \mathcal{O}_F^\times)^2$ and is compatible under $\widetilde{h}$ with the diagonal action on the product $X = S \times_{\mathcal{O}} S$. The conormal bundle $\widetilde{\mathcal{C}}$ of $\widetilde{h}$ is thus equipped with an action of $\mathcal{O}_{F,(p),+}^\times$ over its action on Y, coinciding with the action on $\widetilde{\mathcal{C}}$ obtained from its identification with the pull-back of the conormal bundle of the closed immersion $h : Y_{U_0(\mathfrak{P})} \hookrightarrow Y_U \otimes_{\mathcal{O}} Y_U$, which we denote by $\mathcal{C}$.

Recall also that we have a natural action of $\mathcal{O}_{F,(p),+}^\times$ on the line bundles $\mathcal{L}_\theta$ and $\mathcal{M}_\theta$ over its action on S, under which ν acts as $\theta(\mu)$ if $\nu = \mu^2$ for $\mu \in U \cap \mathcal{O}_F^\times$. It follows that the action factors through $\mathcal{O}_{F,(p),+}^\times/(U \cap \mathcal{O}_F^\times)^2$ and hence defines descent data on the line bundles $\widetilde{\mathcal{D}}_\theta$; we let $\mathcal{D}_\theta$ denote the resulting line bundle on $Y_{U_0(\mathfrak{P})}$. It is straightforward to check that the morphisms constructed in the proof of Theorem 3.1 are compatible with the actions of $\mathcal{O}_{F,(p),+}^\times$ on the bundles $\widetilde{\mathcal{D}}_\theta$ and $\widetilde{\mathcal{C}}$, so that the decomposition and the filtrations on resulting components of $\widetilde{\mathcal{C}}$ descend to ones over $Y_{U_0(\mathfrak{P})}$, as do the isomorphisms between the graded pieces and the line bundles $\widetilde{\mathcal{D}}_\theta$.

Similarly, the constructions in the proof of Theorem 3.1 are compatible with the natural action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$. More precisely, suppose that $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ is such that $g^{-1}Ug \subset U'$, where U, U' are sufficiently small open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$ (and such that Proposition 2.1 also holds for U'), and let $\widetilde{\rho}_g$ denote the morphisms $Y \to Y' := \widetilde{Y}_{U_0'(\mathfrak{P})}$ and $S \to S' = \widetilde{Y}_{U'}$ defined by the action on level structures as in §2.2. Systematically annotating with $'$ the corresponding objects for U', so for example writing $\widetilde{\mathcal{C}}'$ for the conormal bundle of $\widetilde{h}' : Y' \hookrightarrow X'$, Theorem 3.1 yields a decomposition $\widetilde{\mathcal{C}}' = \oplus_{\tau\in\Sigma_0}\widetilde{\mathcal{C}}'_\tau$, filtrations $\mathrm{Fil}^j(\widetilde{\mathcal{C}}'_\tau)$ and isomorphisms $\mathrm{gr}^j(\widetilde{\mathcal{C}}_\tau) \cong \widetilde{\mathcal{D}}'_\theta$, where as usual, $\theta = \theta_{\mathfrak{p},i,j}$ if $\tau = \tau_{\mathfrak{p},i}$. In addition to the isomorphisms $\widetilde{\rho}_g^*\widetilde{\mathcal{D}}'_\theta \xrightarrow{\sim} \widetilde{\mathcal{D}}$ defined as in §2.3, we have the isomorphism $\widetilde{\rho}_g^*\widetilde{\mathcal{C}}' \xrightarrow{\sim} \widetilde{\mathcal{C}}$ obtained from the Cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{\rho}_g"'] & X \arrow[d, "(\widetilde{\rho}_g{,}\widetilde{\rho}_g)"] \\ Y' \arrow[r, "\widetilde{h}'"'] & X'. \end{tikzcd}$$

We then find that this isomorphism restricts to give $\widetilde{\rho}_g^*\mathrm{Fil}^j(\widetilde{\mathcal{C}}'_\tau) \xrightarrow{\sim} \mathrm{Fil}^j(\widetilde{\mathcal{C}}_\tau)$ for all τ and j, and that the resulting diagram

$$\begin{tikzcd} \widetilde{\rho}_g^*\widetilde{\mathcal{D}}'_\theta \arrow[r, "\sim"] \arrow[d, "\wr"'] & \widetilde{\rho}_g^*\mathrm{gr}^j(\widetilde{\mathcal{C}}'_\tau) \arrow[d, "\wr"] \\ \widetilde{\mathcal{D}}_\theta \arrow[r, "\sim"] & \mathrm{gr}^j(\widetilde{\mathcal{C}}_\tau) \end{tikzcd}$$

commutes, where as usual, $\theta = \theta_{\mathfrak{p},i,j}$ if $\tau = \tau_{\mathfrak{p},i}$. Combining this with the compatibility of these isomorphisms with the descent data defined by the $\mathcal{O}_{F,(p),+}^\times$-actions, we obtain analogous results for the vector bundles on $Y_{U_0'(\mathfrak{P})}$ and $Y_{U_0(\mathfrak{P})}$ with respect to the morphism $\rho_g : Y_{U_0(\mathfrak{P})} \to Y_{U_0'(\mathfrak{P})}$.

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Summing up, we have now proved the following:

Theorem 3.2. *Letting $\mathcal{C}$ denote the conormal bundle of the closed immersion $h : Y_{U_0(\mathfrak{P})} \hookrightarrow Y_U \otimes_{\mathcal{O}} Y_U$, there exists a decomposition $\mathcal{C} = \bigoplus_{\tau\in\Sigma_0} \mathcal{C}_\tau$, together with an increasing filtration*

$$0 = \mathrm{Fil}^0(\mathcal{C}_\tau) \subset \mathrm{Fil}^1(\mathcal{C}_\tau) \subset \cdots \subset \mathrm{Fil}^{e_\mathfrak{p}-1}(\mathcal{C}_\tau) \subset \mathrm{Fil}^{e_\mathfrak{p}}(\mathcal{C}_\tau) = \mathcal{C}_\tau$$

for each $\tau = \tau_{\mathfrak{p},i}$, such that $\mathcal{D}_\theta \cong \mathrm{gr}^j(\mathcal{C}_\tau)$ for each $j = 1, \ldots, e_\mathfrak{p}$, $\theta = \theta_{\mathfrak{p},i,j}$. Furthermore, if $g^{-1}Ug \subset U'$, then the canonical isomorphism $\rho_g^*\mathcal{C}' \xrightarrow{\sim} \mathcal{C}$ restricts to $\rho_g^*\mathrm{Fil}^j(\mathcal{C}'_\tau) \xrightarrow{\sim} \mathrm{Fil}^j(\mathcal{C}_\tau)$ for all τ and j as above, and the resulting diagram

$$\begin{array}{ccc}
\rho_g^*\mathcal{D}'_\theta & \xrightarrow{\sim} & \rho_g^*\mathrm{gr}^j(\mathcal{C}'_\tau) \\
\wr\downarrow & & \downarrow\wr \\
\mathcal{D}_\theta & \xrightarrow{\sim} & \mathrm{gr}^j(\mathcal{C}_\tau)
\end{array}$$

commutes.

Corollary 3.3. *If U is sufficiently small that Proposition 2.1 holds, then there is an isomorphism*

$$\pi_1^*\delta^{-1} \otimes \pi_1^*\omega \otimes \pi_2^*\omega \xrightarrow{\sim} \bigwedge^d \mathcal{C}, \tag{3.2}$$

where $\mathcal{C}$ is the conormal bundle of the closed immersion $(\pi_1, \pi_2) : Y_{U_0(\mathfrak{P})} \hookrightarrow Y_U \times Y_U$. Furthermore, the isomorphisms are compatible with the action of $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$, in the sense that if U' is also sufficiently small and $g^{-1}Ug \subset U'$, then the resulting diagram

$$\begin{array}{ccc}
\rho_g^*(\pi_1'^*\delta'^{-1} \otimes \pi_1'^*\omega' \otimes \pi_2'^*\omega') & \xrightarrow{\sim} & \rho_g^* \bigwedge^d \mathcal{C}' \\
\| & & \| \\
\pi_1^*\rho_g^*\delta'^{-1} \otimes \pi_1^*\rho_g^*\omega' \otimes \pi_2^*\rho_g^*\omega' & & \bigwedge^d \rho_g^*\mathcal{C}' \\
\wr\downarrow & & \downarrow\wr \\
\pi_1^*\delta^{-1} \otimes \pi_1^*\omega \otimes \pi_2^*\omega & \xrightarrow{\sim} & \bigwedge^d \mathcal{C}
\end{array}$$

commutes.

Remark 3.4. We comment briefly on the lack of symmetry between the two degeneracy maps π_1 and π_2 in the statements. Exchanging $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ in the definition of $\widetilde{\mathcal{D}}_\theta$ gives the line bundle

$$\widetilde{\mathcal{E}}_\theta := \widetilde{\pi}_2^*\mathcal{M}_\theta^{-1} \otimes \widetilde{\pi}_1^*\mathcal{L}_\theta = \widetilde{\pi}_2^*\mathcal{N}_\theta^{-1} \otimes \widetilde{\pi}_2^*\mathcal{L}_\theta \otimes \widetilde{\pi}_1^*\mathcal{L}_\theta$$

on Y, descending to a line bundle $\mathcal{E}_\theta$ on $Y_{U_0(\mathfrak{P})}$. We may then use the isomorphism

$$\theta(\varpi_\mathfrak{P})^{-1}(\wedge^2\psi_\theta^*) : \widetilde{\pi}_2^*\mathcal{N}_\theta \xrightarrow{\sim} \widetilde{\pi}_1^*\mathcal{N}_\theta$$

to define an isomorphism $\widetilde{\mathcal{D}}_\theta \xrightarrow{\sim} \widetilde{\mathcal{E}}_\theta$ compatible with descent data, and hence inducing an isomorphism $\mathcal{D}_\theta \cong \mathcal{E}_\theta$. The isomorphisms are furthermore compatible with the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$, but are dependent (up to an element of $\mathcal{O}^\times$) on the choices of $\varpi_\mathfrak{p}$. There is, however, a canonical isomorphism

$$\pi_2^*\delta \;\cong\; \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{P}) \otimes_\mathbb{Z} \pi_1^*\delta \;\cong\; \pi_1^*\delta$$

making π_1 and π_2 interchangeable in the statement of Corollary 3.3.

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3.2. Dualizing sheaves

We now combine the Kodaira–Spencer isomorphism over Y_U with the description of the conormal bundle obtained in the preceding section in order to establish a Kodaira–Spencer isomorphism over $Y_{U_0(\mathfrak{P})}$.

Recall that the Kodaira–Spencer filtration on $\Omega^1_{Y_U/\mathcal{O}}$ yields an isomorphism

$$\wedge^d \Omega^1_{Y_U/\mathcal{O}} \cong \delta^{-1} \otimes \omega^{\otimes 2}$$

(see [RX17, §2.8] and [Dia23, Thm. 3.3.1]), which is furthermore compatible with the action of $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ in the same sense as the isomorphism of Corollary 3.3. The identification

$$\Omega^1_{(Y_U \times_{\mathcal{O}} Y_U)/\mathcal{O}} = p_1^*\Omega^1_{Y_U/\mathcal{O}} \otimes_{\mathcal{O}_{Y_U \times_{\mathcal{O}} Y_U}} p_2^*\Omega^1_{Y_U/\mathcal{O}},$$

where $p_1, p_2 : Y_U \times_{\mathcal{O}} Y_U \to Y_U$ are the projection maps, therefore yields an isomorphism

$$\wedge^{2d} \Omega^1_{(Y_U \times_{\mathcal{O}} Y_U)/\mathcal{O}} \cong p_1^*(\delta^{-1} \otimes \omega^{\otimes 2}) \otimes p_2^*(\delta^{-1} \otimes \omega^{\otimes 2}). \tag{3.3}$$

We recall a few general facts about dualizing sheaves (see, for example, [StaX, Ch. 0DWE]). If a scheme Y is Cohen–Macaulay (of constant dimension) over a base R, then it admits a relative dualizing sheaf $\mathcal{K}_{Y/R}$, which is invertible if and only if Y is Gorenstein over R. If $\rho : Y \to Y'$ is étale and Y' is Cohen–Macaulay over R, then $\mathcal{K}_{Y/R}$ is canonically identified with $\rho^*\mathcal{K}_{Y'/R}$, and with $\wedge^n_{\mathcal{O}_Y} \Omega^1_{Y/R}$ if Y is smooth over R of dimension n. More generally, if X is smooth over R of dimension n, Y is syntomic over R of dimension $n - d$, and $i : Y \hookrightarrow X$ is a closed immersion with conormal bundle $\mathcal{C}_{Y/X}$, then

$$\mathcal{K}_{Y/R} \;\cong\; (\wedge^d_{\mathcal{O}_Y} \mathcal{C}_{Y/X})^{-1} \otimes_{\mathcal{O}_Y} i^*\mathcal{K}_{X/R} \;\cong\; (\wedge^d_{\mathcal{O}_Y} \mathcal{C}_{Y/X})^{-1} \otimes_{\mathcal{O}_Y} i^*(\wedge^n_{\mathcal{O}_X} \Omega^1_{X/R}). \tag{3.4}$$

Furthermore, the isomorphism is compatible in the obvious sense with étale base-change $X \to X'$, and if Y itself is smooth over R, then it may be identified with the isomorphism arising from the canonical exact sequence

$$0 \longrightarrow \mathcal{C}_{Y/X} \longrightarrow i^*\Omega^1_{X/R} \longrightarrow \Omega^1_{Y/R} \longrightarrow 0. \tag{3.5}$$

Combining the isomorphisms (3.2), (3.3) and 3.4), we conclude that if U is sufficiently small that Proposition 2.1 holds, then we have an isomorphism

$$\mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}} \;\cong\; \pi_2^*\delta^{-1} \otimes \pi_1^*\omega \otimes \pi_2^*\omega. \tag{3.6}$$

Furthermore, the compatibility of (3.2) and (3.3) with the action of $\mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ and that of (3.4) with étale base-change implies that if U' is also sufficiently small for Proposition 2.1 to hold and $g \in \mathrm{GL}_2(\mathbb{A}^{(p)}_{F,\mathbf{f}})$ is such that $g^{-1}Ug \subset U'$, then the diagram

$$\begin{array}{ccc} \rho_g^*\mathcal{K}_{Y_{U_0'(\mathfrak{P})}/\mathcal{O}} & \xrightarrow{\sim} & \pi_2^*\rho_g^*\delta'^{-1} \otimes \pi_1^*\rho_g^*\omega' \otimes \pi_2^*\rho_g^*\omega' \\ \wr\downarrow & & \downarrow\wr \\ \mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}} & \xrightarrow{\sim} & \pi_2^*\delta^{-1} \otimes \pi_1^*\omega \otimes \pi_2^*\omega \end{array} \tag{3.7}$$

commutes, where we have written ρ_g for both morphisms $Y_{U_0(\mathfrak{P})} \to Y_{U_0'(\mathfrak{P})}$ and $Y_U \to Y_{U'}$ as usual, the top arrow is the pull-back of (3.6) for U', and the right arrow is induced by the isomorphisms $\rho_g^*\delta' \xrightarrow{\sim} \delta$ and $\rho_g^*\omega' \xrightarrow{\sim} \omega$ defined in §2.3.

Note that we may therefore remove the assumption that $h : Y_{U_0(\mathfrak{P})} \to Y_U \times Y_U$ is a closed immersion. Indeed, for any U'' sufficiently small, we may choose U normal in U'' so that Proposition 2.1 holds, and the commutativity of (3.7) for $g \in U''$ and $U = U'$ implies that (3.6) descends to such an isomorphism over $Y_{U_0''(\mathfrak{P})}$, which is furthermore compatible with the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ in the usual sense. We have now proved the following:

Theorem 3.5. *If U is a sufficiently small open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$, then there is an isomorphism*

$$\mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}} \;\cong\; \pi_2^*\delta^{-1} \otimes \pi_1^*\omega \otimes \pi_2^*\omega;$$

furthermore, the isomorphisms for varying U are compatible with the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ in the sense that diagram (3.7) commutes.

Remark 3.6. Recall also from Remark 3.4 that $\pi_1^*\delta \cong \pi_2^*\delta$, so that we may exchange π_1 and π_2 in the statement of the theorem, and deduce also the existence of isomorphisms

$$\pi_1^*\mathcal{K}_{Y_U/\mathcal{O}} \otimes \pi_2^*\mathcal{K}_{Y_U/\mathcal{O}} \;\cong\; \mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}}^{\otimes 2},$$

compatible in the obvious sense with the action $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$.

We will also need to relate the Kodaira–Spencer isomorphisms at level U and level $U_0(\mathfrak{P})$. Consider the morphisms

$$\psi_{\theta,\mathcal{L}}^* : \widetilde{\pi}_2^*\mathcal{L}_\theta \longrightarrow \widetilde{\pi}_1^*\mathcal{L}_\theta \quad \text{and} \quad \psi_{\theta,\mathcal{M}}^{*\vee} : \widetilde{\pi}_1^*\mathcal{M}_\theta^{-1} \longrightarrow \widetilde{\pi}_2^*\mathcal{M}_\theta^{-1}$$

induced by the universal isogeny ψ over $\widetilde{Y}_{U_0(\mathfrak{P})}$. Tensoring over all $\theta \in \Sigma$ then yields morphisms which descend to define

$$\pi_2^*\omega \longrightarrow \pi_1^*\omega \quad \text{and} \quad \pi_1^*(\delta^{-1} \otimes \omega) \longrightarrow \pi_2^*(\delta^{-1} \otimes \omega).$$

Combining these with (3.6) and the isomorphisms $\pi_j^*\mathcal{K}_{Y_U/\mathcal{O}} \cong \pi_j^*(\delta^{-1} \otimes \omega) \otimes \pi_j^*\omega$ obtained by pull-back along π_j (for $j = 1, 2$) of the Kodaira–Spencer isomorphism at level U (i.e. the case $\mathfrak{P} = \mathcal{O}_F$), we thus obtain morphisms

$$\pi_1^*\mathcal{K}_{Y_U/\mathcal{O}} \longrightarrow \mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}} \quad \text{and} \quad \pi_2^*\mathcal{K}_{Y_U/\mathcal{O}} \longrightarrow \mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}} \tag{3.8}$$

(whose tensor product is $\mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{P})$ times the isomorphism of Remark 3.6).

We claim that the morphisms (3.8) extend the canonical isomorphisms

$$\pi_{1,K}^*\mathcal{K}_{Y_{U,K}/K} \stackrel{\sim}{\longrightarrow} \mathcal{K}_{Y_{U_0(\mathfrak{P}),K}/K} \quad \text{and} \quad \pi_{2,K}^*\mathcal{K}_{Y_{U,K}/K} \stackrel{\sim}{\longrightarrow} \mathcal{K}_{Y_{U_0(\mathfrak{P}),K}/K}$$

induced by the étale morphisms $\pi_{1,K}$ and $\pi_{2,K}$ (writing $\cdot_K$ for base-changes from $\mathcal{O}$ to K).

We can again reduce to the case that U is sufficiently small that h is a closed immersion, and replace $Y_{U,K}$ replaced by $\widetilde{Y}_{U,K}$, $Y_{U_0(\mathfrak{P}),K}$ by $\widetilde{Y}_{U_0(\mathfrak{P}),K}$, etc. Furthermore, it suffices to prove the desired equality holds on fibres at geometric closed points of $\widetilde{Y}_{U_0(\mathfrak{P}),K}$.

To that end, we first recall the description on fibres of the Kodaira–Spencer isomorphism over $\widetilde{Y}_{U,K}$. Since we are now working in characteristic zero, the filtration on $\Omega^1_{\widetilde{Y}_{U,K}/K}$ obtained from [Dia23, Thm. 3.3.1] has a canonical splitting giving an isomorphism

$$\bigoplus_{\theta \in \Sigma} \left(\mathcal{M}_{\theta,K}^{-1} \otimes \mathcal{L}_{\theta,K} \right) \stackrel{\sim}{\longrightarrow} \Omega^1_{\widetilde{Y}_{U,K}/K}. \tag{3.9}$$

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It follows from its construction that it is dual to the isomorphism whose fibre at the point $y \in \widetilde{Y}_U(\overline{K})$, corresponding to a tuple $\underline{A}$, is the map

$$d : \mathrm{Tan}_y(\widetilde{Y}_{U,K}) \longrightarrow \bigoplus_{\theta\in\Sigma} \mathrm{Hom}_{\overline{K}}(H^0(A, \Omega^1_{A/\overline{K}})_\theta, H^1(A, \mathcal{O}_A)_\theta)$$

with θ-component d_θ described as follows: Let $\widetilde{y} \in \widetilde{Y}_U(T)$ be a lift of y to $T := \overline{K}[\epsilon]/(\epsilon^2)$ corresponding to data $\underline{\widetilde{A}}$ lifting $\underline{A}$, and let α_θ denote the canonical isomorphism

$$H^1_{\mathrm{dR}}(\widetilde{A}/T)_\theta \cong H^1_{\mathrm{crys}}(A/T)_\theta \cong H^1_{\mathrm{dR}}(A/\overline{K})_\theta \otimes_{\overline{K}} T.$$

For $e_\theta \in H^0(A, \Omega^1_{A/\overline{K}})_\theta$, let $\widetilde{e}_\theta$ be a lift of e_θ to $H^0(\widetilde{A}, \Omega^1_{\widetilde{A}/T})_\theta$. Any two such lifts differ by an element of $\epsilon H^0(\widetilde{A}, \Omega^1_{\widetilde{A}/T})_\theta$, which corresponds to $H^0(A, \Omega^1_{A/\overline{K}})_\theta \otimes_{\overline{K}} \epsilon T$ under the above isomorphism. Therefore,

$$\alpha_\theta(\widetilde{e}_\theta) - e_\theta \otimes 1 = f_\theta \otimes \epsilon$$

for some $f_\theta \in H^1_{\mathrm{dR}}(A/\overline{K})_\theta$ whose image in $H^1(A, \mathcal{O}_A)_\theta$ is independent of the choice of $\widetilde{e}_\theta$, and this image is $d_\theta(\widetilde{y})(e_\theta)$.

Similarly, the filtration obtained from Theorem 3.1 on the conormal bundle $\widetilde{\mathcal{C}}_K$ of

$$\widetilde{h} : \widetilde{Y}_{U_0(\mathfrak{P}),K} \hookrightarrow \widetilde{Y}_{U,K} \times \widetilde{Y}_{U,K}$$

has a canonical splitting giving an isomorphism

$$\bigoplus_{\theta\in\Sigma} \left(\widetilde{\pi}_1^* \mathcal{M}^{-1}_{\theta,K} \otimes \widetilde{\pi}_2^* \mathcal{L}_{\theta,K}\right) \xrightarrow{\sim} \widetilde{\mathcal{C}}_K, \tag{3.10}$$

which combined with (3.4) gives an exact sequence

$$0 \longrightarrow \bigoplus_{\theta\in\Sigma} \left(\widetilde{\pi}_1^* \mathcal{M}^{-1}_{\theta,K} \otimes \widetilde{\pi}_2^* \mathcal{L}_{\theta,K}\right) \longrightarrow \widetilde{\pi}_1^* \Omega^1_{\widetilde{Y}_{U,K}/K} \otimes \widetilde{\pi}_2^* \Omega^1_{\widetilde{Y}_{U,K}/K} \longrightarrow \Omega^1_{\widetilde{Y}_{U_0(\mathfrak{P}),K}/K} \longrightarrow 0.$$

It follows from the construction (and in particular the choice of sign) in the proof of Theorem 3.1 that the first morphism is dual to the one whose fibre at the point $z \in \widetilde{Y}_{U_0(\mathfrak{P})}(\overline{K})$ corresponding to a tuple $(\underline{A}_1, \underline{A}_2, \psi)$ is the map

$$\mathrm{Tan}_{\widetilde{\pi}_1(z)}(\widetilde{Y}_{U,K}) \times \mathrm{Tan}_{\widetilde{\pi}_2(z)}(\widetilde{Y}_{U,K}) \longrightarrow \bigoplus_{\theta\in\Sigma} \mathrm{Hom}_{\overline{K}}(H^0(A_2, \Omega^1_{A_2/\overline{K}})_\theta, H^1(A_1, \mathcal{O}_{A_1})_\theta)$$

with θ-component described as follows: Letting $y_1 = \widetilde{\pi}_1(z)$ correspond to $\underline{A}_1$ and $y_2 = \widetilde{\pi}_2(z)$ correspond to $\underline{A}_2$, the lift $(\widetilde{y}_1, \widetilde{y}_2)$ is sent to

$$d_\theta(\widetilde{y}_1) \circ \psi^*_{\theta,\mathcal{L}} - \psi^*_{\theta,\mathcal{M}} \circ d_\theta(\widetilde{y}_2),$$

where we continue to write

$$\psi^*_{\theta,\mathcal{L}} : H^0(A_2, \Omega^1_{A_2/\overline{K}})_\theta \to H^0(A_1, \Omega^1_{A_1/\overline{K}})_\theta \quad \text{and} \quad \psi^*_{\theta,\mathcal{M}} : H^1(A_2, \mathcal{O}_{A_2})_\theta \to H^1(A_1, \mathcal{O}_{A_1})_\theta$$

for the maps induced by ψ. It follows that the diagram

$$\begin{tikzcd}
\bigoplus_{\theta\in\Sigma}\left(\widetilde{\pi}_1^*\mathcal{M}_{\theta,K}^{-1}\otimes\widetilde{\pi}_2^*\mathcal{L}_{\theta,K}\right) \arrow[rr, "\sim"] \arrow[d] & & \widetilde{\mathcal{C}}_K \arrow[d] \\
\bigoplus_{\theta\in\Sigma}\left((\widetilde{\pi}_1^*\mathcal{M}_{\theta,K}^{-1}\otimes\widetilde{\pi}_1^*\mathcal{L}_{\theta,K})\oplus(\widetilde{\pi}_2^*\mathcal{M}_{\theta,K}^{-1}\otimes\widetilde{\pi}_2^*\mathcal{L}_{\theta,K})\right) \arrow[r, "\sim"] & \widetilde{\pi}_1^*\Omega^1_{\widetilde{Y}_{U,K}/K}\bigoplus\widetilde{\pi}_2^*\Omega^1_{\widetilde{Y}_{U,K}/K} &
\end{tikzcd}$$

commutes, where the horizontal isomorphisms are those of (3.9) and (3.10), the left vertical map respects the decomposition and is defined by $(1\otimes\psi^*_{\theta,\mathcal{L}},-\psi^{*\vee}_{\theta,\mathcal{M}}\otimes 1)$ in the θ-component, and the right vertical map is the inclusion in (3.4). Taking top exterior powers in the resulting exact sequences (and tracking signs) then yields the desired compatibility, which we can state as follows:

Proposition 3.7. *The canonical isomorphism $\pi_1^*\mathcal{K}_{Y_{U,K}/K}\longrightarrow\mathcal{K}_{Y_{U_0(\mathfrak{P}),K}/K}$ extends uniquely to the morphism $\pi_1^*\mathcal{K}_{Y_U/\mathcal{O}}\longrightarrow\mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}}$ corresponding under the Kodaira–Spencer isomorphisms to the morphism $\pi_1^*(\delta^{-1}\otimes\omega)\longrightarrow\pi_2^*(\delta^{-1}\otimes\omega)$ induced by ψ; similarly the canonical isomorphism $\pi_2^*\mathcal{K}_{Y_{U,K}/K}\longrightarrow\mathcal{K}_{Y_{U_0(\mathfrak{P}),K}/K}$ extends uniquely to the morphism $\pi_2^*\mathcal{K}_{Y_U/\mathcal{O}}\longrightarrow\mathcal{K}_{Y_{U_0(\mathfrak{P})}/\mathcal{O}}$ corresponding to the morphism $\pi_2^*\omega\longrightarrow\pi_1^*\omega$ induced by ψ.*

Note that the uniqueness in the statement follows from the flatness over $\mathcal{O}$ of the schemes Y_U and $Y_{U_0(\mathfrak{P})}$ (and the invertibility of their dualizing sheaves).

4. Degeneracy fibres

4.1. Irreducible components

We now turn our attention to Hilbert modular varieties in characteristic p (i.e., over the residue field k of $\mathcal{O}$). We fix an open compact subgroup $U\subset\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$, as usual sufficiently small and containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$, and we let $\overline{Y}=Y_{U,k}$ and $\overline{Y}_0(\mathfrak{P})=Y_{U_0(\mathfrak{P}),k}$.

Our aim is to describe the fibres of the degeneracy map[14] $\overline{\pi}_1=\pi_{1,k}:\overline{Y}_0(\mathfrak{P})\longrightarrow\overline{Y}$, or, more precisely, its restrictions to irreducible components of $\overline{Y}_0(\mathfrak{P})$. We accomplish this by generalizing the arguments in [DKS23, §7], where this was carried out under the assumption that p is unramified in F. See also [ERX17a, §4] for partial results in this direction without this assumption.

As usual, we first carry out various constructions in the context of their étale covers by fine moduli schemes. Recall that the scheme $T:=\widetilde{Y}_{U,k}$ is equipped with a stratification defined by the vanishing of partial Hasse invariants, constructed in the generality of Pappas–Rapoport models by Reduzzi and Xiao in [RX17, §3]. More precisely, for $\theta\in\Sigma$, define $n_\theta=1$ unless $\theta=\theta_{\mathfrak{p},i,1}$ for some $\mathfrak{p}$, i, in which case $n_\theta=p$, and recall that σ denotes the 'right shift' permutation of Σ (see §2.1). We then have a surjective morphism

$$\widetilde{h}_\theta:\mathcal{P}_\theta\longrightarrow\mathcal{L}_{\sigma^{-1}\theta}^{\otimes n_\theta}\tag{4.1}$$

induced by u if $n_\theta=1$, and by the pair of surjections

$$\mathcal{P}_\theta\xleftarrow{u^{e_\mathfrak{p}-1}}\mathcal{H}^1_{\mathrm{dR}}(A/T)_{\tau_{\mathfrak{p},i}}\xrightarrow{\mathrm{Ver}_A^*}\mathcal{L}_{\sigma^{-1}\theta}^{\otimes p}\tag{4.2}$$

if $n_\theta=p$, and for each $J\subset\Sigma$, the vanishing locus of the restriction of $\widetilde{h}_\theta$ to $\mathcal{L}_\theta$ for $\theta\in J$ defines a smooth subscheme of T of codimension $|J|$, which we denote T_J (see [Dia23, §4.1]).

[14] Analogous results hold for $\overline{\pi}_2=\pi_{2,k}$, as can be shown using similar arguments or by deducing them from the results for $\overline{\pi}_1$ using a $w_\mathfrak{P}$-operator.

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Now consider $S := \widetilde{Y}_{U_0(\mathfrak{P}),k}$, and let $(\underline{A}, \underline{A}', \psi)$ denote the universal triple over S. Recall that for each $\theta \in \Sigma$, the isogeny ψ induces an $\mathcal{O}_S$-linear morphism $\psi_\theta^* : \mathcal{P}'_\theta \to \mathcal{P}_\theta$; its image is a rank one subbundle of $\mathcal{P}_\theta$ if $\theta \in \Sigma_\mathfrak{P}$ and it is an isomorphism if $\theta \notin \Sigma_\mathfrak{P}$. Furthermore, ψ_θ^* restricts to a morphism $\mathcal{L}'_\theta \to \mathcal{L}_\theta$ of line bundles over S, which we denote $\psi^*_{\theta,\mathcal{L}}$, and hence also induces one we denote $\psi^*_{\theta,\mathcal{M}} : \mathcal{M}'_\theta \to \mathcal{M}_\theta$.

For $J, J' \subset \Sigma_\mathfrak{P}$, consider the closed subschemes $S_{J,J'}$ of S defined[15] by the vanishing of $\psi^*_{\theta,\mathcal{M}}$ for all $\theta \in J$, and $\psi^*_{\theta,\mathcal{L}}$ for all $\theta \in J'$. Note that a closed point y of S is in $S_{J,J'}$ if and only if $J \subset \Sigma_y$ and $J' \subset \Sigma'_y$, where Σ_y and Σ'_y were defined in §2.6. Furthermore, in terms of the parameters and basis elements chosen in §2.6 for stalks at y, we see that

$$\begin{aligned}
\text{if } \theta \in \Sigma_y, \text{ then } \psi^*_{\theta,\mathcal{M}}(f'_\theta + L'_\theta) &= \begin{cases} s_\theta(f_\theta + L_\theta), & \text{if } \theta \in \Sigma'_y, \\ \theta(\varpi_\mathfrak{P})(f_\theta + L_\theta), & \text{if } \theta \notin \Sigma'_y, \end{cases} \\
\text{and if } \theta \in \Sigma'_y, \text{ then } \psi^*_{\theta,\mathcal{L}}(e'_\theta - s'_\theta f'_\theta) &= \begin{cases} -s'_\theta(e_\theta - s_\theta f_\theta), & \text{if } \theta \in \Sigma_y, \\ \theta(\varpi_\mathfrak{P})(e_\theta - s_\theta f_\theta), & \text{if } \theta \notin \Sigma_y, \end{cases}
\end{aligned}$$

It follows that if $y \in S_{J,J'}$, then $S_{J,J'}$ is defined in a neighborhood of y by the ideal generated by

$$\{\varpi\} \quad \cup \quad \{\, s_\theta \,|\, \theta \in J\,\} \quad \cup \quad \{\, s'_\theta \,|\, \theta \in J'\}.$$

We have the following generalization[16] of [DKS23, Cor. 4.3.2]:

Proposition 4.1. *The scheme $S_{J,J'}$ is a reduced local complete intersection of dimension $d - |J \cap J'|$, and is smooth over k if $J \cup J' = \Sigma_\mathfrak{P}$.*

Proof. Let y be a closed point of $S_{J,J'}$. The description of the completion $\mathcal{O}_{\widetilde{Y}_0(\mathfrak{P}),y}$ at the end of §2.6 gives an isomorphism

$$\mathcal{O}^\wedge_{S,y} \cong k(y)[[X_\theta, X'_\theta]]_{\theta\in\Sigma}/\langle \overline{g}_\theta\rangle_{\theta\in\Sigma}$$

under which $s_\theta \mapsto X_\theta$ and $s'_\theta \mapsto X'_\theta$, where

$$\overline{g}_\theta = \begin{cases} X_\theta & \text{if } \theta \in \Sigma_y - \Sigma'_y, \\ X'_\theta & \text{if } \theta \in \Sigma'_y - \Sigma_y, \\ X_\theta X'_\theta & \text{if } \theta \in \Sigma_y \cap \Sigma'_y, \\ X_\theta - \overline{\theta}(\varpi_\mathfrak{P})X'_\theta, & \text{if } \theta \notin \Sigma_\mathfrak{P}. \end{cases}$$

It follows that $\mathcal{O}^\wedge_{S_{J,J'},y}$ is isomorphic to the quotient of this ring by the ideal $\langle X_\theta\rangle_{\theta\in J} + \langle X'_\theta\rangle_{\theta\in J'}$, and therefore to $\widehat{\bigotimes}_{\theta\notin J\cap J'} R_\theta$, where

$$R_\theta = \begin{cases} k(y)[[X_\theta, X'_\theta]]/(X_\theta X'_\theta), & \text{if } \theta \in (\Sigma_y - J) \cap (\Sigma'_y - J'), \\ k(y)[[X_\theta]] \text{ or } k(y)[[X'_\theta]], & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{O}^\wedge_{S_{J,J'},y}$ is a reduced complete intersection of dimension $d - |J \cap J'|$, and is regular if $J \cup J' = \Sigma_\mathfrak{P}$. □

If $J' = \Sigma_\mathfrak{P} - J$, then $S_{J,J'}$ is a union of irreducible components of S, which we write simply as S_J; furthermore, each irreducible component of S is a connected component of S_J for a unique $J \subset \Sigma_\mathfrak{P}$.

For each $J \subset \Sigma_\mathfrak{P}$, we consider the restriction of $\widetilde{\pi}_{1,k}$ to S_J. We first note that the morphism factors through $T_{J'}$, where $J' = \{\, \theta \in J \,|\, \sigma^{-1}\theta \notin J\,\}$. (Recall that $T_{J'}$ is the intersection of the vanishing loci of the partial Hasse invariants $\mathcal{L}_\theta \xrightarrow{\widetilde{h}_\theta} \mathcal{L}^{\otimes n_\theta}_{\sigma^{-1}\theta}$ for $\theta \in J'$.) Indeed, since S_J is reduced, it suffices to check

[15] Note the deviation here from the notational conventions of [DKS23] and [DK23], where these would be written as $S_{\sigma(J'),J}$ and $Z_{\sigma(J'),J}$.

[16] Note, however, that the descriptions of the cases in the proof of [DKS23, Cor. 4.3.2] are incorrect: the first case there should be $\theta \in (I_Q - I) \cap (J_Q - J)$, the second $(J_Q - I_Q) \cup (J - I)$, and the third $(I_Q - J_Q) \cup (I - J)$.

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the assertion on geometric closed points, which reduces to the statement that if $(\underline{A}, \underline{A}', \psi)$ corresponds to an $\overline{\mathbb{F}}_p$-point of S such that $\psi^*_{\theta,\mathcal{M}} : M'_\theta \to M_\theta$ and $\psi^*_{\sigma^{-1}\theta,\mathcal{L}} : L'_{\sigma^{-1}\theta} \to L_{\sigma^{-1}\theta}$ both vanish, then so does $\widetilde{h}_\theta : L_\theta \longrightarrow L^{\otimes n_\theta}_{\sigma^{-1}\theta}$ (with the obvious notation). If $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j}$ for some $j > 1$, then the vanishing of $\psi^*_{\theta,\mathcal{M}}$ means that

$$\psi^*_\tau(u^{-1}F'^{(j-1)}_\tau) = \psi^*_\tau(G'^{(j)}_\tau) \subset F^{(j)}_\tau,$$

and hence, equality holds by comparing dimensions. However, the vanishing of $\psi^*_{\sigma^{-1}\theta,\mathcal{L}}$ means that $\psi^*_\tau(F'^{(j-1)}_\tau) = F^{(j-2)}_\tau$, and it follows that $uF^{(j)}_\tau = F^{(j-2)}_\tau$, so $\widetilde{h}_\theta$ vanishes. Similarly, if $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,1}$, then we find that

$$\psi^*_\tau(u^{e_\mathfrak{p}-1}H^1_{\mathrm{dR}}(A'/\overline{\mathbb{F}}_p)_\tau) = \psi^*_\tau(G'^{(1)}_\tau) = F^{(1)}_\tau$$

and $\psi^*_{\phi^{-1}\tau}(F'^{(e_\mathfrak{p})}_{\phi^{-1}\tau}) = F^{(e_\mathfrak{p}-1)}_{\phi^{-1}\tau}$. Since $\mathrm{Ver}^*_A \circ \psi^*_\tau = \psi^*_{\phi^{-1}\tau} \circ \mathrm{Ver}^*_{A'}$ and

$$\mathrm{Ver}^*_{A'}(H^1_{\mathrm{dR}}(A'/\overline{\mathbb{F}}_p)_\tau) = (F'^{(e_\mathfrak{p})}_{\phi^{-1}\tau})^{(p)},$$

it follows that $\mathrm{Ver}^*_A(F^{(1)}_\tau) = u^{e_\mathfrak{p}-1}(F^{(e_\mathfrak{p}-1)}_{\phi^{-1}\tau})^{(p)}$, which again means that $\widetilde{h}_\theta$ vanishes. Similarly, one finds that the restriction of $\widetilde{\pi}_{2,k}$ to S_J factors through $T_{J''}$, where $J'' = \{\, \theta \notin J \mid \sigma^{-1}\theta \in J \,\}$.

Now consider the scheme

$$P_J := \prod_{\theta \in J''} \mathbb{P}_{T_{J'}}(\mathcal{P}_\theta),$$

where the product is fibred over $T_{J'}$ and $\mathbb{P}_{T_{J'}}(\mathcal{P}_\theta)$ denotes the projective bundle parametrizing rank one subbundles of $\mathcal{P}_\theta$ over $T_{J'}$. (We will freely use $\mathcal{P}_\theta$ to denote the pull-back of the rank two vector bundle associated to the universal object over T when the base and morphism to T are clear from the context.) The image of $\psi^*_\theta : \mathcal{P}'_\theta \to \mathcal{P}_\theta$ is a rank one subbundle of $\mathcal{P}_\theta$ over S_J, so it determines a morphism $S_J \to \mathbb{P}_{T_{J'}}(\mathcal{P}_\theta)$, and we let

$$\widetilde{\xi}_J : S_J \longrightarrow P_J$$

denote their product over $\theta \in J''$.

The proof of [ERX17a, Prop. 4.5] shows that $\widetilde{\xi}_J$ is bijective on closed points, and hence is a Frobenius factor, in the sense that *some* power of the Frobenius endomorphism on P_J factors through it. We will instead use relative Dieudonné theory, as in [DKS23, §7.1], to geometrize the pointwise construction in [ERX17a] of the inverse map, showing in particular that a *single* power of Frobenius suffices.

Before we carry this out in the next section, let us remark that the closed subschemes T_J of T descend to closed subschemes of $\overline{Y}$, which we denote by $\overline{Y}_J$, and similarly, the subschemes S_J (or more generally $S_{J,J'}$) descend to subschemes of $\overline{Y}_0(\mathfrak{P})$, which we denote $\overline{Y}_0(\mathfrak{P})_J$ (or $\overline{Y}_0(\mathfrak{P})_{J,J'}$). Furthermore, since we are now working in characteristic p, we may choose U sufficiently small that the natural action $U \cap \mathcal{O}_F^\times$ on the vector bundles $\mathcal{P}_\theta$ is trivial. The action of $\mathcal{O}^\times_{F,(p),+}$ therefore defines descent data on the $\mathcal{P}_\theta$ to a vector bundle over $\overline{Y}$, and denoting this too by $\mathcal{P}_\theta$, we obtain a morphism

$$\xi_J : \overline{Y}_0(\mathfrak{P})_J \longrightarrow \prod_{\theta \in J''} \mathbb{P}_{\overline{Y}_{J'}}(\mathcal{P}_\theta).$$

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4.2. Frobenius factorization

We will define a morphism $P_J \longrightarrow S$ corresponding to an isogeny from $A^{(p)}$, where A is the pull-back to P_J of the universal abelian scheme over $T_{J'}$. In order to do this, we first construct a certain Raynaud $(\mathcal{O}_F/\mathfrak{P})$-module scheme over $T_{J'}$. To that end, we need to define a line bundle $\mathcal{A}_\tau$ over P_J for each $\tau \in \Sigma_{\mathfrak{P},0}$, along with morphisms

$$s_\tau : \mathcal{A}_\tau^{\otimes p} \longrightarrow \mathcal{A}_{\phi\circ\tau} \quad \text{and} \quad t_\tau : \mathcal{A}_{\phi\circ\tau} \longrightarrow \mathcal{A}_\tau^{\otimes p}$$

such that $s_\tau t_\tau = 0$ for all τ.

Suppose then that $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$, let $\theta = \theta_{\mathfrak{p},i,1}$. and consider the exact sequence

$$0 \longrightarrow \mathcal{A}_\tau \longrightarrow \mathcal{P}_\theta \longrightarrow \mathcal{A}'_\tau \longrightarrow 0 \tag{4.3}$$

over P_J, where $\mathcal{A}_\tau$ and $\mathcal{A}'_\tau$ are the line bundles defined by

- $\mathcal{A}_\tau = \mathcal{L}_\theta$ and $\mathcal{A}'_\tau = \mathcal{M}_\theta$ if $\theta \in J$;
- $\mathcal{A}_\tau = \ker(\widetilde{h}_\theta) \xleftarrow{\sim} \mathcal{M}^{\otimes p}_{\sigma^{-1}\theta}$ and $\mathcal{A}'_\tau = \mathcal{P}_\theta/\mathcal{A}_\tau \xrightarrow{\sim} \mathcal{L}^{\otimes p}_{\sigma^{-1}\theta}$ if $\sigma^{-1}\theta = \theta_{\mathfrak{p},i-1,e_\mathfrak{p}} \notin J$, with $\widetilde{h}_\theta : \mathcal{P}_\theta \twoheadrightarrow \mathcal{L}^{\otimes p}_{\sigma^{-1}\theta}$ given by (4.2) and the isomorphism $\mathcal{M}^{\otimes p}_{\sigma^{-1}\theta} \xrightarrow{\sim} \mathcal{A}_\tau$ induced by $\mathrm{Frob}_A^* : (\mathcal{G}^{(e_\mathfrak{p})}_{\phi^{-1}\circ\tau})^{(p)} \to \mathcal{P}_\theta$;
- the tautological exact sequence over $\mathbb{P}_{T_{J'}}(\mathcal{P}_\theta)$ if $\theta \in J''$.

Note that the first two conditions are both satisfied if $\theta \in J'$, in which case the definitions coincide since $\ker(\widetilde{h}_\theta) = \mathcal{L}_\theta$ on $T_{J'}$.

We now show that $\mathrm{Frob}_A^* : \mathcal{P}^{(p)}_{\theta_{\mathfrak{p},i,1}} \to \mathcal{P}_{\theta_{\mathfrak{p},i+1,1}}$ and $\mathrm{Ver}_A^* : \mathcal{P}_{\theta_{\mathfrak{p},i+1,1}} \to \mathcal{P}^{(p)}_{\theta_{\mathfrak{p},i,1}}$ restrict to morphisms $\mathcal{A}_\tau^{\otimes p} \to \mathcal{A}_{\phi\circ\tau}$ and $\mathcal{A}_{\phi\circ\tau} \to \mathcal{A}_\tau^{\otimes p}$, and hence induce morphisms $\mathcal{A}'^{\otimes p}_\tau \to \mathcal{A}'_{\phi\circ\tau}$ and $\mathcal{A}'_{\phi\circ\tau} \to \mathcal{A}'^{\otimes p}_\tau$,

- If $\theta = \theta_{\mathfrak{p},i,1} \in J$, then $\mathcal{A}_\tau = \mathcal{L}_\theta = \mathrm{Ver}_A^*(u^{1-e_\mathfrak{p}}\mathcal{P}_{\theta_{\mathfrak{p},i+1,1}})$, so $\mathrm{Frob}_A^*(\mathcal{A}_\tau^{\otimes p}) = 0$ and
$$\mathrm{Ver}_A^*(\mathcal{A}_{\phi\circ\tau}) \subset \mathrm{Ver}_A^*(\mathcal{P}_{\theta_{\mathfrak{p},i+1,1}}) = u^{e_\mathfrak{p}-1}(\mathcal{F}_\tau^{(e_\mathfrak{p})})^{(p)} \subset \mathcal{L}_\theta^{\otimes p} = \mathcal{A}_\tau^{\otimes p}.$$
- If $\theta_{\mathfrak{p},i,e_\mathfrak{p}} \notin J$, then $\mathcal{A}_{\phi\circ\tau} = \mathrm{Frob}_A^*((\mathcal{G}_\tau^{(e_\mathfrak{p})})^{(p)})$, so $\mathrm{Ver}_A^*(\mathcal{A}_{\phi\circ\tau}) = 0$ and
$$\mathrm{Frob}_A^*(\mathcal{A}_\tau^{\otimes p}) \subset \mathrm{Frob}_A^*(\mathcal{P}^{(p)}_{\theta_{\mathfrak{p},i,1}}) \subset \mathrm{Frob}_A^*((\mathcal{G}_\tau^{(e_\mathfrak{p})})^{(p)}) = \mathcal{A}_{\phi\circ\tau}.$$
- If $\theta = \theta_{\mathfrak{p},i,1} \notin J$ and $\theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J$, then $\theta_{\mathfrak{p},i,j} \in J'$ for some $j \in \{2,\dots,e_\mathfrak{p}\}$, which implies that $u\mathcal{F}_\tau^{(j)} = \mathcal{F}_\tau^{(j-2)}$. We deduce that
$$\mathcal{P}_\theta \subset \mathcal{G}_\tau^{(j-1)} = \mathcal{F}_\tau^{(j)} \subset \mathcal{F}_\tau^{(e_\mathfrak{p})} \subset u\mathcal{H}^1_{\mathrm{dR}}(A/P_J)_\tau$$
(where the last inclusion holds since $u^{e_\mathfrak{p}-1}\mathcal{F}_\tau^{(e_\mathfrak{p})} \subset u^{j-1}\mathcal{F}_\tau^{(j)} = u^{j-1}\mathcal{G}_\tau^{(j-1)} = 0$), and hence that $\mathrm{Frob}_A^*(\mathcal{A}_\tau^{\otimes p}) \subset \mathrm{Frob}_A^*(\mathcal{P}_\theta^{(p)}) \subset \mathrm{Frob}_A^*((\mathcal{F}_\tau^{(e_\mathfrak{p})})^{(p)} = 0$ and $\mathrm{Ver}_A^*(\mathcal{A}_{\phi\circ\tau}) \subset \mathrm{Ver}_A^*(\mathcal{P}_{\theta_{\mathfrak{p},i+1,1}}) = u^{e_\mathfrak{p}-1}(\mathcal{F}_\tau^{(e_\mathfrak{p})})^{(p)} = 0$.

We now define $s_\tau : \mathcal{A}_\tau^{\otimes p} \to \mathcal{A}_{\phi\circ\tau}$ and $t_\tau : \mathcal{A}_{\phi\circ\tau} \to \mathcal{A}_\tau^{\phi\circ\tau}$ as the restrictions of Frob_A^* and Ver_A^*, yielding Raynaud data $(\mathcal{A}_\tau, s_\tau, t_\tau)_{\tau\in\Sigma_{\mathfrak{P},0}}$, and we let C denote the corresponding $(\mathcal{O}_F/\mathfrak{P})$-module scheme over P_J. We similarly obtain Raynaud data from the morphisms $s'_\tau : \mathcal{A}'^{\otimes p}_\tau \to \mathcal{A}'_{\phi\circ\tau}$ and $t_\tau : \mathcal{A}'_{\phi\circ\tau} \to \mathcal{A}'^{\otimes p}_\tau$ induced by Frob_A^* and Ver_A^*, and we let C' denote the resulting $(\mathcal{O}_F/\mathfrak{P})$-module scheme.

Recall from [DKS23, §7.1.2] that the Dieudonné crystal $\mathbb{D}(C)$ is isomorphic to $\Pi^*(\mathcal{A})$, with $\Phi = \Pi^*(s)$ and $V = \Pi^*(t)$, where $\mathcal{A} = \oplus\mathcal{A}_\tau$, $s = \oplus s_\tau$, $t = \oplus t_\tau$ (the direct sums being over $\tau \in \Sigma_{\mathfrak{P},0}$) and Π^* is the functor denoted Φ^* in [BBM82, §4.3.4]. (The assertion of [DKS23, Prop.7.1.3] is for $(\mathcal{O}_F/p)$-module schemes, but the analogous one for $(\mathcal{O}_F/\mathfrak{P})$-module schemes is an immediate consequence.)

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Similarly, we have that $\mathbb{D}(C')$ is isomorphic to $\Pi^*(\mathcal{A}')$ with $\Phi = \Pi^*(s')$ and $V = \Pi^*(t')$ (again letting $\mathcal{A}' = \oplus\mathcal{A}'_\tau$, etc.). However, we have the canonical isomorphisms

$$\mathbb{D}(A^{(p)}[\mathfrak{P}]) \cong \Pi^*\mathbb{D}(A[\mathfrak{P}])_{P_J} \cong \Pi^*\mathcal{H}^1_{\mathrm{dR}}(A/P_J) \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\mathfrak{P})$$

provided by [BBM82, (4.3.7.1), (3.3.7.3)], under which Φ (resp. V) corresponds to $\Pi^*\mathrm{Frob}_A$ (resp. $\Pi^*\,\mathrm{Ver}_A$). Combined with the isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(A/P_J) \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\mathfrak{P}) = \bigoplus_{\mathfrak{p}|\mathfrak{P}} \mathcal{H}^1_{\mathrm{dR}}(A/P_J) \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\mathfrak{p}) \xrightarrow{\sim} \bigoplus_{\mathfrak{p}|\mathfrak{P}} \bigoplus_{i\in\mathbb{Z}/f_\mathfrak{p}\mathbb{Z}} \mathcal{P}_{\theta_{\mathfrak{p},i,1}},$$

induced by multiplication by $u^{e_\mathfrak{p}-1}$ in the $\mathfrak{p}$-component, it follows that (4.3) yields an exact sequence of Dieudonné crystals

$$0 \longrightarrow \mathbb{D}(C) \longrightarrow \mathbb{D}(A^{(p)}[\mathfrak{P}]) \longrightarrow \mathbb{D}(C') \longrightarrow 0$$

over P_J. Since P_J is smooth over k, we may apply [BM90, Thm. 4.1.1] to conclude that this arises from an exact sequence of $(\mathcal{O}_F/\mathfrak{P})$-module schemes

$$0 \longrightarrow C' \longrightarrow A^{(p)}[\mathfrak{P}] \longrightarrow C \longrightarrow 0.$$

We now define $A'' = A^{(p)}/C'$, and let $\alpha : A^{(p)} \to A''$ denote the resulting isogeny and $s'' : A'' \to P_J$ the structure morphism.

We now equip A'' with the required auxiliary data so that the triple $(\underline{A}^{(p)}, \underline{A}'', \alpha)$ will correspond to a morphism $P_J \to S$. First, note that A'' inherits an $\mathcal{O}_F$-action from $A^{(p)}$, as well as a unique U^p-level structure with which α is compatible.

The existence of a quasi-polarization λ'' satisfying the required compatibility amounts to $\lambda^{(p)}$ inducing an isomorphism $\mathfrak{P}\mathfrak{c}\mathfrak{d} \otimes_{\mathcal{O}_F} A'' \xrightarrow{\sim} (A'')^\vee$ over each connected component of P_J. The corresponding argument in [Dia23, §6.1] carries over *mutatis mutandis* (with A replaced by $A^{(p)}$, A' by A'', H by C', $\mathfrak{p}$ by $\mathfrak{P}$ and $\mathcal{I}_i$ by $\mathcal{A}_\tau^{\otimes p}$, now running over $\tau \in \Sigma_{\mathfrak{P},0}$ instead of $i \in \mathbb{Z}/f\mathbb{Z}$) to reduce this to the orthogonality of $\mathcal{A}_\tau^{\otimes p}$ and $u^{1-e_\mathfrak{p}}\mathcal{A}_\tau^{\otimes p}$ under the pairing on $\mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/P_J)_\tau$ induced by $\lambda^{(p)}$ (via [Dia23, (4)]). This in turn is equivalent to the orthogonality of $\mathcal{A}_\tau$ and $u^{1-e_\mathfrak{p}}\mathcal{A}_\tau$ under the pairing induced by λ, which follows from [Dia23, Lemma 3.1.1] via the implication that the resulting perfect pairing on $\mathcal{P}_{\theta_{\mathfrak{p},i,1}}$ is alternating.

We now define Pappas–Rapoport filtrations for A'' – that is, a chain of $\mathcal{O}_{P_J}[u]/(u^{e_\mathfrak{p}})$-modules

$$0 =: \mathcal{F}''^{(0)}_\tau \subset \mathcal{F}''^{(1)}_\tau \subset \mathcal{F}''^{(2)}_\tau \subset \cdots \mathcal{F}''^{(e_\mathfrak{p}-1)}_\tau \subset \mathcal{F}''^{(e_\mathfrak{p})}_\tau := (s''_*\Omega^1_{A''/P_J})_\tau$$

for each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$, such that $\mathcal{F}''^{(j)}_\tau/\mathcal{F}^{(j-1)}_\tau$ is an invertible $\mathcal{O}_{P_J}$-module annihilated by u for $j = 1, \dots, e_\mathfrak{p}$.

For each $\tau \in \Sigma_0$, consider the morphism

$$\alpha^*_\tau : \mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A^{(p)}/P_J)_\tau = \mathcal{H}^1_{\mathrm{dR}}(A/P_J)^{(p)}_{\phi^{-1}\circ\tau}$$

Zariski-locally free rank two $\mathcal{O}_{P_J}[u]/(u^{e_\mathfrak{p}})$-modules. Note that if $\tau \notin \Sigma_{\mathfrak{P},0}$, then α^*_τ is an isomorphism sending $(s''_*\Omega^1_{A''/P_J})_\tau$ to $(s_*\Omega^1_{A/P_J})^{(p)}_{\phi^{-1}\circ\tau}$, and we define $\mathcal{F}''^{(j)}_\tau = (\alpha^*_\tau)^{-1}((\mathcal{F}^{(j)}_{\phi^{-1}\tau})^{(p)})$ for $j = 1, \dots, e_\mathfrak{p} - 1$.

We may therefore assume that $\tau = \tau_{\mathfrak{p},i}$ for some $\mathfrak{p}|\mathfrak{P}$, so the image of α^*_τ is the $\mathcal{O}_{P_J}$-subbundle $u^{1-e_\mathfrak{p}}\mathcal{A}^{\otimes p}_{\phi^{-1}\circ\tau}$ of rank $2e_\mathfrak{p} - 1$, and the kernel of α^*_τ is a rank one subbundle of $\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau$. For each $j = 1, \dots, e_\mathfrak{p} - 1$, we will define $\mathcal{F}''^{(j)}_\tau$ as an $\mathcal{O}_{P_J}$-subbundle of $\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau$ of rank j, and then verify the required inclusions on fibres at closed points.

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Suppose first that $\theta_{\mathfrak{p},i-1,j} \notin J$. Since $u^{e_\mathfrak{p}-1}\mathcal{F}^{(j-1)}_{\phi^{-1}\circ\tau} = 0$, it follows that $\mathcal{F}^{(j-1)}_{\phi^{-1}\circ\tau}$ is contained in $u^{1-e_\mathfrak{p}}\mathcal{A}_{\phi^{-1}\circ\tau}$, so $(\mathcal{F}^{(j-1)}_{\phi^{-1}\circ\tau})^{(p)} \subset \mathrm{im}(\alpha^*_\tau)$. Therefore, $(\alpha^*_\tau)^{-1}(\mathcal{F}^{(j-1)}_{\phi^{-1}\circ\tau})^{(p)}$ is an $\mathcal{O}_{P_J}$-subbundle of $\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau$ of rank j, which we define to be $\mathcal{F}''^{(j)}_\tau$.

Suppose next that $\theta_{\mathfrak{p},i-1,j} \in J$. We claim that

$$\mathcal{A}_{\phi^{-1}\circ\tau} \subset \mathcal{F}^{(j)}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(j+1)}_{\phi^{-1}\circ\tau} \subset u^{1-e_\mathfrak{p}}\mathcal{A}_{\phi^{-1}\circ\tau}.$$

Indeed, if $\theta_{\mathfrak{p},i-1,1} \in J$, then $\mathcal{A}_{\phi^{-1}\circ\tau} = \mathcal{F}^{(1)}_{\phi^{-1}\circ\tau}$ and

$$\mathcal{F}^{(1)}_{\phi^{-1}\circ\tau} \subset \mathcal{F}^{(j)}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(j+1)}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(e_\mathfrak{p})}_{\phi^{-1}\circ\tau} \subset u^{1-e_\mathfrak{p}}\mathcal{F}^{(1)}_{\phi^{-1}\circ\tau}.$$

However, if $\theta_{\mathfrak{p},i-1,1} \notin J$, then $\theta_{\mathfrak{p},i-1,\ell} \in J'$ for some $\ell \in \{2,\ldots,j\}$, so that $\mathcal{F}^{(\ell)}_{\phi^{-1}\circ\tau} = \mathcal{G}^{(\ell-1)}_{\phi^{-1}\circ\tau} = u^{-1}\mathcal{F}^{(\ell-2)}_{\phi^{-1}\circ\tau}$, which implies that $u^{e_\mathfrak{p}-1}\mathcal{G}^{(e_\mathfrak{p})}_{\phi^{-1}\circ\tau} = u^{e_\mathfrak{p}-2}\mathcal{F}^{(e_\mathfrak{p}-1)}_{\phi^{-1}\circ\tau} \subset u^{\ell-1}\mathcal{F}^{(\ell)} = u^{\ell-2}\mathcal{F}^{(\ell-2)} = 0$, and hence,

$$\mathcal{A}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(1)}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(\ell-1)}_{\phi^{-1}\circ\tau} = \mathcal{F}^{(\ell)}_{\phi^{-1}\circ\tau} \subset \mathcal{F}^{(j)}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(j+1)}_{\phi^{-1}\circ\tau} \subset \mathcal{G}^{(e_\mathfrak{p})}_{\phi^{-1}\circ\tau} \subset u^{1-e_\mathfrak{p}}\mathcal{A}_{\phi^{-1}\circ\tau}.$$

Now define $\mathcal{T}^{(j+1)}_{\phi^{-1}\circ\tau}$ to be $\mathcal{F}^{(j+1)}_{\phi^{-1}\circ\tau}$ if $\theta_{\mathfrak{p},i-1,j+1} \in J$, and to be the preimage in $\mathcal{G}^{(j+1)}_{\phi^{-1}\circ\tau}$ of the pull-back to P_J of the tautological subbundle of $\mathcal{P}_{\theta_{\mathfrak{p},i-1,j+1}}$ over $\mathbb{P}_{T_{J'}}(\mathcal{P}_{\theta_{\mathfrak{p},i-1,j+1}})$ if $\theta_{\mathfrak{p},i-1,j+1} \notin J$ (in which case $\theta_{\mathfrak{p},i-1,j+1} \in J''$). In either case, $(\mathcal{T}^{(j+1)}_{\phi^{-1}\circ\tau})^{(p)}$ is contained in the image of α^*_τ, so that $(\alpha^*_\tau)^{-1}(\mathcal{T}^{(j+1)}_{\phi^{-1}\circ\tau})^{(p)}$ is an $\mathcal{O}_{P_J}$-subbundle of $\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)$ of rank $j+2$. Furthermore, we have

$$\alpha^*_\tau(u^{e_\mathfrak{p}-1}\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau) = \mathcal{A}^{\otimes p}_{\phi^{-1}\circ\tau} \subset (\mathcal{T}^{(j+1)}_{\phi^{-1}\circ\tau})^{(p)},$$

so that $u^{e_\mathfrak{p}-1}\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau$ is a rank two $\mathcal{O}_{P_J}$-subbundle of $(\alpha^*_\tau)^{-1}(\mathcal{T}^{(j+1)}_{\phi^{-1}\circ\tau})^{(p)}$. We conclude that

$$\mathcal{F}''^{(j)}_\tau := u(\alpha^*_\tau)^{-1}(\mathcal{T}^{(j+1)}_{\phi^{-1}\circ\tau})^{(p)}$$

is a rank j subbundle of $u\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau$, and hence of $\mathcal{H}^1_{\mathrm{dR}}(A''/P_J)_\tau$.

To prove that the vector bundles $\mathcal{F}''^{(j)}_\tau$ define a Pappas–Rapoport filtration, we must show that

$$u\mathcal{F}''^{(j)}_\tau \subset \mathcal{F}''^{(j-1)}_\tau \subset \mathcal{F}''^{(j)}_\tau$$

for all $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$ and $j = 1,\ldots,e_\mathfrak{p}$, and the desired compatibility with α is the statement that $\alpha^*_{\phi\circ\tau}(\mathcal{F}''^{(j)}_{\phi\circ\tau}) \subset (\mathcal{F}^{(j)}_\tau)^{(p)}$ for all such τ and j. These inclusions are immediate from the definitions if $\tau \notin \Sigma_{\mathfrak{P},0}$, so we assume $\tau \in \Sigma_{\mathfrak{P},0}$.

Note that since P_J is smooth, it suffices to prove the corresponding inclusions hold on fibres at every geometric closed point $y \in P_J(\overline{\mathbb{F}}_p)$. To that end, let $F^{(j)}_\tau$ (resp. $F''^{(j)}_\tau$) denote the fibre at y of $\mathcal{F}^{(j)}_\tau$ (resp. $\mathcal{F}''^{(j)}_\tau$), and for each $\tau = \tau_{\mathfrak{p},i}$ and j such that $\theta = \theta_{\mathfrak{p},i,j} \in J''$, let $T^{(j)}_\tau$ denote the fibre of the the preimage in $\mathcal{G}^{(j)}_\tau$ of the tautological line bundle in $\mathcal{P}_{\theta_{\mathfrak{p},i,j}}$ over P_J.

We also let $W = W(\overline{\mathbb{F}}_p)$ and consider the Dieudonné module

$$\mathbb{D}(A_y[\mathfrak{p}^\infty]) = H^1_{\mathrm{crys}}(A_y/W)_\mathfrak{p} = \bigoplus_{\tau\in\Sigma_{\mathfrak{p},0}} H^1_{\mathrm{crys}}(A_y/W)_\tau$$

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for each $\mathfrak{p}|\mathfrak{P}$. Thus, the modules $H^1_{\mathrm{crys}}(A_y/W)_\tau$ are free of rank two over $W[u]/(E_\tau)$, equipped with $W[u]/(E_\tau)$-linear maps

$$H^1_{\mathrm{crys}}(A_y/W)^{\phi}_{\phi^{-1}\circ\tau} \xrightarrow{\Phi} H^1_{\mathrm{crys}}(A_y/W)_\tau \quad \text{and} \quad H^1_{\mathrm{crys}}(A_y/W)_\tau \xrightarrow{V} H^1_{\mathrm{crys}}(A_y/W)^{\phi}_{\phi^{-1}\circ\tau}$$

induced by Frob_A and Ver_A and satisfying $\Phi V = V\Phi = p$, where we use $\cdot^{\phi}$ to denote $\cdot\otimes_{W,\phi} W$. Consider also the Dieudonné module $\mathbb{D}(A''_y[\mathfrak{p}^\infty]) = \oplus_{\tau\in\Sigma_{\mathfrak{p},0}} H^1_{\mathrm{crys}}(A''_y/W)_\tau$, similarly equipped with morphisms Φ and V, as well as the injective $W[u]/(E_\tau)$-linear maps

$$\widetilde{\alpha}^*_\tau : H^1_{\mathrm{crys}}(A''_y/W)_\tau \longrightarrow H^1_{\mathrm{crys}}(A_y/W)^{\phi}_{\phi^{-1}\circ\tau},$$

compatible with Φ and V. Furthermore, the morphisms Φ, V and $\widetilde{\alpha}^*_\tau$ are compatible under the canonical isomorphisms $H^1_{\mathrm{crys}}(A_y/W)_\tau \otimes_W \overline{\mathbb{F}}_p \cong H^1_{\mathrm{dR}}(A_y/\overline{\mathbb{F}}_p)_\tau$ with the maps induced by the corresponding isogenies on de Rham cohomology. Thus, letting $\widetilde{F}^{(j)}_\tau$ denote the preimage of $F^{(j)}_\tau$ in $H^1_{\mathrm{crys}}(A_y/W)_\tau$, and similarly defining $\widetilde{F}''^{(j)}_\tau$, we are reduced to proving that

$$u\widetilde{\alpha}^*_\tau(\widetilde{F}''^{(j)}_\tau) \subset \widetilde{\alpha}^*_\tau(\widetilde{F}''^{(j-1)}_\tau) \subset \widetilde{\alpha}^*_\tau(\widetilde{F}''^{(j)}_\tau) \subset (\widetilde{F}^{(j)}_{\phi^{-1}\circ\tau})^{\phi}$$

for all $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$ and $j = 1,\dots,e_{\mathfrak{p}}$, or equivalently that

$$u\widetilde{F}'^{(j)}_\tau \subset \widetilde{E}^{(j-1)}_\tau \subset \widetilde{E}^{(j)}_\tau \subset \widetilde{F}^{(j)}_\tau, \tag{4.4}$$

where $\widetilde{E}^{(j)}_\tau := (\widetilde{\alpha}^*_{\phi\circ\tau}(\widetilde{F}''^{(j)}_{\phi\circ\tau}))^{\phi^{-1}}$.

We use the following description of $\widetilde{E}^{(j)}_\tau$ for $\tau = \tau_{\mathfrak{p},i}$ and $j = 0,1,\dots,e_{\mathfrak{p}}$, where we let $\widetilde{T}^{(j)}_\tau$ denote the preimage of $T^{(j)}_\tau$ in $H^1_{\mathrm{crys}}(A_y/W)_\tau$ for $\theta_{\mathfrak{p},i,j} \in J''$.

- Since $\widetilde{\alpha}^*_\tau(\widetilde{F}''^{(0)}_\tau) = p\widetilde{\alpha}^*_\tau(H^1_{\mathrm{crys}}(A''_y/W)_\tau) = pu^{1-e_{\mathfrak{p}}}\widetilde{A}^{\phi}_{\phi^{-1}\circ\tau} = u\widetilde{A}^{\phi}_{\phi^{-1}\circ\tau}$, where

$$\widetilde{A}_\tau = \begin{cases} \widetilde{F}^{(1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,1} \in J;\\ u^{e_{\mathfrak{p}}-1}V^{-1}(\widetilde{F}^{(e_{\mathfrak{p}}-1)}_{\phi^{-1}\circ\tau})^{\phi}, & \text{if } \theta_{\mathfrak{p},i-1,e_{\mathfrak{p}}} \in J;\\ \widetilde{T}^{(1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,1} \in J''; \end{cases}$$

  we have

$$\widetilde{E}^{(0)}_\tau = \begin{cases} \Phi((\widetilde{F}^{(e_{\mathfrak{p}}-1)}_{\phi^{-1}\circ\tau})^{\phi}), & \text{if } \theta_{\mathfrak{p},i-1,e_{\mathfrak{p}}} \notin J;\\ u\widetilde{F}^{(1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,1} \in J;\\ u\widetilde{T}^{(1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,1} \in J''. \end{cases}$$

- If $1 \le j \le e_{\mathfrak{p}} - 1$, then it follows from the definition of $\mathcal{F}''^{(j)}_{\phi\circ\tau}$ that

$$\widetilde{E}^{(j)}_\tau = \begin{cases} \widetilde{F}^{(j-1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,j} \notin J;\\ u\widetilde{F}^{(j+1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,j+1} \in J;\\ u\widetilde{T}^{(j+1)}_\tau, & \text{if } \theta_{\mathfrak{p},i,j+1} \in J''. \end{cases}$$

  (Note that if $\theta_{\mathfrak{p},i,j} \notin J$ and $\theta_{\mathfrak{p},i,j+1} \in J$, then $\theta_{\mathfrak{p},i,j+1} \in J'$, so that $u\widetilde{F}^{(j+1)}_\tau = \widetilde{F}^{(j-1)}_\tau$.)

- Since $\Phi((\widetilde{F}''^{(e_\mathfrak{p})}_\tau)^\phi) = \widetilde{F}''^{(0)}_{\phi\circ\tau}$, we have $(\widetilde{E}^{(e_\mathfrak{p})}_\tau) = p^{-1}(V(\widetilde{E}^{(0)}_{\phi\circ\tau}))^{\phi^{-1}}$, so the formulas in the case $j=0$ imply that

$$\widetilde{E}^{(e_\mathfrak{p})}_\tau = \begin{cases} \widetilde{F}^{(e_\mathfrak{p}-1)}_\tau & \text{if } \theta_{\mathfrak{p},i,e_\mathfrak{p}} \notin J; \\ (V(u^{1-e_\mathfrak{p}}\widetilde{F}^{(1)}_{\phi\circ\tau}))^{\phi^{-1}}, & \text{if } \theta_{\mathfrak{p},i+1,1} \in J; \\ (V(u^{1-e_\mathfrak{p}}\widetilde{T}^{(1)}_{\phi\circ\tau}))^{\phi^{-1}}, & \text{if } \theta_{\mathfrak{p},i+1,1} \in J''. \end{cases}$$

Note in particular that

$$u\widetilde{F}^{(j)}_\tau \subset \widetilde{E}^{(j)}_\tau \subset \widetilde{F}^{(j)}_\tau$$

for all $\tau = \tau_{\mathfrak{p},i}$, $j = 0,\ldots,e_\mathfrak{p}$ (the case of $j = e_\mathfrak{p}$ for $\tau = \tau_{\mathfrak{p},i}$ being equivalent by an application of Φ to that of $j=0$ for $\tau = \tau_{\mathfrak{p},i+1}$). The inclusions in (4.4) are then immediate from the formulas

$$\widetilde{E}^{(j-1)}_\tau = u\widetilde{F}^{(j)}_\tau \quad \text{if } \theta_{\mathfrak{p},i,j} \in J, \quad \text{and} \quad \widetilde{E}^{(j)}_\tau = \widetilde{F}^{(j-1)}_\tau \quad \text{if } \theta_{\mathfrak{p},i,j} \notin J. \tag{4.5}$$

We have now shown that the triple $(\underline{A}^{(p)}, \underline{A}'', \alpha)$ defines a morphism $\widetilde{\zeta}_J : P_J \to S$. Furthermore, rewriting equations (4.5) in the form

$$\begin{array}{rcl} \widetilde{\alpha}^*_{\phi\circ\tau}(u^{-1}\widetilde{F}''^{(j-1)}_{\phi\circ\tau}) &=& (\widetilde{F}^{(j)}_\tau)^\phi, \ \text{if } \theta_{\mathfrak{p},i,j} \in J, \\ \text{and} \quad \widetilde{\alpha}^*_{\phi\circ\tau}(\widetilde{F}''^{(j)}_{\phi\circ\tau}) &=& (\widetilde{F}^{(j-1)}_\tau)^\phi \ \text{if } \theta_{\mathfrak{p},i,j} \notin J, \end{array}$$

shows that $\widetilde{\zeta}_J(y) \in S_{\phi(J)}(\overline{\mathbb{F}}_p)$ for all $y \in P_J(\overline{\mathbb{F}}_p)$ (where we define $\phi : \Sigma_\mathfrak{P} \to \Sigma_\mathfrak{P}$ by $\phi(\theta_{\mathfrak{p},i,j}) = \theta_{\mathfrak{p},i+1,j}$). Since P_J is reduced, it follows that $\widetilde{\zeta}_J$ factors through $S_{\phi(J)}$. Note furthermore that if $(\underline{A}, \underline{A}', \psi)$ is the universal triple over S_J, then its pull-back by $\phi \in \mathrm{Aut}\,(k)$ is isomorphic to the universal triple over $S_{\phi(J)}$, yielding an identification of $S^{(p)}_J$ with $S_{\phi(J)}$ under which the relative Frobenius morphism $S_J \to S^{(p)}_J = S_{\phi(J)}$ corresponds to the triple $(\underline{A}, \underline{A}', \psi)^{(p)}$ over S_J. Note also that $\phi(J)' = \phi(J')$ and $\phi(J)'' = \phi(J'')$, and we may similarly identify $T^{(p)}_{J'}$ with $T_{\phi(J')}$ and $P^{(p)}_J$ with $P_{\phi(J)}$, under which $\widetilde{\xi}^{(p)}_J$ corresponds to $\widetilde{\xi}_{\phi(J)}$ and $\widetilde{\zeta}^{(p)}_J$ to $\widetilde{\zeta}_{\phi(J)}$.

Finally, we verify that the composites

$$P_J \xrightarrow{\widetilde{\zeta}_J} S_{\phi(J)} \xrightarrow{\widetilde{\xi}_{\phi(J)}} P_{\phi(J)} \quad \text{and} \quad S_J \xrightarrow{\widetilde{\xi}_J} P_J \xrightarrow{\widetilde{\zeta}_J} S_{\phi(J)}$$

are the Frobenius morphisms (relative to $\phi \in \mathrm{Aut}\,(k)$).

Indeed, it suffices to check this on geometric closed points, so suppose first that $(\underline{A}, (T_\theta)_{\theta\in J''})$ corresponds to an element $y \in P_J(\overline{\mathbb{F}}_p)$, where $\underline{A}$ corresponds to an element of $T_{J'}(\overline{\mathbb{F}}_p)$ and each T_θ to a line in $P_\theta = u^{-1}\widetilde{F}^{(j-1)}_\tau / \widetilde{F}^{(j-1)}_\tau$ (using the above notation with y suppressed, so $\tau = \tau_{\mathfrak{p},i}$, $\theta = \theta_{\mathfrak{p},i,j}$, and $\widetilde{F}^\bullet_\tau$ is the preimage in $H^1_{\mathrm{crys}}(A/W)_\tau$ of the Pappas–Rapoport filtration on $F^{(e_\mathfrak{p})}_\tau = H^0(A, \Omega^1_{A/\overline{\mathbb{F}}_p})_\tau \subset H^1_{\mathrm{dR}}(A/\overline{\mathbb{F}}_p)_\tau$). Its image $\widetilde{\zeta}_J(y)$ then corresponds to the triple $(\underline{A}^{(p)}, \underline{A}'', \alpha)$ produced by the construction above, and $\widetilde{\xi}_{\phi(J)}(\widetilde{\zeta}_J(y))$ is given by $(\underline{A}^{(p)}, \alpha^*_{\phi(\theta)}(P''_{\phi(\theta)}))$, where $P''_{\phi(\theta)} = u^{-1}\widetilde{F}''^{(j-1)}_{\phi\circ\tau} / \widetilde{F}''^{(j-1)}_{\phi\circ\tau}$ and $\alpha^*_{\phi(\theta)} : P''_{\phi(\theta)} \to P^{(p)}_\theta$ is induced by $\widetilde{\alpha}^*_{\phi\circ\tau}$. From the formulas above, we have

$$\widetilde{\alpha}^*_{\phi\circ\tau}(\widetilde{F}''^{(j-1)}_{\phi\circ\tau}) = u^{-1}(\widetilde{E}^{(j-1)}_\tau)^\phi = (\widetilde{T}^{(j)}_\tau)^\phi$$

for $\theta_{\mathfrak{p},i,j} \in J''$, where $\widetilde{T}^{(j)}_\tau$ is the preimage of T_θ in $H^1_{\mathrm{crys}}(A/W)_\tau$, and it follows that $\alpha^*_{\phi(\theta)}(P''_{\phi(\theta)}) = T^{(p)}_\theta$, as required.

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Suppose now that $(\underline{A}, \underline{A}', \psi)$ corresponds to an element $x \in S_J(\overline{\mathbb{F}}_p)$, so that if $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$ and $\theta = \theta_{\mathfrak{p},i,j}$, then $\widetilde{\psi}_\tau^*(u^{-1}\widetilde{F}_\tau'^{(j-1)}) = \widetilde{F}_\tau^{(j)}$ if $\theta \in J$ and $\widetilde{\psi}_\tau^*(\widetilde{F}_\tau'^{(j)}) = \widetilde{F}_\tau^{(j-1)}$ if $\theta \notin J$ (with the evident notation). Its image $\widetilde{\xi}_J(x)$ then corresponds to $(\underline{A}, (T_\theta)_{\theta \in J''})$, where

$$T_\theta = \psi_\theta^*(P_\theta') = \widetilde{\psi}_\tau^*(u^{-1}\widetilde{F}_\tau'^{(j-1)})/\widetilde{F}_\tau^{(j-1)},$$

and we must show that the triple $(\underline{A}^{(p)}, \underline{A}'', \alpha)$ obtained from it by applying $\widetilde{\zeta}_J$ is isomorphic to $(\underline{A}^{(p)}, \underline{A}'^{(p)}, \psi^{(p)})$. To that end, note that the formulas for $\widetilde{E}_\tau^{(j)}$ imply that $\widetilde{\psi}_\tau^*(\widetilde{F}_\tau'^{(j)}) = \widetilde{E}_\tau^{(j)}$ for all $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$ and $j = 0, 1, \ldots, e_\mathfrak{p}$ (using the equivalence provided by Φ between the cases of $j = e_\mathfrak{p}$ for τ and $j = 0$ for $\phi \circ \tau$). In particular, it follows from the case of $j = 0$ that

$$\begin{aligned} (\widetilde{\psi}_\tau^*)^\phi :\ & H^1_{\mathrm{crys}}(A'/W)_\tau^\phi \longrightarrow H^1_{\mathrm{crys}}(A/W)_\tau^\phi \\ \text{and}\ \ \widetilde{\alpha}_{\phi\circ\tau}^* :\ & H^1_{\mathrm{crys}}(A''/W)_{\phi\circ\tau} \longrightarrow H^1_{\mathrm{crys}}(A/W)_\tau^\phi \end{aligned}$$

have the same image for all $\tau \in \Sigma_{\mathfrak{P},0}$. Combined with the fact that $\widetilde{\psi}_\tau^*$ and $\widetilde{\alpha}_\tau^*$ are isomorphisms for $\tau \notin \Sigma_{\mathfrak{P},0}$, we conclude that the images of

$$\mathbb{D}(A'^{(p)}[p^\infty]) \longrightarrow \mathbb{D}(A^{(p)}[p^\infty]) \quad \text{and} \quad \mathbb{D}(A''[p^\infty]) \longrightarrow \mathbb{D}(A^{(p)}[p^\infty])$$

under the morphisms induced by $\psi^{(p)}$ and α are the same. It follows that there is an isomorphism $A'^{(p)} \xrightarrow{\sim} A''$ such that the diagram

$$\begin{tikzcd} & A^{(p)} \arrow[dl, "\psi^{(p)}"'] \arrow[dr, "\alpha"] & \\ A'^{(p)} \arrow[rr, "\sim"] & & A'' \end{tikzcd}$$

commutes. Its compatibility with all auxiliary structures is automatic, with the exception of the Pappas–Rapoport filtrations for $\tau \in \Sigma_{\mathfrak{P},0}$, where the compatibility is implied by the equality

$$(\widetilde{\psi}_{\tau'}^*(\widetilde{F}_{\tau'}'^{(j)}))^\phi = (\widetilde{E}_{\tau'}^{(j)})^\phi = \widetilde{\alpha}_{\phi\circ\tau}^*(\widetilde{F}_{\phi\circ\tau}''^{(j)})$$

for $\tau' = \phi^{-1} \circ \tau$ and $j = 1, \ldots, e_\mathfrak{p} - 1$.

We have now proved the following generalization of [DKS23, Lemma 7.1.5]:

Lemma 4.2. *There is a morphism $\widetilde{\zeta}_J : P_J \to S_J^{(p)}$ such that the composites $\widetilde{\xi}_J^{(p)} \circ \widetilde{\zeta}_J : P_J \to P_J^{(p)}$ and $\widetilde{\zeta}_J \circ \widetilde{\xi}_J : S_J \to S_J^{(p)}$ are the Frobenius morphisms (relative to $\phi \in \mathrm{Aut}(k)$).*

4.3. First-order deformations

In this section, we compute the effect of the morphisms

$$S_J \xrightarrow{\widetilde{\xi}_J} P_J \longrightarrow T_{J'}$$

on tangent spaces with a view to generalizing the results of [DKS23, §7.1.4].

We let $\mathbb{F}$ denote an algebraically closed field of characteristic p. First, recall that if $x \in T(\mathbb{F})$ corresponds to the data $\underline{A} = (A, \iota, \lambda, \eta, F^\bullet)$, then the Kodaira–Spencer filtration on $\Omega^1_{T/k}$ (as described in [Dia23, §3.3]) equips the tangent space of T at x with a decomposition

$$\mathrm{Tan}_x(T) = \bigoplus_{\tau \in \Sigma_0} \mathrm{Tan}_x(T)_\tau$$

and a decreasing filtration on each component such that

$$\bigoplus_{\tau\in\Sigma_0} \mathrm{Fil}^{\,j_\tau}(\mathrm{Tan}_x(T)_\tau)$$

is identified with the set of lifts $\widetilde{\underline{A}}$ of $\underline{A}$ to $T(\mathbb{F}[\epsilon])$ with the property that for all $\tau\in\Sigma_0$ and $1\le j\le j_\tau$, $\widetilde{F}^{(j)}_\tau$ corresponds to $F^{(j)}_\tau\otimes_{\mathbb{F}}\mathbb{F}[\epsilon]$ under the canonical isomorphism

$$H^1_{\mathrm{dR}}(\widetilde{A}/\mathbb{F}[\epsilon]/(\epsilon^2))_\tau \cong H^1_{\mathrm{crys}}(\widetilde{A}/\mathbb{F})_\tau \cong H^1_{\mathrm{dR}}(A/\mathbb{F})_\tau\otimes_{\mathbb{F}}\mathbb{F}[\epsilon].$$

In particular, for each $\tau=\tau_{\mathfrak{p},i}$ and $j=1,\dots,e_{\mathfrak{p}}$, the one-dimensional space

$$\mathrm{gr}^{\,j-1}(\mathrm{Tan}_x(T)_\tau) := \mathrm{Fil}^{\,j-1}(\mathrm{Tan}_x(T)_\tau)/\mathrm{Fil}^{\,j}(\mathrm{Tan}_x(T)_\tau)$$

is identified with the set of lines in $P_\theta\otimes_{\mathbb{F}}\mathbb{F}[\epsilon]$ lifting L_θ, or equivalently with $\mathrm{Hom}_{\mathbb{F}}(L_\theta,M_\theta)$, where $\theta=\theta_{\mathfrak{p},i,j}$ and $M_\theta=P_\theta/L_\theta$.

Furthermore, if $x\in T_{J'}(\mathbb{F})$, then $\widetilde{\underline{A}}$ corresponds to an element of $T_{J'}(\mathbb{F}[\epsilon])$ if and only if the Pappas–Rapoport filtrations $\widetilde{F}^{\bullet}_\tau$ have the property that

$$\widetilde{F}^{(j)}_\tau = \begin{cases} F^{(1)}_\tau\otimes_{\mathbb{F}}\mathbb{F}[\epsilon], & \text{if } j=1;\\ u^{-1}\widetilde{F}^{(j-2)}_\tau, & \text{if } j\ge 2;\end{cases}$$

for all j such that $\theta=\theta_{\mathfrak{p},i,j}\in J'$ (via the canonical isomorphism if $j=1$). It follows that $\mathrm{Tan}_x(T_{J'})$ inherits a decomposition into components $\mathrm{Tan}_x(T_{J'})_\tau$ equipped with filtrations such that

$$\mathrm{gr}^{\,j-1}(\mathrm{Tan}_x(T_{J'})_\tau) = \begin{cases} \mathrm{gr}^{\,j-1}(\mathrm{Tan}_x(T)_\tau), & \text{if } \theta_{\mathfrak{p},i,j}\notin J';\\ 0, & \text{if } \theta_{\mathfrak{p},i,j}\in J'.\end{cases}$$

Suppose now that $y\in P_J(\mathbb{F})$ corresponds to the data $(\underline{A},(L''_\theta)_{\theta\in J''})$, where $\underline{A}$ corresponds to a point $x\in T_{J'}(\mathbb{F})$ and each L''_θ is a line in P_θ. We may then decompose

$$\mathrm{Tan}_y(P_J) = \bigoplus_{\tau\in\Sigma_0}\mathrm{Tan}_y(P_J)_\tau$$

and filter each component so that

$$\bigoplus_{\tau}\mathrm{Fil}^{\,j_\tau}(\mathrm{Tan}_y(P_J)_\tau)$$

is identified with the set of lifts $(\widetilde{\underline{A}},(\widetilde{L}''_\theta)_{\theta\in J''})$ of $(\underline{A},(L''_\theta)_{\theta\in J''})$ to $P_J(\mathbb{F}[\epsilon])$ such that the following hold for all $\tau=\tau_{\mathfrak{p},i}\in\Sigma_0$ and $1\le j\le j_\tau$:

- ∘ $\widetilde{F}^{(j)}_\tau$ corresponds to $F^{(j)}_\tau\otimes_{\mathbb{F}}\mathbb{F}[\epsilon]$ under the canonical isomorphism;
- ∘ if $\theta=\theta_{\mathfrak{p},i,j}\in J''$, then $\widetilde{L}''_\theta$ corresponds to $L''_\theta\otimes_{\mathbb{F}}\mathbb{F}[\epsilon]$ under the resulting isomorphism

$$\widetilde{P}_\theta = u^{-1}\widetilde{F}^{(j-1)}_\tau/\widetilde{F}^{j-1}_\tau \cong P_\theta\otimes_{\mathbb{F}}\mathbb{F}[\epsilon].$$

We then have that

$$\dim_{\mathbb{F}}\mathrm{gr}^{\,j-1}(\mathrm{Tan}_y(P_J)_\tau) = \begin{cases} 0, & \text{if } \theta_{\mathfrak{p},i,j}\in J';\\ 2, & \text{if } \theta_{\mathfrak{p},i,j}\in J'';\\ 1, & \text{otherwise.}\end{cases}$$

Furthermore the projection $\mathrm{Tan}_y(P_J) \to \mathrm{Tan}_x(T_{J'})$ respects the decomposition and restricts to surjections $\mathrm{Fil}^j(\mathrm{Tan}_y(P_J)_\tau) \to \mathrm{Fil}^j(\mathrm{Tan}_x(T_{J'})_\tau)$ inducing an isomorphism

$$\mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau) \to \mathrm{gr}^{j-1}(\mathrm{Tan}_x(T_{J'})_\tau)$$

unless $\theta = \theta_{\mathfrak{p},i,j} \in J''$, in which case we have a decomposition

$$\mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau) = V_\theta \oplus V''_\theta \tag{4.6}$$

where V_θ (resp. V''_θ) is identified with the set of lines in $P_\theta \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ lifting L_θ (resp. L''_θ), so that the map to $\mathrm{gr}^{j-1}(\mathrm{Tan}_x(T_{J'})_\tau)$ has kernel V''_θ and restricts to an isomorphism on V_θ.

Finally, suppose that $z \in S_J(\mathbb{F})$ corresponds to the data $(\underline{A}, \underline{A}', \psi)$ and identify the tangent space $\mathrm{Tan}_z(S_J)$ with the set of lifts $(\underline{\widetilde{A}}, \underline{\widetilde{A}}', \widetilde{\psi})$ corresponding to elements of $S_J(\mathbb{F}[\epsilon])$. We may then decompose

$$\mathrm{Tan}_z(S_J) = \bigoplus_{\tau \in \Sigma_0} \mathrm{Tan}_z(S_J)_\tau$$

and filter each component so that

$$\bigoplus_{\tau} \mathrm{Fil}^{j_\tau}(\mathrm{Tan}_z(S_J)_\tau)$$

is identified with the set of lifts $(\underline{\widetilde{A}}, \underline{\widetilde{A}}', \widetilde{\psi})$ of $(\underline{A}, \underline{A}', \psi)$ to $S_J(\mathbb{F}[\epsilon])$ such that the canonical isomorphisms identify $\widetilde{F}_\tau^{(j)}$ with $F_\tau^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ and $\widetilde{F}_\tau'^{(j)}$ with $F_\tau'^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ for all $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$ and $1 \le j \le j_\tau$. Note that if $\tau \notin \Sigma_{\mathfrak{P},0}$, then the isomorphism induced by ψ renders the conditions for $\widetilde{F}_\tau^{(j)}$ and $\widetilde{F}_\tau'^{(j)}$ equivalent. Suppose, however, that $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$. If $\theta = \theta_{\mathfrak{p},i,j} \in J$ and $\widetilde{F}_\tau'^{(j-1)}$ corresponds to $F_\tau'^{(j-1)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$, then the functoriality of the isomorphisms between crystalline and de Rham cohomology implies that $\widetilde{G}_\tau'^{(j)} = u^{-1}\widetilde{F}_\tau'^{(j)}$ corresponds to $G_\tau'^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$, and consequently that $\widetilde{F}_\tau^{(j)}$ corresponds to $F_\tau^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$. It follows that in this case, the $\mathbb{F}$-linear map $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \to \mathrm{gr}^{j-1}(\mathrm{Tan}_x(T)_\tau)$ is trivial (where $x = \widetilde{\pi}_1$ corresponds to $\underline{A}$). Similarly, if $\theta = \theta_{\mathfrak{p},i,j} \notin J$ and $\widetilde{F}_\tau^{(j-1)}$ corresponds to $F_\tau^{(j-1)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$, then we find that $\widetilde{F}_\tau'^{(j)}$ corresponds to $F_\tau'^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$, so that the map

$$\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \to \mathrm{gr}^{j-1}(\mathrm{Tan}_{x'}(T)_\tau)$$

is trivial (where $x' = \widetilde{\pi}_2(z)$ corresponds to $\underline{A}'$). The degeneracy maps $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ therefore induce injective maps from $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau)$ to

- $\mathrm{gr}^{j-1}(\mathrm{Tan}_{x'}(T)_\tau) = \mathrm{gr}^{j-1}(\mathrm{Tan}_{x'}(T_{J''})_\tau)$ if $\theta \in J$;
- $\mathrm{gr}^{j-1}(\mathrm{Tan}_{x}(T)_\tau) = \mathrm{gr}^{j-1}(\mathrm{Tan}_{x}(T_{J'})_\tau)$ if $\theta \in \Sigma_{\mathfrak{P}}$ but $\theta \notin J$;
- either of the above if $\theta \notin \Sigma_{\mathfrak{P}}$.

Since $\mathrm{Tan}_z(S_J)$ has dimension d, it follows that all these maps are isomorphisms.

We now describe the map induced by $\widetilde{\xi}_J : S_J \to P_J$ on tangent spaces in terms of its effect on filtrations and graded pieces.

Lemma 4.3. *Let $z \in S_J(\mathbb{F})$ be a geometric point of S_J, and let $y = \widetilde{\xi}_J(z)$. Then the $\mathbb{F}$-linear homomorphism*

$$t : \mathrm{Tan}_z(S_J) \to \mathrm{Tan}_y(P_J)$$

respects the decompositions and filtrations. Furthermore, if $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j}$, then the following hold:

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1. *If $\theta, \sigma^{-1}\theta \notin J$, then t induces an isomorphism*

$$\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \xrightarrow{\sim} \mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau).$$

2. *If $\theta \in J$, then $t(\mathrm{Fil}^{j-1}(\mathrm{Tan}_z(S_J)_\tau)) \subset \mathrm{Fil}^j(\mathrm{Tan}_y(P_J)_\tau)$.*
3. *If $\theta, \sigma^{-1}\theta \in J$ and $j \geq 2$, then t induces an isomorphism*

$$\mathrm{gr}^{j-2}(\mathrm{Tan}_z(S_J)_\tau) \xrightarrow{\sim} \mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau).$$

4. *If $\theta \in J''$, then t induces an injection*

$$\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \hookrightarrow \mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau)$$

with image V_θ (under the decomposition (4.6)), extending to an isomorphism

$$\mathrm{Fil}^{j-2}(\mathrm{Tan}_z(S_J)_\tau)/\mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau) \xrightarrow{\sim} \mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau)$$

if $j \geq 2$.

Proof. Suppose that $(\underline{\widetilde{A}}, \underline{\widetilde{A}}', \widetilde{\psi})$ corresponds to an element $\widetilde{z} \in \mathrm{Fil}^{j-1}(\mathrm{Tan}_z(S_J)_\tau)$, where $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j}$, and let $(\underline{\widetilde{A}}, (\widetilde{L}_{\theta'})_{\theta' \in J''})$ correspond to its image $\widetilde{y} \in \mathrm{Tan}_y(P_J)$. We then have that $\widetilde{F}_{\tau'}^{(j')}$ (resp. $\widetilde{F}_{\tau'}'^{(j')}$) corresponds to $F_{\tau'}^{(j')} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ (resp. $F_{\tau'}'^{(j')} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$) under the canonical isomorphisms

$$\begin{aligned} H^1_{\mathrm{dR}}(\widetilde{A}/\mathbb{F}[\epsilon])_{\tau'} &\cong H^1_{\mathrm{dR}}(A/\mathbb{F})_{\tau'} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon] \\ (\text{resp. } H^1_{\mathrm{dR}}(\widetilde{A}'/\mathbb{F}[\epsilon])_{\tau'} &\cong H^1_{\mathrm{dR}}(A'/\mathbb{F})_{\tau'} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]) \end{aligned}$$

for $\tau' = \tau$ and $j' \leq j-1$, and for all $\tau' = \tau_{\mathfrak{p}',i'} \neq \tau$ and $j' \leq e_{\mathfrak{p}'}$. It follows also that $\widetilde{L}''_{\theta'}$ corresponds to $L''_{\theta'} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ for all $\theta' = \theta_{\mathfrak{p}',i',j'} \in J''$ such that $\tau' \neq \tau$, or $j' \leq j$ if $\tau' = \tau$. This proves that t respects the decompositions and filtrations, and that $\theta \in J''$, then the image of $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \to \mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau)$ is contained in V_θ. Furthermore, we have already seen that $\widetilde{F}_\tau^{(j)}$ corresponds to $F_\tau^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ if $\theta \in J$, so that 2) holds.

For the injectivity in 1) and the first part of 4), recall that $\widetilde{F}_\tau'^{(j)}$ corresponds to $F_\tau'^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ if $\theta \notin J$, so if $\widetilde{y} \in \mathrm{Fil}^j(\mathrm{Tan}_y(P_J)_\tau)$, then $\widetilde{z} \in \mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau)$. The assertions concerning the image then follow from the fact that each space is one-dimensional.

Again, comparing dimensions, it suffices to prove injectivity in 3) and the second part of 4). Applying the injectivity in the first part of 4), and then shifting indices to maintain the assumption that $\widetilde{z} \in \mathrm{Fil}^{j-1}(\mathrm{Tan}_z(S_J)_\tau)$, both assertions reduce to the claim that if $j < e_{\mathfrak{p}}$, $\theta \in J$ and $\widetilde{y} \in \mathrm{Fil}^{j+1}(\mathrm{Tan}_y(P_J)_\tau)$, then $\widetilde{z} \in \mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau)$.

To prove the claim, let $\widetilde{F}_\tau''^{(j+1)}$ denote the image of $\widetilde{G}_\tau'^{(j+1)} = u^{-1}\widetilde{F}_\tau'^{(j)}$ in $\widetilde{G}_\tau^{(j+1)} = u^{-1}\widetilde{F}_\tau^{(j)}$ under the morphism induced by $\widetilde{\psi}$. Note that if $\sigma\theta \in J$, then $\widetilde{F}_\tau''^{(j+1)} = \widetilde{F}_\tau^{(j+1)}$, whereas if $\sigma\theta \notin J$, then $\widetilde{F}_\tau''^{(j+1)}/\widetilde{F}_\tau^{(j)} = \widetilde{L}''_{\sigma\theta}$. Thus, in either case, the hypothesis that $\widetilde{y} \in \mathrm{Fil}^{j+1}(\mathrm{Tan}_y(P_J)_\tau)$ implies that $\widetilde{F}_\tau''^{(j+1)}$ corresponds to $F_\tau''^{(j+1)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$. Functoriality of the canonical isomorphism between crystalline and de Rham cohomology then implies that their preimages under the maps induced by $\widetilde{\psi}$ and ψ correspond, that is, that $\widetilde{G}_\tau'^{(j+1)}$ corresponds to $G_\tau'^{(j+1)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$, and hence that $\widetilde{F}_\tau'^{(j)}$ corresponds to $F_\tau'^{(j)} \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$, so $\widetilde{z} \in \mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau)$. ∎

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Lemma 4.4. *Let $t_\tau : \mathrm{Tan}_z(S_J)_\tau \longrightarrow \mathrm{Tan}_y(P_J)_\tau$ denote the τ-component of the $\mathbb{F}$-linear map t, with notation as in Lemma 4.3. Then*

$$\ker(t_\tau) = \begin{cases} \mathrm{Fil}^{e_\mathfrak{p}-1}(\mathrm{Tan}_z(S_J)_\tau), & \text{if } \theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J; \\ 0, & \text{if } \theta_{\mathfrak{p},i,e_\mathfrak{p}} \notin J; \end{cases}$$
$$\mathrm{im}(t_\tau) = \begin{cases} \mathrm{Tan}_y(P_J)_\tau, & \text{if } \theta_{\mathfrak{p},i-1,e_\mathfrak{p}} \notin J; \\ \mathrm{Fil}^1(\mathrm{Tan}_y(P_J)_\tau), & \text{if } \theta_{\mathfrak{p},i,1} \in J; \\ \widetilde{V}_{\theta_{\mathfrak{p},i,1}}, & \text{if } \theta = \theta_{\mathfrak{p},i,1} \in J'', \end{cases}$$

where $\widetilde{V}_{\theta_{\mathfrak{p},i,1}}$ is the preimage of $V_{\theta_{\mathfrak{p},i,1}} \subset \mathrm{gr}^0(\mathrm{Tan}_y(P_J)_\tau)$ under t he projection from $\mathrm{Tan}_y(P_J)_\tau$.

In particular $\ker(t)$ has dimension $\#\{\,(\mathfrak{p},i) \mid \theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J\,\}$.

Proof. It is immediate from part 2) of Lemma 4.3 that if $\widetilde{z} \in \mathrm{Fil}^{e_\mathfrak{p}-1}(\mathrm{Tan}_z(S_J)_\tau)$, then $t(\widetilde{z}) = 0$.

Suppose, however, that $\widetilde{z} \in \ker(t_\tau)$. We prove by induction on j that $\widetilde{z} \in \mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau)$ for $j = 1, \dots, e_\mathfrak{p} - 1$, as well as $j = e_\mathfrak{p}$ if $\theta_{\mathfrak{p},i,e_\mathfrak{p}} \notin J$. Indeed, if $\widetilde{z} \in \mathrm{Fil}^{j-1}(\mathrm{Tan}_z(S_J)_\tau)$ and $\theta_{\mathfrak{p},i,j} \notin J$, then since $t(\widetilde{z}) \in \mathrm{Fil}^j(\mathrm{Tan}_y(P_J)_\tau)$, parts 1) and 4) of Lemma 4.3 imply that $\widetilde{z} \in \mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau)$. However, if $\widetilde{z} \in \mathrm{Fil}^{j-1}(\mathrm{Tan}_z(S_J)_\tau)$, $\theta_{\mathfrak{p},i,j} \in J$ and $j < e_\mathfrak{p}$, then since $t(\widetilde{z}) \in \mathrm{Fil}^{j+1}(\mathrm{Tan}_y(P_J)_\tau)$, parts 3) and 4) of Lemma 4.3 imply that $\widetilde{z} \in \mathrm{Fil}^j(\mathrm{Tan}_z(S_J)_\tau)$.

This completes the proof of the description of the kernel of t_τ, which immediately implies the formula for the dimension of $\ker(t)$.

The fact that the image of t_τ is contained in the described space is also immediate from Lemma 4.3. (Note that the first two cases in the description are not exclusive, but are consistent since $\mathrm{gr}^1(\mathrm{Tan}_y(P_J)_\tau) = 0$ if $\theta_{\mathfrak{p},i,1} \in J'$.) Equality then follows on noting that their direct sum over all τ has dimension equal to the rank of t. □

Now let Z denote the fibre at x of $\pi_J : S_J \to T_{J'}$, defined as the restriction of $\widetilde{\pi}_{1,k}$, and consider the tangent space $\mathrm{Tan}_z(Z)$, which may be identified with the kernel of the $\mathbb{F}$-linear map $\mathrm{Tan}_z(S_J) \to \mathrm{Tan}_x(T_{J'})$. We denote the map by s and write it as the composite

$$\mathrm{Tan}_z(S_J) \xrightarrow{t} \mathrm{Tan}_y(P_J) \longrightarrow \mathrm{Tan}_x(T_{J'}),$$

where the second map is the natural projection. Since each map respects the decomposition over $\tau \in \Sigma_0$, we may similarly decompose $\mathrm{Tan}_z(Z) = \bigoplus_\tau \mathrm{Tan}_z(Z)_\tau$ where $\mathrm{Tan}_z(Z)_\tau$ is the kernel of the composite

$$s_\tau : \mathrm{Tan}_z(S_J)_\tau \longrightarrow \mathrm{Tan}_y(P_J)_\tau \longrightarrow \mathrm{Tan}_x(T_{J'})_\tau.$$

We may furthermore consider the filtration on $\mathrm{Tan}_z(Z)_\tau$ induced by the one on $\mathrm{Tan}_z(S_J)_\tau$, so that if z corresponds to $(\underline{A}, \underline{A}', \psi)$, then $\bigoplus_\tau \mathrm{Fil}^{j_\tau}(\mathrm{Tan}_z(Z)_\tau)$ is identified with the set of its lifts $(\widetilde{\underline{A}}, \widetilde{\underline{A}}', \widetilde{\psi})$ to $S_J(\mathbb{F}[\epsilon])$ such that $\widetilde{\underline{A}} = \underline{A} \otimes_\mathbb{F} \mathbb{F}[\epsilon]$ and $\widetilde{F}'^{(j)}_\tau$ corresponds to $F^{(j)}_\tau \otimes_\mathbb{F} \mathbb{F}[\epsilon]$ for all $\tau \in \Sigma_0$ and $j \le j_\tau$. (Note that this condition is automatic for $\tau \notin \Sigma_\mathfrak{P}$, in which case $\mathrm{Tan}_z(Z)_\tau = 0$.)

Lemma 4.5. *The $\mathbb{F}$-vector space $\mathrm{Tan}_z(Z)$ has dimension $|J'| + \delta = |J''| + \delta$, where*

$$\delta = \#\{\,\tau = \tau_{\mathfrak{p},i} \mid \theta_{\mathfrak{p},i,e_\mathfrak{p}}, \theta_{\mathfrak{p},i+1,1} \in J\,\}.$$

More precisely, if $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j}$, then

$$\dim_\mathbb{F} \mathrm{gr}^{j-1}(\mathrm{Tan}_z(Z)_\tau) = \begin{cases} 1, & \text{if } \theta = \theta_{\mathfrak{p},i,j} \in J \text{ and either } \sigma\theta \notin J \text{ or } j = e_\mathfrak{p}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It follows from the description of the image of t_τ in Lemma 4.4 that if $\tau = \tau_{\mathfrak{p},i}$, then s_τ is surjective unless $\theta_{\mathfrak{p},i-1,e_\mathfrak{p}}, \theta_{\mathfrak{p},i,1} \in J$, in which case the image of s_τ is $\mathrm{Fil}^1(\mathrm{Tan}_x(T_{J'})_\tau)$, which has codimension one in $\mathrm{Tan}_x(T_{J'})_\tau$. Therefore, the image of s has codimension δ in $\mathrm{Tan}_x(T_{J'})$, and the kernel of s has dimension

$$\dim_{\mathbb{F}}(\mathrm{Tan}_z(S_J)) - \dim_{\mathbb{F}}(\mathrm{Tan}_x(T_{J'})) + \dim_{\mathbb{F}}(\mathrm{coker}\, s) = |J'| + \delta.$$

Since $\dim_{\mathbb{F}} \mathrm{gr}^{j-1}(\mathrm{Tan}_z(Z)_\tau) \le \dim_{\mathbb{F}} \mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) = 1$ for all $\tau = \tau_{\mathfrak{p},i}$ and $j = 1, \dots, e_{\mathfrak{p}}$, and δ is the total number of τ and j such that if $\theta = \theta_{\mathfrak{p},i,j} \in J$ and either $\sigma\theta \notin J$ or $j = e_{\mathfrak{p}}$, it suffices to prove that $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(Z)_\tau) = 0$ if either $\theta \notin J$, or $j < e_{\mathfrak{p}}$ and $\theta_{\mathfrak{p},i,j}, \theta_{\mathfrak{p},i,j+1} \in J$. This is immediate from parts 1), 3) and 4) of Lemma 4.3, which show that the following maps induced by s_τ are isomorphisms:

- $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \longrightarrow \mathrm{gr}^{j-1}(\mathrm{Tan}_y(P_J)_\tau) \xrightarrow{\sim} \mathrm{gr}^{j-1}(\mathrm{Tan}_x(T_{J'})_\tau)$ in the case $\sigma^{-1}\theta, \theta \notin J$;
- $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \longrightarrow \mathrm{gr}^{j}(\mathrm{Tan}_y(P_J)_\tau) \xrightarrow{\sim} \mathrm{gr}^{j}(\mathrm{Tan}_x(T_{J'})_\tau)$ in the case $\theta, \sigma\theta \in J$, $j < e_{\mathfrak{p}}$;
- $\mathrm{gr}^{j-1}(\mathrm{Tan}_z(S_J)_\tau) \longrightarrow V_\theta \xrightarrow{\sim} \mathrm{gr}^{j-1}(\mathrm{Tan}_x(T_{J'})_\tau)$ in the case $\theta \in J''$.

□

4.4. Unobstructed deformations

We maintain the notation of the preceding section, so that in particular, J is a subset of $\Sigma_{\mathfrak{P}}$ and Z is the fibre of $\pi_J : S_J \to T_{J'}$ at a geometric point $x \in T_{J'}(\mathbb{F})$, where $J' = \{\, \theta \in J \mid \sigma^{-1}\theta \notin J \,\}$. Following on from the description of the tangent space in Lemma 4.5, we continue the study of the local structure of Z at each closed point z in order to prove a generalization of [DKS23, Lemma 7.1.6], showing in particular that the reduced subscheme is smooth. We remark, however, that if $\mathfrak{P}$ is divisible by a prime $\mathfrak{p}$ ramified in F, then the fibres over $T_{J'}$ of $\widetilde{\xi}_J$ need not be totally inseparable, so the desired conclusions do not immediately follow as in [DKS23] from the commutative algebra results of [KN82] and the description of the morphisms on tangent spaces. We therefore resort to a more detailed analysis of the deformation theory.

Fix $m \ge 0$ and let $R = \mathbb{F}[[X_1, \dots, X_m]]$ and $R_n = R/\mathfrak{m}^{n+1}$ for $n \ge 0$, where $\mathfrak{m} = \mathfrak{m}_R = (X_1, \dots, X_m)$. Similarly, let $\widetilde{R} = W_2[[X_1, \dots, X_m]]$ and $\widetilde{R}_n = \widetilde{R}/I^{n+1}$ where $W_2 = W_2(\mathbb{F})$ and $I = (X_1, \dots, X_m)$, so that $\widetilde{R}_n$ is flat over W_2 and $\widetilde{R}_n/p\widetilde{R}_n = R_n$. We let $\widetilde{\phi} = \widetilde{\phi}_n : \widetilde{R}_n \to \widetilde{R}_n$ denote the lift of the absolute Frobenius $\phi = \phi_n$ on R_n defined by $\widetilde{\phi}(X_j) = X_j^p$ for $j = 1, \dots, m$. We view $\widetilde{R}_{n+1}$ (as well as $\widetilde{R}_n$ and R_{n+1}) as a divided power thickening of R_n with divided powers defined by

$$(pf + g)^{[i]} = p^{[i]} f^i + p^{[i-1]} f^{i-1} g$$

for $f \in \widetilde{R}_{n+1}$, $g \in I^{n+1}$ and $i \ge 1$.[17]

We now denote the data corresponding to $x \in T_{J'}(\mathbb{F})$ by $\underline{A}_0$, take $(\underline{A}_n, \underline{A}'_n, \psi_n)$ corresponding to an element $z_n \in Z(R_n)$, so that $\underline{A}_n = \underline{A}_0 \otimes_{\mathbb{F}} R_n$, and let $F_n^\bullet = F_0^\bullet \otimes_{\mathbb{F}} R_n$ and $F_n'^\bullet$ denote the associated Pappas–Rapoport filtrations. Suppose that $\tau = \tau_{\mathfrak{p},i}$ is such that $\theta = \theta_{\mathfrak{p},i,e_{\mathfrak{p}}} \in J$, and consider the morphisms

$$\begin{array}{ccc} H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_n)_\tau & \xrightarrow{\widetilde{\psi}^*_{\tau,n}} & H^1_{\mathrm{crys}}(A_n/\widetilde{R}_n)_\tau \\ & & \uparrow{\scriptstyle \widetilde{V}_{\tau,n}} \\ & & H^1_{\mathrm{crys}}(A_n^{(p^{-1})}/\widetilde{R}_n)_\tau \end{array} \tag{4.7}$$

of free rank two $\widetilde{R}_n[u]/(E_\tau)$-modules, where $A_n^{(p^{-1})} := A_0^{(p^{-1})} \otimes_{\mathbb{F}} R_n$ and $\widetilde{V}_{\tau,n}$ is induced by $\mathrm{Ver} : A_0^{(p^{-1})} \to A_0$. Note that $H^1_{\mathrm{crys}}(A_n/\widetilde{R}_n)_\tau = H^1_{\mathrm{crys}}(A_0/W_2)_\tau \otimes_{W_2} \widetilde{R}_n$ and $H^1_{\mathrm{crys}}(A_n^{(p^{-1})}/\widetilde{R}_n)_\tau = H^1_{\mathrm{crys}}(A_0^{(p^{-1})}/W_2)_\tau \otimes_{W_2} \widetilde{R}_n = H^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau} \otimes_{W_2} \widetilde{R}_n$, identifying $\widetilde{V}_{\tau,n}$ with $\widetilde{V}_{\tau,0} \otimes_{W_2} \widetilde{R}_n$. Note in

[17] Note that $p^{[i]} = 0$ in W_2 if $i > 1$ and $p > 2$.

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particular that $\widetilde{V}_{\tau,n}$ has kernel and cokernel annihilated by p, and its image is $\widetilde{F}^{(e_\mathfrak{p})}_{\tau,0} \otimes_{W_2} \widetilde{R}_n$, where $\widetilde{F}^{(e_\mathfrak{p})}_{\tau,0}$ is the preimage in $H^1_{\mathrm{crys}}(A_0/W_2)_\tau$ of $F^{(e_\mathfrak{p})}_{\tau,0} = H^0(A_0, \Omega^1_{A_0/\mathbb{F}})$ under the canonical projection to $H^1_{\mathrm{crys}}(A_0/\mathbb{F})_\tau \cong H^1_{\mathrm{dR}}(A_0/\mathbb{F})_\tau$. However, the kernel and cokernel of $\widetilde{\psi}^*_{\tau,n}$ are annihilated by u, and the assumption that $\theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J$ implies that ψ_n induces a surjection

$$G'^{(e_\mathfrak{p})}_{\tau,n} = u^{-1} F'^{(e_\mathfrak{p}-1)}_{\tau,n} \longrightarrow F^{(e_\mathfrak{p})}_{\tau,0} \otimes_\mathbb{F} R_n.$$

It follows that $\widetilde{\psi}^*_{\tau,n}$ induces a surjection

$$\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n} \longrightarrow \widetilde{F}^{(e_\mathfrak{p})}_{\tau,0} \otimes_{W_2} \widetilde{R}_n,$$

where $\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n}$ is the preimage in $H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_n)_\tau$ of $G'^{(e_\mathfrak{p})}_{\tau,n}$ under the canonical projection to $H^1_{\mathrm{crys}}(A'_n/R_n)_\tau \cong H^1_{\mathrm{dR}}(A'_n/R_n)_\tau$. We therefore obtain from (4.7) a commutative diagram of $\widetilde{R}_n[u]/(E_\tau)$-linear isomorphisms

$$\begin{tikzcd}
\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n}/\ker(\widetilde{\psi}^*_{\tau,n}) \arrow[r, "\sim"] \arrow[dr, "\sim"'] & \widetilde{F}^{(e_\mathfrak{p})}_{\tau,0} \otimes_{W_2} \widetilde{R}_n \\
& (H^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau}/\ker(\widetilde{V}_{\tau,0})) \otimes_{W_2} \widetilde{R}_n. \arrow[u, "\wr"']
\end{tikzcd}$$

Furthermore, since $\ker(\widetilde{\psi}^*_{\tau,n}) \subset pH^1_{\mathrm{crys}}(A'_n/R_n)_\tau \subset u\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n}$, we in turn obtain R_n-linear isomorphisms

$$\begin{tikzcd}
\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n}/u\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n} \arrow[r, "\sim"] \arrow[dr, "\sim"'] & (\widetilde{F}^{(e_\mathfrak{p})}_{\tau,0}/u\widetilde{F}^{(e_\mathfrak{p})}_{\tau,0}) \otimes_\mathbb{F} R_n \\
& (H^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau}/uH^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau}) \otimes_\mathbb{F} R_n. \arrow[u, "\wr"']
\end{tikzcd}$$

Finally, noting that the projection $\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n} \to G'^{(e_\mathfrak{p})}_{\tau,n}$ identifies $\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n}/u\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n}$ with $P'_{\theta,n} := G'^{(e_\mathfrak{p})}_{\tau,n}/F'^{(e_\mathfrak{p}-1)}_{\tau,n}$, and that $u^{e_\mathfrak{p}-1}$ induces

$$\begin{aligned}
H^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau}/uH^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau} &= H^1_{\mathrm{dR}}(A_0/\mathbb{F})^{(p^{-1})}_{\phi\circ\tau}/uH^1_{\mathrm{dR}}(A_0/\mathbb{F})^{(p^{-1})}_{\phi\circ\tau} \\
&\xrightarrow{\sim} H^1_{\mathrm{dR}}(A_0/\mathbb{F})^{(p^{-1})}_{\phi\circ\tau}[u] = P^{(p^{-1})}_{\sigma\theta,0}
\end{aligned}$$

(where $P_{\sigma\theta,0} = G^{(1)}_{\phi\circ\tau,0}$), we obtain from (4.7) an isomorphism

$$\varepsilon_\tau(z_n) : P'_{\theta,n} \xrightarrow{\sim} P^{(p^{-1})}_{\sigma\theta,0} \otimes_\mathbb{F} R_n$$

of free rank two R_n-modules.

We say that z_n is *unobstructed* if $\varepsilon_\tau(z_n)$ sends $L'_{\theta,n}$ to $L^{(p^{-1})}_{\sigma\theta,0} \otimes_\mathbb{F} R_n$ for all $\tau = \tau_{\mathfrak{p},i}$ such that $\theta = \theta_{\mathfrak{p},i,e_\mathfrak{p}}$ and $\sigma\theta = \theta_{\mathfrak{p},i+1,1}$ are both in J, where as usual $L'_{\theta,n} = F'^{(e_\mathfrak{p})}_{\tau,n}/F'^{(e_\mathfrak{p}-1)}_{\tau,n}$ and $L_{\sigma\theta,0} = F^{(1)}_{\phi\circ\tau,0}$. Note that the condition on $\varepsilon_\tau(z_n)$ is equivalent to the vanishing of the induced homomorphism

$$L'_{\theta,n} \longrightarrow (P_{\sigma\theta,0}/L_{\sigma\theta,0})^{(p^{-1})} \otimes_\mathbb{F} R_n$$

of free rank one R_n-modules. We let $Z(R_n)^{\mathrm{uno}}$ denote the set of unobstructed elements in $Z(R_n)$.

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Note firstly that every element $z = z_0$ of $Z(\mathbb{F})$ is unobstructed. Indeed, the condition that $\sigma\theta \in J$ implies that the image of

$$\widetilde{\psi}^*_{\phi\circ\tau} : H^1_{\mathrm{crys}}(A'_0/W_2) \longrightarrow H^1(A_0/W_2)$$

is $u^{1-e_{\mathfrak{p}}}\widetilde{F}^{(1)}_{\phi\circ\tau,0}$, so the claim follows from the commutativity of the diagram

$$\begin{tikzcd} H^1_{\mathrm{crys}}(A'_0/W_2)^{\phi^{-1}}_{\phi\circ\tau} \arrow[r, "(\widetilde{\psi}^*_{\phi\circ\tau})^{\phi^{-1}}"] \arrow[d, "\widetilde{V}'_{\tau,0}"'] & H^1_{\mathrm{crys}}(A_0/W_2)^{\phi^{-1}}_{\phi\circ\tau} \arrow[d, "\widetilde{V}_{\tau,0}"] \\ H^1_{\mathrm{crys}}(A'_0/W_2)_\tau \arrow[r, "\widetilde{\psi}^*_\tau"] & H^1_{\mathrm{crys}}(A_0/W_2)_\tau . \end{tikzcd}$$

Furthermore, the construction of $\varepsilon_\tau(z_n)$ is functorial in R_n for varying m and n, in the sense that if

$$\begin{array}{ccc} \alpha : \mathbb{F}[[X_1,\ldots,X_m]]/(X_1,\ldots,X_m)^{n+1} & \longrightarrow & \mathbb{F}[[X_1,\ldots,X_{m'}]]/(X_1,\ldots,X_{m'})^{n'+1} \\ \| & & \| \\ R_n & & R'_{n'} \end{array}$$

is an $\mathbb{F}$-algebra morphism admitting a lift $\widetilde{\alpha} : \widetilde{R}_n \to \widetilde{R}'_{n'}$, then $\varepsilon_\tau(\alpha^* z_n) = \alpha^*\varepsilon_\tau(z_n)$, so that α^* sends $Z(R_n)^{\mathrm{uno}}$ to $Z(R'_{n'})^{\mathrm{uno}}$.[18] In particular, if $z_1 \in \mathrm{Tan}_z(Z)$ (viewed as an element of $Z(R_1)$, with $R_1 = \mathbb{F}[X]/(X^2)$), then $\varepsilon_\tau(z_1) - \varepsilon_\tau(z) \otimes 1$ induces an $\mathbb{F}$-linear map

$$L'_{\theta,0} \cong L'_{\theta,1}/XL'_{\theta,1} \longrightarrow (P^{(p^{-1})}_{\sigma\theta,0}/L^{(p^{-1})}_{\sigma\theta,0}) \otimes_{\mathbb{F}} XR_1.$$

Writing this map as $\partial_\tau(z_1) \otimes X$ for a unique $\partial_\tau(z_1) \in \mathrm{Hom}_{\mathbb{F}}(L'_{\theta,0}, (P_{\sigma\theta,0}/L_{\sigma\theta,0})^{(p^{-1})})$, the condition on $\varepsilon_\tau(z_1)$ in the definition of unobstructedness is then equivalent to the vanishing of $\partial_\tau(z_1)$.

More generally, if $n = 1$ and m is arbitrary, then to give an element $z_1 \in Z(R_1)$ over $z = z_0 \in Z(\mathbb{F})$ is equivalent to giving an m-tuple of tangent vectors $(z_1^{(1)}, \ldots, z_1^{(m)})$ in $\mathrm{Tan}_z(Z)$, or more functorially an element of $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^m, \mathrm{Tan}_z(Z))$, the induced morphism $\mathcal{O}^\wedge_{Z,z} \to \mathcal{O}_{Z,z}/\mathfrak{m}_z^2 \to R_1$ being surjective if and only if the m-tuple is linearly independent. Furthermore, any $\mathbb{F}$-algebra morphism $\alpha : R_1 \to R'_1$ (for arbitrary m, m') admits a lift $\widetilde{\alpha} : \widetilde{R}_1 \to \widetilde{R}'_1$, so the function

$$\partial_\tau : \mathrm{Tan}_z(Z) \longrightarrow \mathrm{Hom}_{\mathbb{F}}(L'_{\theta,0}, (P_{\sigma\theta,0}/L_{\sigma\theta,0})^{(p^{-1})}) \tag{4.8}$$

is $\mathbb{F}$-linear. (Let $z_1 \in Z(R_1)$ correspond to a basis $(z_1^{(1)}, \ldots, z_1^{(m)})$ of $\mathrm{Tan}_z(Z)$, and realize an arbitrary tangent vector $\sum a_i z_1^{(i)}$ as the image of z_1 under $X_i \mapsto a_i X$.) The set of unobstructed elements in $\mathrm{Tan}_z(Z)$ is thus the kernel of the resulting $\mathbb{F}$-linear map

$$\mathrm{Tan}_z(Z) \longrightarrow \bigoplus_{\tau\in\Delta} \mathrm{Hom}_{\mathbb{F}}(L'_{\theta,0}, (P_{\sigma\theta,0}/L_{\sigma\theta,0})^{(p^{-1})}),$$

where $\Delta = \{\, \tau = \tau_{\mathfrak{p},i} \mid \theta_{\mathfrak{p},i,e_{\mathfrak{p}}}, \theta_{\mathfrak{p},i+1,1} \in J \,\}$, and is therefore a vector space of dimension at least $|J'|$ by Lemma 4.5. The functoriality of ε_τ also implies that an element $z_1 \in Z(R_1)$ corresponding to an m-tuple $(z_1^{(1)}, \ldots, z_1^{(m)})$ is unobstructed if and only if each of the $z_1^{(i)}$ is, so it follows that there is a

[18] We could not directly define $Z(R_n)^{\mathrm{uno}}$ as the set of R_n-points of a closed subscheme since the construction of $\varepsilon_\tau(z_n)$ relied on the existence and choice of flat W_2-lifts. An alternative approach would be to prove a suitable generalization of [DKS23, Lemma 7.2.1] (the 'crystallization lemma'), and use it together with the flatness of $\widetilde{\xi}_J$ to define ε_τ over Z_{red}, without knowing a priori that it is smooth. We opted instead here for a bootstrapping argument that will allow us first to establish the smoothness of Z_{red} so that we can apply a more direct generalization of the crystallization lemma proved in [DKS23].

surjective $\mathbb{F}$-algebra homomorphism

$$\widehat{\mathcal{O}}_{Z,z} \longrightarrow R_1 = \mathbb{F}[[X_1,\dots,X_{|J'|}]]/(X_1,\dots,X_{|J'|})^2 \tag{4.9}$$

corresponding to an element $z_1 \in Z(R_1)^{\mathrm{uno}}$.

We shall in fact need the following more precise description of the set of unobstructed tangent vectors.

Lemma 4.6. *If $z \in Z(\mathbb{F})$, then $\mathrm{Tan}_z(Z)^{\mathrm{uno}} = \oplus_\tau \mathrm{Tan}_z(Z)^{\mathrm{uno}}_\tau$, where $\mathrm{Tan}_z(Z)^{\mathrm{uno}}$ (resp. $\mathrm{Tan}_z(Z)^{\mathrm{uno}}_\tau$) is the set of unobstructed elements of $\mathrm{Tan}_z(Z)$ (resp. $\mathrm{Tan}_z(Z)_\tau$). Furthermore, for each $\tau = \tau_{\mathfrak{p},i}$, $\theta = \theta_{\mathfrak{p},i,j}$, we have*

$$\dim_{\mathbb{F}} \mathrm{gr}^{\,j-1}(\mathrm{Tan}_z(Z)^{\mathrm{uno}}_\tau) = \begin{cases} 1, & \text{if } \sigma\theta \in J''; \\ 0, & \text{otherwise,} \end{cases}$$

with respect to the filtration induced by the one on $\mathrm{Tan}_z(Z)_\tau$, so $\dim_{\mathbb{F}}(\mathrm{Tan}_z(Z)^{\mathrm{uno}}) = |J'| = |J''|$.

Proof. The assertion concerning the decomposition follows from the vanishing of the restriction of ∂_τ to $\mathrm{Tan}_z(Z)_{\tau'}$ for $\tau \in \Delta$ and $\tau' \neq \tau$, where ∂_τ is defined in (4.8). The assertion concerning the filtration then follows from Lemma 4.5 and the injectivity of ∂_τ on $\mathrm{Fil}^{\,e_{\mathfrak{p}}-1}(\mathrm{Tan}_z(Z)_\tau)$. The desired vanishing and injectivity are both consequences of the claim that if $\tau \in \Delta$, $R_1 = \mathbb{F}[X]/(X^2)$ and $z_1 \in Z(R_1)$ is a lift of z such that $F'^{(e_{\mathfrak{p}}-1)}_{\tau,1}$ corresponds to $F'^{(e_{\mathfrak{p}}-1)}_{\tau,0} \otimes_{\mathbb{F}} R_1$ under the canonical isomorphism $H^1_{\mathrm{dR}}(A'_1/R_1)_\tau \cong H^1_{\mathrm{crys}}(A'_0/R_1)_\tau \cong H^1_{\mathrm{dR}}(A'_0/\mathbb{F}) \otimes_{\mathbb{F}} R_1$, then $F'^{(e_{\mathfrak{p}})}_{\tau,1}$ corresponds to $F'^{(e_{\mathfrak{p}})}_{\tau,0} \otimes_{\mathbb{F}} R_1$ if and only if $\partial_\tau(z_1) = 0$ (i.e., $L'_{\theta,1}$ corresponds to $L^{(p^{-1})}_{\sigma\theta,0} \otimes_{\mathbb{F}} R_1$ under $\varepsilon_\tau(z_1)$). Indeed, if $z_1 \in \mathrm{Tan}_z(Z)_{\tau'}$ for some $\tau' \neq \tau$, then $F'^{(j)}_{\tau,1}$ corresponds to $F'^{(j)}_{\tau,0} \otimes_{\mathbb{F}} R_1$ for $j = 0,\dots,e_{\mathfrak{p}}$, and if $z_1 \in \mathrm{Fil}^{\,e_{\mathfrak{p}}-1}\mathrm{Tan}_z(Z)_\tau$, then $F'^{(e_{\mathfrak{p}}-1)}_{\tau,1}$ corresponds to $F'^{(e_{\mathfrak{p}}-1)}_{\tau,0} \otimes_{\mathbb{F}} R_1$, and z_1 is trivial if and only if $F'^{(e_{\mathfrak{p}})}_{\tau,1}$ corresponds to $F'^{(e_{\mathfrak{p}})}_{\tau,0} \otimes_{\mathbb{F}} R_1$

The claim follows from the commutativity of the diagram:

$$\begin{tikzcd} P'_{\theta,0} \otimes_{\mathbb{F}} R_1 \arrow[rr, "\sim"] \arrow[dr, "\sim", "\varepsilon_\tau(z_0)\otimes R_1"'] & & P'_{\theta,1} \arrow[dl, "\sim"', "\varepsilon_\tau(z_1)"] \\ & P^{(p^{-1})}_{\sigma\theta,0} \otimes_{\mathbb{F}} R_1 & \end{tikzcd}$$

(where the top arrow is the canonical isomorphism), which in turn follows from that of

$$\begin{tikzcd} \widetilde{G}'^{(e_{\mathfrak{p}})}_{\tau,0} \otimes_{W_2} \widetilde{R}_1 \arrow[rr, "\sim"] \arrow[dr, "\widetilde{\psi}^*_{\tau,0}\otimes\widetilde{R}_1"'] & & \widetilde{G}'^{(e_{\mathfrak{p}})}_{\tau,1} \arrow[dl, "\widetilde{\psi}^*_{\tau,1}"] \\ & \widetilde{F}^{(e_{\mathfrak{p}})}_{\tau,0} \otimes_{W_2} \widetilde{R}_1, & \end{tikzcd}$$

where the top arrow is induced by the morphisms $(\mathbb{F}, W_2) \to (\mathbb{F}, \widetilde{R}_1) \leftarrow (R_1, \widetilde{R}_1)$ of divided power thickenings. This in turn is implied by the commutativity of

$$\begin{tikzcd} H^1_{\mathrm{crys}}(A'_0/W_2)_\tau \otimes_{W_2} \widetilde{R}_1 \arrow[r, "\sim"] \arrow[d, "\widetilde{\psi}^*_{\tau,0}\otimes\widetilde{R}_1"'] & H^1_{\mathrm{crys}}(A'_1/\widetilde{R}_1)_\tau \arrow[d, "\widetilde{\psi}^*_{\tau,1}"] \\ H^1_{\mathrm{crys}}(A_0/W_2)_\tau \otimes_{W_2} \widetilde{R}_1 \arrow[r, "\sim"] & H^1_{\mathrm{crys}}(A_1/\widetilde{R}_1)_\tau, \end{tikzcd}$$

which in turn results from $\widetilde{\psi}^*_{\tau,0}$ and $\widetilde{\psi}^*_{\tau,1}$ being evaluations of the morphism of crystals over R_1 induced by ψ_1. $\square$

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Suppose now that m is fixed (for example, as $|J'|$) and consider the thickening $R_{n+1} \to R_n$ (with trivial divided powers), and consider the natural map $Z(R_{n+1})^{\mathrm{uno}} \to Z(R_n)^{\mathrm{uno}}$. We will prove (Lemma 4.7 below) that this map is surjective, explaining why we call such deformations *unobstructed*. Before doing so, however, we illustrate the construction of the desired liftings with some examples.

Suppose first that $[F:\mathbb{Q}] = 2$. If p splits in F, then $J' = J'' = \emptyset$ and $\Delta = J$. Therefore, Lemma 4.6 implies that all unobstructed deformations are trivial (i.e., $Z(R_n)^{\mathrm{uno}} = Z(\mathbb{F})$), so there is nothing to prove. The same holds if p is inert in F, and either $J = \Sigma$ or $\emptyset$. More explicitly, it is well-known that in the preceding cases, Z is isomorphic to $(\mathrm{Spec}\,(\mathbb{F}[X]/(X^p)))^\delta$ (see, for example, [DKS23, (43)]), so $Z(R_n)^{\mathrm{uno}}$ consists of a single element.

Suppose, however, that p is inert in F and $J = \{\tau_0\}$, where $\mathfrak{P} = p\mathcal{O}_F$ and $\Sigma = \{\tau_0, \tau_1\}$, so $J' = \{\tau_0\}$, $J' = \{\tau_1\}$ and $\Delta = \emptyset$. In particular, the condition in the definition of unobstructed is vacuous, so $Z(R_n)^{\mathrm{uno}} = Z(R_n)$ for all $n \geq 1$. Therefore, the surjectivity of $Z(R_{n+1})^{\mathrm{uno}} \to Z(R_n)^{\mathrm{uno}}$ in this case amounts to the smoothness of Z, which follows for example from [DKS23, Lemma 7.1.4.1] (or alternatively, from Lemmas 4.2 and 4.5 above). To illustrate the proof of Lemma 4.7 more explicitly in this case, suppose that $z_n \in Z(R_n)$ and let $(\underline{A}_n, \underline{A}'_n, \psi_n)$ be the corresponding tuple. So in particular, $\underline{A}_n = \underline{A}_0 \otimes_{\mathbb{F}} R_n$ and $\psi_n : A_n \to A'_n$ is an isogeny such that the morphisms

$$P'_{0,n}/L'_{0,n} \longrightarrow P_{0,n}/L_{0,n} \quad \text{and} \quad L'_{1,n} \longrightarrow L_{1,n}$$

induced by ψ_n are trivial, where

$$P'_{i,n} = H^1_{\mathrm{dR}}(A'_n/R'_n)_{\tau_i} \quad \text{and} \quad P_{i,n} = H^1_{\mathrm{dR}}(A_n/R_n) = H^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\tau_i} \otimes_{\mathbb{F}} R_n$$

for $i = 0, 1$, and similarly, $L'_{i,n} = H^0(A'_n, \Omega^1_{A'_n/R_n})$ and $L_{i,n} = H^0(A_0, \Omega^1_{A_0/\mathbb{F}}) \otimes_{\mathbb{F}} R_n$. By the Grothendieck–Messing Theorem, lifts of $\underline{A}'_n$ to R_{n+1} correspond to lifts of the

$$L'_{i,n} \subset P'_{i,n} \cong H^1_{\mathrm{crys}}(A'_n/R_n)_{\tau_i} \cong H^1_{\mathrm{crys}}(A'_n/R_{n+1})_{\tau_i} \otimes_{R_{n+1}} R_n$$

to free rank one R_{n+1}-submodules $L'_{i,n+1}$ of $P'_{i,n+1} := H^1_{\mathrm{crys}}(A'_n/R_{n+1})_{\tau_i}$ for $i = 0, 1$. Furthermore, ψ_n lifts to an isogeny $\psi_{n+1} : A_{n+1} = A_0 \otimes R_{n+1} \to A'_{n+1}$ (necessarily unique and compatible with the auxiliary data) if and only if the morphisms

$$\psi^*_{i,n+1} : P'_{i,n+1} \longrightarrow P_{i,n+1} := H^1_{\mathrm{crys}}(A_n/R_{n+1})_{\tau_i} \cong P_{i,0} \otimes_{\mathbb{F}} R_{n+1}$$

induced by ψ_n send $L'_{i,n+1}$ to $L_{i,n+1} = L_{i,0} \otimes_{\mathbb{F}} R_{n+1}$ for $i = 0, 1$. Moreover, the resulting point of $Y_{U_0(\mathfrak{P})}(R_{n+1})$ lies in $S_J(R_{n+1})$, and hence in $Z(R_{n+1})$, if and only if

$$\psi^*_{0,n+1}(P'_{0,n+1}) \subset L_{0,n+1} \quad \text{and} \quad \psi^*_{1,n+1}(L'_{1,n+1}) = 0.$$

Choosing an arbitrary lift $L'_{0,n+1}$ of $L'_{0,n}$ and setting $L'_{1,n+1} = \ker(\psi^*_{1,n+1})$ ensures these conditions are satisfied (see the first part of the proof of Lemma 4.7 for the former). Note that these imply that $\psi^*_{i,n+1}(L'_{i,n+1}) \subset L_{i,n+1}$ for $i = 0, 1$, and hence supply the desired lift of z_n to an element $z_{n+1} \in Z(R_{n+1})$

Continue to assume that $[F:\mathbb{Q}] = 2$, but suppose now that p is ramified in F. Let $\mathfrak{P}$ be the unique prime over p, and write $\Sigma_0 = \{\tau\}$ and $\Sigma = \{\theta_1, \theta_2\}$. If $J = \Sigma$ or $\emptyset$, then we again have $J' = J'' = \emptyset$ (and $\Delta = \Sigma_0$ or $\emptyset$, according to whether $J = \Sigma$ or $\emptyset$). Again, it follows from Lemma 4.5 that all unobstructed lifts are trivial, so there is nothing to prove.

However, if $J = \{\theta_1\}$ or $\{\theta_2\}$, then $\Delta = \emptyset$, so again all lifts are unobstructed and the desired surjectivity amounts to the smoothness of Z, which can again be deduced from Lemmas 4.2 and 4.5. To see the surjectivity more explicitly, we can proceed to construct a lift of $z_n \in Z(R_n)$ to $Z(R_{n+1})$ as in the inert case, the task at hand now being to lift the Pappas–Rapoport filtration:

$$0 \subset F'^{(1)}_n \subset F'^{(2)}_n = H^0(A'_n, \Omega^1_{A'_n/R_n}) \subset H^1_{\mathrm{dR}}(A'_n/R_n) \cong H^1_{\mathrm{crys}}(A'_n/R_n)$$

to one in $H^1_{\mathrm{crys}}(A'_n/R_{n+1})$ with the desired properties. In particular, if $J = \{\theta_1\}$, then we require both $G'^{(1)}_{n+1} = uH^1_{\mathrm{crys}}(A'_n/R_{n+1})$ and $F'^{(2)}_{n+1}$ be sent to $F^{(1)}_0 \otimes_{\mathbb{F}} R_{n+1}$ under the induced map

$$\psi^*_{n+1} : H^1_{\mathrm{crys}}(A'_n/R_{n+1}) \to H^1_{\mathrm{crys}}(A_n/R_{n+1}) \cong H^1_{\mathrm{dR}}(A_0/\mathbb{F}) \otimes_{\mathbb{F}} R_{n+1}.$$

Note that in this case, $J'' = \{\theta_2\}$, so $F'^{(2)}_n = uH^1_{\mathrm{dR}}(A'_n/R_n)$; furthermore, the first part of the proof of Lemma 4.7 shows that the image of ψ^*_{n+1} is $u^{-1}F^{(1)}_0 \otimes_{\mathbb{F}} R_{n+1}$. So we obtain a lift with the desired properties by setting $F'^{(2)}_{n+1} = uH^1_{\mathrm{dR}}(A'_n/R_{n+1})$, within which we choose an arbitrary lift $F'^{(1)}_{n+1}$ of $F'^{(1)}_n$. However, if $J = \{\theta_2\}$, then we require ψ^*_{n+1} to send $F'^{(1)}_{n+1}$ to 0 and $G'^{(2)}_{n+1} = u^{-1}F'^{(1)}_{n+1}$ to $F^{(2)}_0 \otimes_{\mathbb{F}} R_{n+1}$. Note that in this case, $J' = \{\theta_2\}$, so $F^{(2)}_0 = uH^1_{\mathrm{dR}}(A_0/\mathbb{F})$, and we obtain a lift with the desired properties by setting $F'^{(1)}_{n+1} = \ker(\psi^*_{n+1})$ and defining $F'^{(2)}_{n+1}$ by arbitrarily lifting $L'_n := F'^{(2)}_n/F'^{(1)}_n \subset P'_n = G'^{(2)}_n/F'^{(1)}_n$ to a line L'_{n+1} in $P'_{n+1} = G'^{(2)}_{n+1}/F'^{(1)}_{n+1}$, where $G'^{(2)}_\ell = u^{-1}F'^{(1)}_\ell$ for $\ell = n, n+1$.

The notion of unobstructedness can only play a more serious role if $[F : \mathbb{Q}] \geq 3$. Suppose, for example, that $[F : \mathbb{Q}] = 4$ and $S_p = \{\mathfrak{p}\}$ with $e_{\mathfrak{p}} = f_{\mathfrak{p}} = 2$. Write $\Sigma_0 = \{\tau_0, \tau_1\}$ and $\Sigma = \{\theta_{0,1}, \theta_{0,2}, \theta_{1,1}, \theta_{1,2}\}$, and consider the case $J = \{\theta_{0,1}, \theta_{1,1}, \theta_{1,2}\}$, so that

$$J' = \{\theta_{1,1}\}, \quad J'' = \{\theta_{0,2}\} \quad \text{and} \quad \Delta = \{\tau_1\}.$$

Let $(\underline{A}_n, \underline{A}'_n, \psi_n)$ correspond to an element $z_n \in Z(R_n)^{\mathrm{uno}}$, and consider the associated Pappas–Rapoport filtrations

$$0 \subset F'^{(1)}_{i,n} \subset F'^{(2)}_{i,n} = H^0(A'_n, \Omega^1_{A'_n/R_n})_{\tau_i} \subset H^1_{\mathrm{dR}}(A'_n/R_n)_{\tau_i} \cong H^1_{\mathrm{crys}}(A'_n/R_n)_{\tau_i}$$

for $i = 0, 1$. The task now is to lift these to filtrations in $H^1_{\mathrm{crys}}(A'_n/R_{n+1})_{\tau_i}$ with the desired properties relative to the $R_{n+1}[u]/(u^2)$-linear maps

$$\psi^*_{i,n+1} : H^1_{\mathrm{crys}}(A'_n/R_{n+1})_{\tau_i} \longrightarrow H^1_{\mathrm{crys}}(A_n/R_{n+1})_{\tau_i} \cong H^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\tau_i} \otimes_{\mathbb{F}} R_{n+1}.$$

More precisely, lifting z_n to $Z(R_{n+1})$ is equivalent (by the Grothendieck–Messing Theorem) to defining chains of free R_{n+1}-submodules

$$0 = F'^{(0)}_{i,n+1} \subset F'^{(1)}_{i,n+1} \subset F'^{(2)}_{i,n+1}$$

of $H^1_{\mathrm{crys}}(A'_n/R_{n+1})_{\tau_i}$ for $i = 0, 1$ such that

- $F'^{(j)}_{i,n+1} \otimes_{R_{n+1}} R_n = F'^{(j)}_{i,n}$ for $i = 0, 1$, $j = 1, 2$;
- $uF'^{(1)}_{i,n+1} = 0$ and $uF'^{(2)}_{i,n+1} \subset F'^{(1)}_{i,n+1}$ for $i = 0, 1$;
- $\psi^*_{n+1}(G'^{(j)}_{i,n+1}) \subset F^{(j)}_{i,0} \otimes_{\mathbb{F}} R_{n+1}$ if $\theta_{i,j} \in J$, where $G'^{(j)}_{i,n+1} := u^{-1}F'^{(j-1)}_{i,n+1}$;
- $\psi^*_{n+1}(F'^{(2)}_{0,n+1}) \subset F^{(1)}_{0,0} \otimes_{\mathbb{F}} R_{n+1}$;

and we must furthermore ensure that the resulting lift $z_{n+1} \in Z(R_{n+1})$ is unobstructed.

Note that since $G'^{(1)}_{i,n+1} = uH^1_{\mathrm{crys}}(A'_n/R_{n+1})$, the cases with $j = 1$ in the third bullet amount to requiring the image of $\psi^*_{i,n+1}$ to be $u^{-1}F^{(1)}_{i,0} \otimes_{\mathbb{F}} R_{n+1}$. This is again a special case of the assertion established in the first part of the proof of Lemma 4.7; we remark that for $i = 0$ (so $\sigma^{-1}\theta, \theta \in J$ and $\phi^{-1} \circ \tau \in \Delta$, where $\theta = \theta_{0,1}$ and $\tau = \tau_0$), the argument makes crucial use of the assumption that z_n is unobstructed. Furthermore, since $\theta_{0,2} \in J''$, we have $F'^{(2)}_{0,n} = uH^1_{\mathrm{dR}}(A'_n/R_n)_{\tau_0}$. It follows that all the conditions with $i = 0$ are satisfied if we set $F'^{(2)}_{0,n+1} = uH^1_{\mathrm{crys}}(A'_n/R_{n+1})_{\tau_0}$ and choose within it an arbitrary lift $F'^{(1)}_{0,n+1}$ of $F'^{(1)}_{0,n}$.

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The third bullet for $i = 1$, $j = 2$ translates into the requirement that

$$G'^{(2)}_{1,n+1} \subset (\psi^*_{1,n+1})^{-1}(F^{(2)}_{1,0} \otimes_{\mathbb{F}} R_{n+1}),$$

which forces equality since each is free of rank 3 over R_{n+1}. Note also that

$$\psi^*_{1,n+1}(uH^1_{\mathrm{crys}}(A'_n/R_{n+1})) = F^{(1)}_{1,0} \otimes_{\mathbb{F}} R_{n+1} \subset F^{(2)}_{1,0} \otimes_{\mathbb{F}} R_{n+1},$$

so we obtain the desired lift of $F'^{(1)}_{1,n}$ by setting $F'^{(1)}_{1,n+1} = u(\psi^*_{1,n+1})^{-1}(F^{(2)}_{1,0} \otimes_{\mathbb{F}} R_{n+1})$. Finally, we could choose $F'^{(2)}_{1,n+1}$ so that $F'^{(2)}_{1,n+1}/F'^{(1)}_{1,n+1}$ is an arbitrary lift of the line $F'^{(2)}_{1,n}/F'^{(1)}_{1,n} \subset G'^{(2)}_{1,n}/F'^{(1)}_{1,n}$ to one in $G'^{(2)}_{1,n+1}/F'^{(1)}_{1,n+1}$, thus defining a lift of z_n to an [element $z_{n+1} \in Z(R_{n+1})$. However, the requirement that it be unobstructed determines a unique such line, specifically via the isomorphism (4.11) constructed in the proof of Lemma 4.7.

We now return to the general setting.

Lemma 4.7. *For all $n \geq 0$, the map $Z(R_{n+1})^{\mathrm{uno}} \to Z(R_n)^{\mathrm{uno}}$ is surjective.*

Proof. Let $(\underline{A}_n, \underline{A}'_n, \psi_n)$ correspond to an element $z_n \in Z(R_n)^{\mathrm{uno}}$. For $\ell = n, n+1$, we let

$$\psi^*_{\tau,\ell} : H^1_{\mathrm{crys}}(A'_n/R_\ell)_\tau \longrightarrow H^1_{\mathrm{crys}}(A_n/R_\ell)_\tau \cong H^1_{\mathrm{crys}}(A_0/\mathbb{F})_\tau \otimes_{\mathbb{F}} R_\ell$$

denote the morphism induced by ψ, so that $\psi^*_{\tau,n+1} \otimes_{R_{n+1}} R_n$ is identified with

$$\psi^*_{\tau,n} : H^1_{\mathrm{dR}}(A'_n/R_n)_\tau \to H^1_{\mathrm{dR}}(A_n/R_n)_\tau$$

under the canonical isomorphisms

$$H^1_{\mathrm{crys}}(B/R_{n+1}) \otimes_{R_{n+1}} R_n \cong H^1_{\mathrm{crys}}(B/R_n) \cong H^1_{\mathrm{dR}}(B/R_n)$$

for $B = A_n, A'_n$. Similarly, let $\widetilde{\psi}^*_{\tau,\ell}$ denote the morphism

$$H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_\ell)_\tau \longrightarrow H^1_{\mathrm{crys}}(A_n/\widetilde{R}_\ell)_\tau \cong H^1_{\mathrm{crys}}(A_0/W_2)_\tau \otimes_{W_2} \widetilde{R}_\ell$$

for $\ell = n, n+1$, so that $\widetilde{\psi}_{\tau,n+1} \otimes_{W_2} \mathbb{F} = \psi^*_{\tau,n+1}$ and $\widetilde{\psi}^*_{\tau,n+1} \otimes_{\widetilde{R}_{n+1}} \widetilde{R}_n = \widetilde{\psi}^*_{\tau,n}$.

Recall that to give an element of $Z(R_{n+1})^{\mathrm{uno}}$ lifting z_n amounts to defining suitable chains of $R_{n+1}[u]/(u^{e_\mathfrak{p}})$-submodules

$$0 = F'^{(0)}_{\tau,n+1} \subset F'^{(1)}_{\tau,n+1} \subset \cdots F'^{(e_\mathfrak{p})}_{\tau,n+1} \subset H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau \tag{4.10}$$

lifting the Pappas–Rapoport filtrations on $F'^{(e_\mathfrak{p})}_{\tau,n}$ for all $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$. Before carrying this out, we prove that if τ is such that $\theta = \theta_{\mathfrak{p},i,1} \in J$, then the image of $\psi^*_{\tau,n+1}$ is $u^{1-e_\mathfrak{p}}F^{(1)}_{\tau,0} \otimes_{\mathbb{F}} R_{n+1}$.

Suppose first that $\sigma^{-1}\theta = \theta_{\mathfrak{p},i-1,e_\mathfrak{p}} \notin J$, so that $\theta \in J'$ and $(F^{(e_\mathfrak{p}-1)}_{\phi^{-1}\circ\tau,0})^{(p)}$ has preimage $u^{1-e_\mathfrak{p}}F^{(1)}_{\tau,0}$ under the morphism

$$H^1_{\mathrm{dR}}(A_0/\mathbb{F})_\tau \to H^1_{\mathrm{dR}}(A_0^{(p)}/\mathbb{F})_\tau \cong H^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\phi^{-1}\circ\tau} \otimes_{\mathbb{F},\phi} \mathbb{F}$$

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induced by $\mathrm{Ver} : A_0^{(p)} \to A_0$. Consider the commutative diagram

$$\begin{array}{ccccc}
H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau & \longrightarrow & H^1_{\mathrm{crys}}(A'^{(p)}_n/R_{n+1})_\tau & \cong & H^1_{\mathrm{dR}}(A'_n/R_n)_{\phi^{-1}\circ\tau} \otimes_{R_n,\phi} R_{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau & \longrightarrow & H^1_{\mathrm{crys}}(A^{(p)}_n/R_{n+1})_\tau & \cong & H^1_{\mathrm{dR}}(A_n/R_n)_{\phi^{-1}\circ\tau} \otimes_{R_n,\phi} R_{n+1} \\
\| \wr & & \| \wr & & \| \wr \\
H^1_{\mathrm{dR}}(A_0/\mathbb{F})_\tau \otimes_{\mathbb{F}} R_{n+1} & \longrightarrow & H^1_{\mathrm{dR}}(A^{(p)}_0/\mathbb{F})_\tau \otimes_{\mathbb{F}} R_{n+1} & \cong & H^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\phi^{-1}\circ\tau} \otimes_{\mathbb{F},\phi} R_{n+1},
\end{array}$$

where the first horizontal maps are induced by the Verschiebung morphisms, the horizontal isomorphisms by base-change relative to absolute Frobenius morphisms (and crystalline-de Rham isomorphisms), top vertical maps by ψ_n (and $\psi_n^{(p)}$) and vertical isomorphisms by base-change (and crystalline-de Rham isomorphisms again). The image across the top of the diagram is $F'^{(e_\mathfrak{p})}_{\phi^{-1}\circ\tau,n} \otimes_{R_n,\phi} R_{n+1}$, and since $z_n \in S_J(R_n)$ and $\sigma^{-1}\theta \notin J$, this maps to $F^{(e_\mathfrak{p}-1)}_{\phi^{-1}\circ\tau,0} \otimes_{\mathbb{F},\phi} R_{n+1}$ along the right side of the diagram. Therefore, the image along the left side of the diagram must be contained in the preimage of this, which is $u^{1-e_\mathfrak{p}} F^{(1)}_{\tau,0} \otimes_{\mathbb{F}} R_{n+1}$, and equality follows on comparing ranks.

Suppose, however, that $\sigma^{-1}\theta \in J$, so that $\phi^{-1} \circ \tau \in \Delta$. In this case, we consider the commutative diagram:

$$\begin{tikzcd}
 & H^1_{\mathrm{crys}}(A'^{(p)}_n/\widetilde{R}_{n+1})_\tau \arrow[dd] & \\
H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_{n+1})_\tau \arrow[ur] \arrow[dd] & & H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_n)_{\phi^{-1}\circ\tau} \otimes_{\widetilde{R}_n,\widetilde{\phi}} \widetilde{R}_{n+1} \arrow[ul, "\sim"'] \arrow[dd] \\
 & H^1_{\mathrm{crys}}(A^{(p)}_n/\widetilde{R}_{n+1})_\tau & \\
H^1_{\mathrm{crys}}(A_n/\widetilde{R}_{n+1})_\tau \arrow[ur] & & H^1_{\mathrm{crys}}(A_n/\widetilde{R}_n)_{\phi^{-1}\circ\tau} \otimes_{\widetilde{R}_n,\widetilde{\phi}} \widetilde{R}_{n+1} \arrow[ul, "\sim"'] \\
 & H^1_{\mathrm{crys}}(A^{(p^{-1})}_n/\widetilde{R}_n)_{\phi^{-1}\circ\tau} \otimes_{\widetilde{R}_n,\widetilde{\phi}} \widetilde{R}_{n+1} \arrow[ul, "\sim"] \arrow[ur] & \\
H^1_{\mathrm{crys}}(A_0/W_2)_\tau \otimes_{W_2} \widetilde{R}_{n+1} \arrow[uu, "\wr"] & & H^1_{\mathrm{crys}}(A_0/W_2)_{\phi^{-1}\circ\tau} \otimes_{W_2,\widetilde{\phi}} \widetilde{R}_{n+1} \arrow[uu, "\wr"'] \\
 & H^1_{\mathrm{crys}}(A^{(p^{-1})}_0/W_2)_{\phi^{-1}\circ\tau} \otimes_{W_2,\widetilde{\phi}} \widetilde{R}_{n+1} \arrow[ul, "\sim"] \arrow[ur] \arrow[uu, "\wr"] &
\end{tikzcd}$$

where the downwards arrows are induced by ψ_n (and its base-changes), the northeastward arrows by Verschiebung morphisms, all the isomorphisms are crystalline transition (and base-change) maps, and $\widetilde{\phi}$ denotes both the restriction of the map $\widetilde{\phi}_{n+1} : \widetilde{R}_{n+1} \to \widetilde{R}_{n+1}$ to W_2 and its factorization through $\widetilde{R}_n \to \widetilde{R}_{n+1}$.

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We have that the image of $H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_{n+1})_\tau$ in $H^1_{\mathrm{crys}}(A'^{(p)}_n/\widetilde{R}_{n+1})$ under the morphism induced by $\mathrm{Ver}: A'^{(p)}_n \longrightarrow A'_n$ corresponds to

$$\widetilde{F}'^{(e_{\mathfrak{p}})}_{\phi^{-1}\circ\tau,n} \otimes_{\widetilde{R}_n,\widetilde{\phi}} \widetilde{R}_{n+1} \subset H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_n)_{\phi^{-1}\circ\tau} \otimes_{\widetilde{R}_n,\widetilde{\phi}} \widetilde{R}_{n+1}$$

(this being an inclusion since $\widetilde{R}_{n+1}$ is flat over W_2 and $H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_n)_{\phi^{-1}\circ\tau}/\widetilde{F}'^{(e_{\mathfrak{p}})}_{\phi^{-1}\circ\tau,n}$ is free over $\widetilde{R}_n/p\widetilde{R}_n = R_n$). Since z_n is unobstructed, this submodule has the same the image along the right-hand side of the diagram as that of

$$u^{1-e_{\mathfrak{p}}}(\widetilde{F}^{(1)}_{\tau,0})^{(p^{-1})} \otimes_{W_2,\widetilde{\phi}} \widetilde{R}_{n+1}$$

under the morphism induced by $\mathrm{Ver}: A_0 \longrightarrow A_0^{(p^{-1})}$. It follows that the image of $H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_{n+1})_\tau$ along the sides of the top left parallelogram is the same as that of $u^{1-e_{\mathfrak{p}}}\widetilde{F}^{(1)}_{\tau,0} \otimes_{W_2} \widetilde{R}_{n+1} \cong u^{1-e_{\mathfrak{p}}}\widetilde{F}^{(1)}_{\tau,n+1}$ under the morphism

$$H^1_{\mathrm{crys}}(A_0/W_2)_\tau \otimes_{W_2} \widetilde{R}_{n+1} \cong H^1_{\mathrm{crys}}(A_n/\widetilde{R}_{n+1})_\tau \longrightarrow H^1_{\mathrm{crys}}(A_n^{(p)}/\widetilde{R}_{n+1})_\tau$$

induced by Verschiebung. Furthermore, the kernel of this morphism is contained in $pH^1_{\mathrm{crys}}(A_n/\widetilde{R}_{n+1})_\tau \subset u^{1-e_{\mathfrak{p}}}\widetilde{F}^{(1)}_{\tau,n+1}$, so we conclude that the image of $\widetilde{\psi}^*_{\tau,n+1}$ is contained in $u^{1-e_{\mathfrak{p}}}\widetilde{F}^{(1)}_{\tau,n+1}$. Therefore, the image of $\psi^*_{\tau,n+1}$ is contained in $u^{1-e_{\mathfrak{p}}}F^{(1)}_{\tau,n+1}$, and equality follows on comparing ranks.

We now proceed to define the lifts of the Pappas-Rapoport filtration as in (4.10). If $\tau \notin \Sigma_{\mathfrak{P},0}$, then $\psi^*_{\tau,n+1}$ is an isomorphism, under which we require that $F'^{(j)}_{\tau,n+1}$ correspond to $F^{(j)}_{\tau,n+1} = F^{(j)}_{\tau,0} \otimes_{\mathbb{F}} R_{n+1}$ for all j. If $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$, then the definition of $F'^{(j)}_{\tau,n+1}$ will depend on whether $\theta = \theta_{\mathfrak{p},i,j}$ and $\sigma\theta$ are in J (and whether $j = e_{\mathfrak{p}}$):

- Let $F'^{(0)}_{\tau,n+1} = 0$ and $G'^{(1)}_{\tau,n+1} = H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau[u] = u^{e_{\mathfrak{p}}-1}H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau$.
- If $\theta \notin J$, then we let $F'^{(j)}_{\tau,n+1}$ denote the preimage of $F^{(j-1)}_{\tau,n+1}$ under $\psi^*_{\tau,n+1}$, and if $j < e_{\mathfrak{p}}$, then let $G'^{(j+1)}_{\tau,n+1} = u^{-1}F'^{(j)}_{\tau,n+1}$. Note that since

  $$F^{(j-1)}_{\tau,n+1} \subset u^{e_{\mathfrak{p}}-j+1}H^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau \subset uH^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau \subset \mathrm{im}(\psi^*_{\tau,n+1})$$

  and $\ker(\psi^*_{\tau,n+1})$ is free of rank one over R_{n+1}, it follows that $F'^{(j)}_{\tau,n+1}$ is free of rank j over R_{n+1}. In particular, $F'^{(j)}_{\tau,n+1}$ is annihilated by u^j, so if $j < e_{\mathfrak{p}}$, then $G'^{(j+1)}_{\tau,n+1}$ is free of rank $j+2$ over R_{n+1}. Furthermore, $F'^{(j)}_{\tau,n+1} \otimes_{R_{n+1}} R_n$ is identified with the preimage of $F^{(j-1)}_{\tau,n}$ under $\psi^*_{\tau,n}$, which is $F'^{(j)}_{\tau,n}$ since $\theta \notin J$, and hence also $G'^{(j+1)}_{\tau,n+1} \otimes_{R_{n+1}} R_n = G'^{(j+1)}_{\tau,n}$ if $j < e_{\mathfrak{p}}$.
- If $\theta \in J$, $\sigma\theta \in J$ and $j < e_{\mathfrak{p}}$, then let $G'^{(j+1)}_{\tau,n+1}$ denote the preimage of $F^{(j+1)}_{\tau,n+1}$ under $\psi^*_{\tau,n+1}$ and let $F'^{(j)}_{\tau,n+1} = uG'^{(j+1)}_{\tau,n+1}$. Note that if $\theta_{\mathfrak{p},i,1} \notin J$, then $\theta_{\mathfrak{p},i,\ell} \in J'$ for some ℓ such that $2 \leq \ell \leq j$, so $uF^{(\ell)}_{\tau,n+1} = F^{(\ell-2)}_{\tau,n+1}$, which implies that

  $$u^{e_{\mathfrak{p}}-1}H^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau \subset F^{(j+1)}_{\tau,n+1} \subset uH^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau \subset \mathrm{im}(\psi^*_{\tau,n+1}).$$

  However, if $\theta_{\mathfrak{p},i,1} \in J$, then we have shown that $\mathrm{im}(\psi^*_{\tau,n+1}) = u^{1-e_{\mathfrak{p}}}F^{(1)}_{\tau,n+1}$, which again contains $F^{(j+1)}_{\tau,n+1}$, and

  $$\psi^*_{\tau,n+1}(u^{e_{\mathfrak{p}}-1}H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau) \subset F^{(1)}_{\tau,n+1} \subset F^{(j+1)}_{\tau,n+1}.$$

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Therefore, in either case, $G'^{(j+1)}_{\tau,n+1}$ is free over R_{n+1} of rank $j+2$ and contains $u^{e_\mathfrak{p}-1}H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau$, so that $F'^{(j)}_{\tau,n+1}$ is free of rank j over R_{n+1}. Furthermore, $G'^{(j+1)}_{\tau,n+1}\otimes_{R_{n+1}}R_n$ is identified with the preimage of $F^{(j+1)}_{\tau,n}$ under $\psi^*_{\tau,n}$, which is $G'^{(j+1)}_{\tau,n}$ since $\sigma\theta\in J$, and hence also $F'^{(j)}_{\tau,n+1}\otimes_{R_{n+1}}R_n=F'^{(j)}_{\tau,n}$.

- If $\theta\in J$ and $\sigma\theta\notin J$, then we have defined $F'^{(j-1)}_{\tau,n+1}$ and $G'^{(j)}_{\tau,n+1}$ above, with the property that $P'_{\theta,n+1}:=G'^{(j)}_{\tau,n+1}/F'^{(j-1)}_{\tau,n+1}$ is free of rank two over R_{n+1} and annihilated by u. Furthermore, $P'_{\theta,n+1}\otimes_{R_{n+1}}R_n$ is identified with $P'_{\theta,n}=G'^{(j)}_{\tau,n}/F'^{(j-1)}_{\tau,n}$, and we define $F'^{(j)}_{\tau,n+1}$ so that

$$F'^{(j-1)}_{\tau,n+1}\subset F'^{(j)}_{\tau,n+1}\subset G'^{(j)}_{\tau,n+1}$$

 and $L'_{\theta,n+1}:=F'^{(j)}_{\tau,n+1}/F'^{(j-1)}_{\tau,n+1}$ is an arbitrary lift of $L'_{\theta,n}=F'^{(j)}_{\tau,n}/F'^{(j-1)}_{\tau,n}$ to a free rank one submodule of $P'_{\theta,n+1}$.

- Finally, if $\theta=\theta_{\mathfrak{p},i,e_\mathfrak{p}}\in J$ and $\sigma\theta\in J$ (so $\tau\in\Delta$), then we have already defined $F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}\subset G'^{(e_\mathfrak{p})}_{\tau,n+1}$ lifting $F'^{(e_\mathfrak{p}-1)}_{\tau,n}\subset G'^{(e_\mathfrak{p})}_{\tau,n}$. Furthermore, we claim that $\psi^*_{\tau,n+1}(G'^{(e_\mathfrak{p})}_{\tau,n+1})=F^{(e_\mathfrak{p})}_{\tau,n+1}$. Note that it suffices to prove $\psi^*_{\tau,n+1}(G'^{(e_\mathfrak{p})}_{\tau,n+1})\subset F^{(e_\mathfrak{p})}_{\tau,n+1}$ since $G'^{(e_\mathfrak{p})}$ is free of rank $e_\mathfrak{p}+1$ over R_{n+1} and contains $\ker(\psi^*_{\tau,n+1})$, which is free of rank one. If $e_\mathfrak{p}=1$, then the desired equality is a special case of the fact that if $\theta_{\mathfrak{p},i,1}\in J$, then $\psi^*_{\tau,n+1}(H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau)=u^{1-e_\mathfrak{p}}F^{(1)}_{\tau,n+1}$, so assume $e_\mathfrak{p}>1$. If $\theta_{\mathfrak{p},i,e_\mathfrak{p}-1}\notin J$, then $\theta\in J'$, so $uF^{(e_\mathfrak{p})}_{\tau,n+1}=F^{(e_\mathfrak{p}-2)}_{\tau,n+1}$, whose preimage under $\psi^*_{\tau,n+1}$ is defined to be $F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}$, so $\psi^*_{\tau,n+1}$ maps $G'^{(e_\mathfrak{p})}_{\tau,n+1}=u^{-1}F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}$ to $u^{-1}F^{(e_\mathfrak{p}-2)}_{\tau,n+1}=F^{(e_\mathfrak{p})}_{\tau,n+1}$. Finally, if $\theta_{\mathfrak{p},i,e_\mathfrak{p}-1}\in J$, then we defined $G'^{(e_\mathfrak{p})}_{\tau,n+1}$ as the preimage of $F^{(e_\mathfrak{p})}_{\tau,n+1}$. This completes the proof of the claim, which implies that the restriction of $\widetilde{\psi}^*_{\tau,n+1}$ defines a surjection $\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n+1}\to\widetilde{F}^{(e_\mathfrak{p})}_{\tau,n+1}=\widetilde{F}^{(e_\mathfrak{p})}_{\tau,0}\otimes_{W_2}\widetilde{R}_{n+1}$, where as usual $\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,n+1}$ denotes the preimage of $G'^{(e_\mathfrak{p})}_{\tau,n+1}$ in $H^1_{\mathrm{crys}}(A'_n/\widetilde{R}_{n+1})_\tau$. We thus obtain an isomorphism

$$P'_{\theta,n+1}:=G'^{(e_\mathfrak{p})}_{\tau,n+1}/F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}\longrightarrow P^{(p^{-1})}_{\sigma\theta,0}\otimes_{\mathbb{F}}R_{n+1}\tag{4.11}$$

 exactly as in the construction of $\varepsilon_\tau(z_n)$, and we define $F'^{(e_\mathfrak{p})}_{\tau,n+1}$ so that

$$F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}\subset F'^{(e_\mathfrak{p})}_{\tau,n+1}\subset G'^{(e_\mathfrak{p})}_{\tau,n+1}$$

 and $F'^{(e_\mathfrak{p})}_{\tau,n+1}/F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}$ corresponds under (4.11) to $L^{(p^{-1})}_{\sigma\theta,0}\otimes_{\mathbb{F}}R_{n+1}$. Note that $F'^{(e_\mathfrak{p})}_{\tau,n+1}$ is free of rank $e_\mathfrak{p}$ over R_{n+1}, and lifts $F'^{(e_\mathfrak{p})}_{\tau,n}$ since (4.11) lifts $\varepsilon_\tau(z_n)$.

We have now defined an $R_{n+1}[u]/u^{e_\mathfrak{p}}$-submodule $F'^{(j)}_{\tau,n+1}$ of $H^1_{\mathrm{crys}}(A'_n/R_{n+1})_\tau$ lifting $F'^{(j)}_{\tau,n}\subset H^1_{\mathrm{dR}}(A'_n/R_n)_\tau\cong H^1_{\mathrm{crys}}(A'_n/R_n)_\tau$ for all $\tau=\tau_{\mathfrak{p},i}$, $j=1,\dots,e_\mathfrak{p}$. Furthermore, each $F'^{(j)}_{\tau,n+1}$ is free of rank j over R_{n+1}, and corresponds to $F^{(j)}_{\tau,n+1}$ under $\psi^*_{\tau,n+1}$ if $\tau\notin\Sigma_{\mathfrak{P},0}$, so to complete the proof of the lemma, it suffices to show that the following hold for all $\tau=\tau_{\mathfrak{p},i}\in\Sigma_{\mathfrak{P},0}$:

- The inclusions (4.10) hold with successive quotients annihilated by u, and hence (by the Grothendieck–Messing Theorem, as explained at the end of §2.4) determine data $\underline{A}'_{n+1}$ corresponding to a point of $\widetilde{Y}(R_{n+1})$ lifting $\underline{A}'_n$ so that (4.10) corresponds to the Pappas–Rapoport filtration under the canonical isomorphism $H^1_{\mathrm{crys}}(A'_{n+1}/R_{n+1})\cong H^1_{\mathrm{dR}}(A'_{n+1}/R_{n+1})$.
- If $1\le j\le e_\mathfrak{p}$ and $\theta_{\mathfrak{p},i,j}\notin J$ (resp. $\theta_{\mathfrak{p},i,j}\in J$), then $\psi^*_{\tau,n+1}$ sends $F'^{(j)}_{\tau,n+1}$ to $F^{(j-1)}_{\tau,n+1}$ (resp. $G'^{(j)}_{\tau,n+1}=u^{-1}F'^{(j-1)}_{\tau,n+1}$ to $F^{(j)}_{\tau,n+1}$), so that ψ_n extends (again by the Grothendieck–Messing Theorem) to an isogeny ψ_{n+1} and $(\underline{A}_{n+1},\underline{A}'_{n+1},\psi_{n+1})$ corresponds to a point $z_{n+1}\in Z(R_{n+1})$ lifting z_n (where $\underline{A}_{n+1}=\underline{A}_0\otimes_{\mathbb{F}}R_{n+1}$).

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- If $\tau \in \Delta$ (i.e., $\theta_{\mathfrak{p},i,e_\mathfrak{p}}$ and $\theta_{\mathfrak{p},i+1,1}$ are both in J), then $F'^{(e_\mathfrak{p})}_{\tau,n+1}/F'^{(e_\mathfrak{p}-1)}_{\tau,n+1}$ corresponds to $(F^{(1)}_{\phi\circ\tau,0})^{(p^{-1})} \otimes_\mathbb{F} R_{n+1}$ under $\varepsilon_\tau(z_{n+1})$, so that $z_{n+1} \in Z^{\mathrm{uno}}(R_{n+1})$.

We start with the second assertion, which is immediate from the definition of $F'^{(j)}_{\tau,n+1}$ if $\theta_{\mathfrak{p},i,j} \notin J$, so suppose that $\theta = \theta_{\mathfrak{p},i,j} \in J$. Recall we proved that if $\theta_{\mathfrak{p},i,1} \in J$, then $\psi^*_{\tau,n+1}(H^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau) = u^{1-e_\mathfrak{p}}F^{(1)}_{\tau,n+1}$, so it follows that

$$\psi^*_{\tau,n+1}(G'^{(1)}_{\tau,n+1}) = \psi^*_{\tau,n+1}(u^{e_\mathfrak{p}-1}H^1_{\mathrm{crys}}(A_n/R_{n+1})_\tau) \subset F^{(1)}_{\tau,n+1}.$$

Therefore, the assertion holds for $j = 1$. Note also that if $j > 1$ and $\sigma^{-1}\theta \in J$, then the assertion is immediate from the definition of $G'^{(j)}_{\tau,n+1}$. However, if $j > 1$ and $\sigma^{-1}\theta \notin J$, then $\theta \in J'$, so $F^{(j)}_{\tau,n+1} = u^{-1}F^{(j-2)}_{\tau,n+1}$, and therefore,

$$\begin{aligned} G'^{(j)}_{\tau,n+1} = u^{-1}F'^{(j-1)}_{\tau,n+1} &= u^{-1}(\psi^*_{\tau,n+1})^{-1}(F^{(j-2)}_{\tau,n+1}) \\ &= (\psi^*_{\tau,n+1})^{-1}(u^{-1}F^{(j-2)}_{\tau,n+1}) = (\psi^*_{\tau,n+1})^{-1}(F^{(j)}_{\tau,n+1}). \end{aligned}$$

For the first assertion, we need to show that if $1 \le j \le e_\mathfrak{p}$, then

$$uF'^{(j)}_{\tau,n+1} \subset F'^{(j-1)}_{\tau,n+1} \subset F'^{(j)}_{\tau,n+1}. \tag{4.12}$$

Suppose that $\theta = \theta_{\mathfrak{p},i,j} \notin J$, so $F'^{(j)}_{\tau,n+1} = (\psi^*_{\tau,n+1})^{-1}(F^{(j-1)}_{\tau,n+1})$. In particular, if $j = 1$, then $F'^{(1)}_{\tau,n+1} = \ker(\psi^*_{\tau,n+1}) \subset G'^{(1)}_{\tau,n+1}$, so we may assume $j \ge 2$. If $\sigma^{-1}\theta \notin J$, then (4.12) is immediate from the fact that $uF^{(j-1)}_{\tau,n+1} \subset F^{(j-2)}_{\tau,n+1} \subset F^{(j-1)}_{\tau,n+1}$, and if $\sigma^{-1}\theta \in J$, then we have shown that $u^{-1}F'^{(j-2)}_{\tau,n+1} = G'^{(j-1)}_{\tau,n+1} = F'^{(j)}_{\tau,n+1}$, so the inclusions in (4.12) are precisely the ones in the definition of $F'^{(j-1)}_{\tau,n+1}$. However, if $\theta = \theta_{\mathfrak{p},i,j} \in J$, then (4.12) is immediate from the definition of $F'^{(j)}_{\tau,n+1}$ if either $\sigma\theta \notin J$ or $j = e_\mathfrak{p}$, so suppose that $\theta, \sigma\theta \in J$ and $j < e_\mathfrak{p}$. Since $G'^{(\ell)}_{\tau,n+1} = (\psi^*_{\tau,n+1})^{-1}(F^{(\ell)}_{\tau,n+1})$ for $\ell = j, j+1$, we have

$$F'^{(j)}_{\tau,n+1} = uG'^{(j+1)}_{\tau,n+1} \subset G'^{(j)}_{\tau,n+1} \subset G'^{(j+1)}_{\tau,n+1},$$

which in turn implies (4.12).

Finally, the third assertion is immediate from the definition of $F'^{(e_\mathfrak{p})}_{\tau,n+1}$ and the fact that (4.11) is identified with $\varepsilon_\tau(z_{n+1})$ via the canonical isomorphism

$$H^1_{\mathrm{crys}}(A_n/R_{n+1}) \cong H^1_{\mathrm{crys}}(A_{n+1}/R_{n+1}) \cong H^1_{\mathrm{dR}}(A_{n+1}/R_{n+1}).$$

□

4.5. *Local structure*

We now use the results of the previous sections to prove a generalization of [DKS23, Lemma 7.1.6], describing the local structure of the fibres of the projection $\pi_J : S_J \to T_{J'}$ at geometric points. As before, we let $z \in S_J(\mathbb{F})$, $x = \pi_J(z) \in T_{J'}(\mathbb{F})$, and Z denote the (geometric) fibre of π_J at x.

Lemma 4.8. *There are isomorphisms*

$$\widehat{\mathcal{O}}_{Z,z} \cong \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/(T_1^p,\dots,T_\delta^p) \quad \textit{and} \quad \widehat{\mathcal{O}}_{Z_{\mathrm{red}},z} \cong \mathbb{F}[[X_1,\dots,X_m]],$$

where $m = |J'| = |J''|$ and $\delta = |\Delta|$. In particular, Z_{red} is smooth of dimension m over $\mathbb{F}$. Moreover, the closed immersion $Z_{\mathrm{red}} \hookrightarrow Z$ identifies $\mathrm{Tan}_z(Z_{\mathrm{red}})$ with $\mathrm{Tan}_z(Z)^{\mathrm{uno}}$.

Proof. First, consider the morphism $\chi_J = \widetilde{\zeta}_J^{(p^{-1})} : P_J^{(p^{-1})} \to S_J$, where ζ_J is the morphism of Lemma 4.2; thus χ_J is finite, flat, and bijective on closed points. Let W denote the fibre of $\pi_J \circ \chi_J$ at x, so that the fibre of χ_J at x is a finite, flat morphism $W \to Z$ which is bijective on closed points, and let $w = \widetilde{\xi}_J(z)^{(p^{-1})}$ denote the unique element of $W(\mathbb{F})$ mapping to z. Since the completion of W at w is the same as its completion at the fibre of χ_J at z, the resulting morphism $\widehat{\mathcal{O}}_{Z,z} \longrightarrow \widehat{\mathcal{O}}_{W,w}$ is finite, flat, and in particular injective. Furthermore, choosing parameters on $P_{J,\mathbb{F}}$ at $y := \widetilde{\xi}_J(z) = w^{(p)}$ so that the morphism on completions induced by the projection $P_J \to T_{J'}$ takes the form

$$\widehat{\mathcal{O}}_{T_{J'},\mathbb{F},x} \cong \mathbb{F}[[Y_{m+1},\dots,Y_d]] \hookrightarrow \mathbb{F}[[Y_1,\dots,Y_d]] \cong \widehat{\mathcal{O}}_{P_{J,\mathbb{F}},y},$$

we see that $\widehat{\mathcal{O}}_{W,w} \cong \mathbb{F}[[Y_1^{1/p},\dots,Y_d^{1/p}]]/(Y_{m+1},\dots,Y_d)$. Note in particular that $\widehat{\mathcal{O}}^{\mathrm{red}}_{W,w}$ is a domain of dimension m and that if r is in the nilradical of $\widehat{\mathcal{O}}_{W,w}$, then $r^p = 0$. It follows that the same assertions hold with $\widehat{\mathcal{O}}_{W,w}$ replaced by $\widehat{\mathcal{O}}_{Z,z}$.

Now we apply the results of §4.4 to obtain a surjective $\mathbb{F}$-algebra morphism

$$\widehat{\mathcal{O}}_{Z,z} \longrightarrow R = \mathbb{F}[[X_1,\dots,X_m]]. \tag{4.13}$$

More precisely, by Lemma 4.7 there exists an element of

$$\varprojlim Z^{\mathrm{uno}}(R_n) \subset \varprojlim Z(R_n) = Z(R)$$

lifting the morphism (4.9), whose surjectivity implies that of the morphism (4.13) induced by $\mathrm{Spec}\,(R) \to Z$. Since the morphism (4.13) factors through $\widehat{\mathcal{O}}^{\mathrm{red}}_{Z,z}$, which is a domain of dimension m, the resulting morphism $\widehat{\mathcal{O}}^{\mathrm{red}}_{Z,z} \to R$ must be injective, and hence an isomorphism. Furthermore, since Z is Nagata, it follows that

$$\widehat{\mathcal{O}}_{Z_{\mathrm{red}},z} = \widehat{\mathcal{O}}^{\mathrm{red}}_{Z,z} \cong \mathbb{F}[[X_1,\dots,X_m]]$$

is formally smooth of dimension m, and hence, Z^{red} is smooth of dimension m. Note also that the image of $\mathrm{Tan}_z(Z_{\mathrm{red}})$ in $\mathrm{Tan}_z(Z)$ is the same as that of the tangent space of $\mathrm{Spec}\,(R)$ at its closed point under the morphism induced by (4.13), which is $\mathrm{Tan}_z(Z)^{\mathrm{uno}}$ by construction of (4.9).

Turning to the task of describing $\widehat{\mathcal{O}}_{Z,z}$, let $\mathfrak{m}$ denote its maximal ideal and $\mathfrak{n}$ its nilradical, and consider the exact sequence

$$0 \longrightarrow \mathfrak{n}/(\mathfrak{n}\cap\mathfrak{m}^2) \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/(\mathfrak{n},\mathfrak{m}^2) \longrightarrow 0.$$

Recall that $\mathfrak{m}/\mathfrak{m}^2$ has dimension $m+\delta$ (over $\mathbb{F}$) by Lemma 4.5, and we have just seen that $\mathfrak{m}/(\mathfrak{n},\mathfrak{m}^2)$ has dimension m, so $\mathfrak{n}/(\mathfrak{n}\cap\mathfrak{m}^2)$ has dimension δ. We may therefore choose elements $r_1,\dots,r_\delta \in \mathfrak{n}$ and $s_1,\dots,s_m \in \mathfrak{m}$ so that $(r_1,\dots,r_\delta,s_1,\dots,s_m)$ lifts a basis of $\mathfrak{m}/\mathfrak{m}^2$, and consider the surjective $\mathbb{F}$-algebra morphism

$$\mu : \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]] \twoheadrightarrow \widehat{\mathcal{O}}_{Z,z}$$

sending T_i to r_i for $i = 1,\dots,\delta$ and X_i to s_i for $i = 1,\dots,m$. Note that the kernel of the composite $\mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]] \overset{\mu}{\twoheadrightarrow} \widehat{\mathcal{O}}_{Z,z} \twoheadrightarrow \widehat{\mathcal{O}}^{\mathrm{red}}_{Z,z}$ contains $I := (T_1,\dots,T_\delta)$, so must in fact equal I, and therefore, $\ker(\mu) \subset I$. However, from the first paragraph of the proof, we have $r_i^p = 0$ for $i = 1,\dots,\delta$, so $I^{(p)} := (T_1^p,\dots,T_\delta^p) \subset \ker(\mu)$. We have now shown that

$$\widehat{\mathcal{O}}_{Z,z} \cong \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/J$$

for some ideal J of $\mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]$ such that $I^{(p)} \subset J \subset I$.

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It remains to prove that $J = I^{(p)}$. To that end, consider the morphism $\widehat{\mathcal{O}}_{P_{J,\mathbb{F}},y} \to \widehat{\mathcal{O}}_{S_{J,\mathbb{F}},z}$ induced by ξ_J. As it is a morphism of regular local $\mathbb{F}$-algebras of the same dimension, namely d, and its image contains that of the Frobenius endomorphism, it is (by [KN82, Cor. 2]) finite flat of degree p^n where n is the dimension of the kernel of the induced map on tangent spaces, which by Lemma 4.4 is $\#\{\,(\mathfrak{p}, i) \mid \theta_{\mathfrak{p},i,e_{\mathfrak{p}}} \in J\,\}$. It follows that the induced morphism

$$\widehat{\mathcal{O}}_{Y,y} \longrightarrow \widehat{\mathcal{O}}_{Z,z}$$

is also finite flat of degree p^n, where $Y \cong (\mathbb{P}^1_{\mathbb{F}})^m$ is the fibre of P_J over x. Similarly, the composite

$$\widehat{\mathcal{O}}_{Y,y} \longrightarrow \widehat{\mathcal{O}}_{Z,z} \longrightarrow \widehat{\mathcal{O}}_{Z_{\mathrm{red}},z}$$

is also a morphism of regular local $\mathbb{F}$-algebras of the same dimension (now m) whose image contains that of the Frobenius endomorphism, so it is finite flat of degree $p^{n'}$ where n' is the dimension of the kernel of the composite

$$\mathrm{Tan}_z(Z_{\mathrm{red}}) \hookrightarrow \mathrm{Tan}_z(Z) \to \mathrm{Tan}_y(Y) \hookrightarrow \mathrm{Tan}_y(P_J).$$

Since the first inclusion identifies $\mathrm{Tan}_z(Z_{\mathrm{red}})$ with $\mathrm{Tan}_z(Z)^{\mathrm{uno}} \subset \mathrm{Tan}_z(Z)$, it follows from Lemmas 4.4 and 4.6 that $n' = n - \delta$. Now consider the commutative diagram:

$$\begin{tikzcd}
 & \mathbb{F}[[Y_1,\dots,Y_m]] \arrow[r, "\sim"] \arrow[d] \arrow[dl, dashed] & \widehat{\mathcal{O}}_{Y,y} \arrow[d] \\
Q \arrow[r, two heads] \arrow[dr, two heads] & \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/J \arrow[r, "\sim"] \arrow[d, two heads] & \widehat{\mathcal{O}}_{Z,z} \arrow[d, two heads] \\
 & \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/I \arrow[r, "\sim"] & \widehat{\mathcal{O}}_{Z_{\mathrm{red}},z},
\end{tikzcd}$$

where $Q = \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/I^{(p)}$ and the dashed arrow is any $\mathbb{F}$-algebra morphism lifting $\mathbb{F}[[Y_1,\dots,Y_m]] \to \mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/J$. Note that $\mathrm{gr}_I Q$ naturally has the structure of a finite $\widehat{\mathcal{O}}_{Z_{\mathrm{red}},z} \cong Q/IQ$-algebra, compatible with its structure as an $\widehat{\mathcal{O}}_{Y,y}$-algebra. Furthermore, $\mathrm{gr}_I Q$ is free of rank p^δ over $\widehat{\mathcal{O}}_{Z_{\mathrm{red}},z}$, and hence free of rank $p^{n'}p^\delta = p^n$ over $\widehat{\mathcal{O}}_{Y,y}$; therefore, Q is also free of rank p^n over $\widehat{\mathcal{O}}_{Y,y}$. Since the rank is the same as that of $\widehat{\mathcal{O}}_{Z,z}$, it follows that the surjection $Q \to \widehat{\mathcal{O}}_{Z,z}$ is an isomorphism. □

Remark 4.9. It would in fact suffice for our purposes to know that $\widehat{\mathcal{O}}_{Z,z}$ is isomorphic to $\mathbb{F}[[T_1,\dots,T_\delta,X_1,\dots,X_m]]/J$ for some J such that $I^{(p)} \subset J \subset I$, and the equality $J = I^{(p)}$ would follow from later considerations, but we find it interesting and satisfying to recognize it also as a consequence of the tangent space calculations via the commutative algebra argument in the proof of the lemma.

4.6. Reduced fibres

In this section, we give a complete description of Z_{red}, the reduced subscheme of the fibre Z of $\pi_J : S_J \to T_{J'}$ at any geometric point x of $T_{J'}$. This generalizes Theorem 7.2.2 of [DKS23], where it is assumed p is unramified in F. With the ingredients put in place in the preceding sections, we may now proceed by an argument similar to the one in [DKS23, §7.2].

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Recall from §4.2 that we have a morphism

$$\widetilde{\xi}_J : S_J \longrightarrow P_J = \prod_{\theta \in J''} \mathbb{P}_{T_{J'}}(\mathcal{P}_\theta)$$

which is bijective on closed points, where the product is a fibre product over $T_{J'}$. Taking fibres over $x \in T_{J'}(\mathbb{F})$ therefore yields such a morphism

$$Z_{\mathrm{red}} \longrightarrow \prod_{\theta \in J''} \mathbb{P}(P_{\theta,0}), \tag{4.14}$$

where the product is over $\mathbb{F}$, $P_{\theta,0} = G^{(j)}_{\tau,0}/F^{(j-1)}_{\tau,0} = u^{-1}F^{(j-1)}_{\tau,0}/F^{(j-1)}_{\tau,0}$, $\theta = \theta_{\mathfrak{p},i,j}$, $\tau = \tau_{\mathfrak{p},i}$ and $F^\bullet_0$ is the Pappas–Rapoport filtration in the data associated to x. We saw, however (in the proof of Lemma 4.8, where the target is denoted Y) that (4.14) may not be an isomorphism, but is in fact finite flat of degree $p^{n'}$ where $n' = \#\{\, (\mathfrak{p}, i) \mid \theta_{p,i,1} \in J'' \,\}$. We will use the approach in [DKS23, §7.2.2] to remove a factor of Frobenius from the relevant projections $Z_{\mathrm{red}} \to \mathbb{P}^1$. We will, however, need the following slight generalization of the crystallization lemma stated in [DKS23, §7.2.1], now allowing p to be ramified in F and requiring only an inclusion relation (rather than equality) between images of components of Dieudonné modules. We omit the proof, which is essentially the same as that of [DKS23, Lemma 7.2.1].

Lemma 4.10. *Suppose that S is a smooth scheme over an algebraically closed field $\mathbb{F}$ of characteristic p, A, B_1 and B_2 are abelian schemes over S with O_F-action, and $\tau \in \Sigma_{\mathfrak{p},0}$. Let $\alpha_i : A \to B_i$ for $i = 1, 2$ be O_F-linear isogenies such that*

- $\ker(\alpha_i) \cap A[p^\infty] \subset A[p]$ *for $i = 1, 2$, and*
- $\widetilde{\alpha}^*_{1,s}\mathbb{D}(B_{1,s}[p^\infty])_\tau \subset \widetilde{\alpha}^*_{2,s}\mathbb{D}(B_{2,s}[p^\infty])_\tau$ *for all $s \in S(\mathbb{F})$*

Then there is a unique morphism $\mathcal{H}^1_{\mathrm{dR}}(B_1/S)_\tau \to \mathcal{H}^1_{\mathrm{dR}}(B_2/S)_\tau$ of $\mathcal{O}_S[u]/(u^{e_\mathfrak{p}})$-modules whose fibres are compatible with the injective maps

$$\mathbb{D}(B_{1,s}[p^\infty])_\tau \hookrightarrow \mathbb{D}(B_{2,s}[p^\infty])_\tau$$

induced by $(\widetilde{\alpha}^*_{2,s})^{-1}\widetilde{\alpha}^*_{1,s}$ for all $s \in S(\mathbb{F})$. Furthermore, if $j : \mathrm{Spec}\,(R_1) = S_1 \to S$ denotes the first infinitesimal neighborhood of s, then the morphism is also compatible with the isomorphisms

$$H^1_{\mathrm{dR}}(B_{i,s}/\mathbb{F}) \otimes_{\mathbb{F}} R_1 \cong H^1_{\mathrm{dR}}(j^*B_i/R_1)$$

induced by their canonical isomorphisms with $H^1_{\mathrm{crys}}(B_{i,s}/R_1)$ for $i = 1, 2$.

Now let $(\underline{A}, \underline{A}', \psi)$ denote the restriction to Z_{red} of the universal triple, and suppose that $\tau = \tau_{\mathfrak{p},i}$ is such that $\theta = \theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J$. Then for any closed point z of Z_{red}, we have

$$\mathrm{Ver}^*(H^1_{\mathrm{dR}}(A_0^{(p^{-1})}/\mathbb{F})_\tau) = F^{(e_\mathfrak{p})}_{\tau,0} = \psi^*_{\tau,z}(G'^{(e_\mathfrak{p})}_{\tau,z}) \subset \psi^*_{\tau,z}(H^1_{\mathrm{dR}}(A'_z/\mathbb{F})_\tau), \tag{4.15}$$

where as usual, we write $\underline{A}_0$ for the fibre of $\underline{A}$ at z (or x) and $F^{(j)}_{\tau,0}$ for (the sections of) $\mathcal{F}^\bullet_{\tau,0}$, and similarly $\underline{A}'_z$ for the fibre of $\underline{A}'$, $F'^{(j)}_{\tau,z}$ for $\mathcal{F}'^{(j)}_{\tau,z}$ and $G'^{(j)}_{\tau,z} = u^{-1}F'^{(j-1)}_{\tau,z}$. We may therefore apply Lemma 4.10 with $S = Z_{\mathrm{red}}$, $B_1 = A_0^{(p^{-1})} \times_{\mathbb{F}} Z_{\mathrm{red}}$, $B_2 = A'$, $\alpha_2 = \psi$ and $\alpha_1 = \mathrm{Ver} \times_{\mathbb{F}} Z_{\mathrm{red}}$ (where $\mathrm{Ver} : A_0 \to A_0^{(p^{-1})}$) to obtain an $\mathcal{O}_{Z_{\mathrm{red}}}[u]/u^{e_\mathfrak{p}}$-linear morphism

$$\beta : (H^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\phi\circ\tau})^{(p^{-1})} \otimes_{\mathbb{F}} \mathcal{O}_{Z_{\mathrm{red}}} = \mathcal{H}^1_{\mathrm{dR}}(B_1/Z_{\mathrm{red}})_\tau \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A'/Z_{\mathrm{red}})_\tau$$

which at closed points is compatible in the evident sense with the injective morphisms induced by ψ and Ver on Dieudonné modules and with the isomorphisms over first-order thickenings provided by crystalline-de Rham comparisons (cf. diagrams (39) and (40) of [DKS23]). In particular, the fibre β_z at each closed point z is given by the reduction mod p of the unique injective homomorphism $\widetilde{\beta}_z$ of free rank two $W(\mathbb{F})[u]/(E_\tau)$-modules making the diagram

$$\begin{tikzcd}
\mathbb{D}(A_0[p^\infty])^{\phi^{-1}}_{\phi\circ\tau} \arrow[rr, "\widetilde{\beta}_z"] \arrow[dr, "V"'] & & \mathbb{D}(A'_z[p^\infty])_\tau \arrow[dl, "\widetilde{\psi}^*_{\tau,z}"] \\
& \mathbb{D}(A_0[p^\infty])_\tau &
\end{tikzcd}$$

commute. It follows from (4.15) that the image of $\widetilde{\beta}_z$ is the preimage $\widetilde{G}'^{(e_\mathfrak{p})}_{\tau,z}$ of $G'^{(e_\mathfrak{p})}_{\tau,z}$ under reduction mod p, and therefore that of β_z is $G'^{(e_\mathfrak{p})}_{\tau,z}$. Since this holds for each closed pont z, the image of β is $\mathcal{G}'^{(e_\mathfrak{p})}_{\tau}$, and hence, it induces an isomorphism

$$\begin{tikzcd}
(H^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\phi\circ\tau}/uH^1_{\mathrm{dR}}(A_0/\mathbb{F})_{\phi\circ\tau})^{(p^{-1})}\otimes_{\mathbb{F}}\mathcal{O}_{Z_{\mathrm{red}}} \arrow[r, "\sim"] \arrow[d, "u^{e_\mathfrak{p}-1}"', "\wr"] & \mathcal{G}'^{(e_\mathfrak{p})}_{\tau}/\mathcal{F}'^{(e_\mathfrak{p}-1)}_{\tau} =: \mathcal{P}'_\theta. \\
P^{(p^{-1})}_{\sigma\theta,0}\otimes_{\mathbb{F}}\mathcal{O}_{Z_{\mathrm{red}}} &
\end{tikzcd} \tag{4.16}$$

We can thus define the morphism

$$\mu_{\sigma\theta} : Z_{\mathrm{red}} \longrightarrow \mathbb{P}(P^{(p^{-1})}_{\sigma\theta,0})$$

such that the tautological line bundle pulls back to the subbundle corresponding to $\mathcal{L}'_\theta$ under (4.16). The commutativity of the diagram

$$\begin{tikzcd}
(\widetilde{G}'^{(1)}_{\phi\circ\tau,z})^{\phi^{-1}} \arrow[d, hook, "(\widetilde{\psi}^*_{\phi\circ\tau},z)^{\phi^{-1}}"'] & \mathbb{D}(A'_z[p^\infty])^{\phi^{-1}}_{\phi\circ\tau} \arrow[l, "\sim"', "u^{e_\mathfrak{p}-1}"] \arrow[r, "\sim", "V"'] & \widetilde{F}'^{(e_\mathfrak{p})}_{\tau,z} \arrow[r, hook] & \widetilde{G}'^{(e_\mathfrak{p})}_{\tau,z} \arrow[d, "\wr"', "\widetilde{\psi}^*_{\tau,z}"] \\
(\widetilde{G}^{(1)}_{\phi\circ\tau,0})^{\phi^{-1}} & \mathbb{D}(A_0[p^\infty])^{\phi^{-1}}_{\phi\circ\tau} \arrow[l, "\sim"', "u^{e_\mathfrak{p}-1}"] \arrow[rr, "\sim", "V"'] & & \widetilde{F}^{(e_\mathfrak{p})}_{\tau,0}
\end{tikzcd}$$

implies that on the fibre at z, the line $L'_{\theta,z}$ in $P'_{\theta,z}$ corresponds under (4.16) to the image of the morphism

$$(\psi^*_{\sigma\theta,z})^{(p^{-1})} : P'^{(p^{-1})}_{\sigma\theta,z} \longrightarrow P^{(p^{-1})}_{\sigma\theta,0}.$$

(cf. the proof of [DKS23, Thm. 7.2.2.1]). If $\sigma\theta = \theta_{\mathfrak{p},i+1,1} \in J$, then this image is $L^{(p^{-1})}_{\sigma\theta,0}$ for all z, so $\mu_{\sigma\theta}$ is a constant morphism. However, if $\sigma\theta \notin J$, so $\sigma\theta \in J''$, then the composite of $\mu_{\sigma\theta}$ with the Frobenius morphism

$$\mathbb{P}(P^{(p^{-1})}_{\sigma\theta,0}) \longrightarrow \mathbb{P}(P_{\sigma\theta,0})$$

is precisely the factor indexed by $\sigma\theta$ in (4.14).

Now for any $\theta = \theta_{\mathfrak{p},i,j} \in \Sigma$, let

$$P^{(n_\theta^{-1})}_{\theta,0} = \begin{cases} P^{(p^{-1})}_{\theta,0}, & \text{if } j = 1; \\ P_{\theta,0}, & \text{if } j > 1. \end{cases}$$

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Replacing the θ-factor in (4.14) with μ_θ for each $\theta = \theta_{\mathfrak{p},i,j} \in J''$ such that $j = 1$ therefore yields a commutative diagram

$$\begin{tikzcd} Z_{\mathrm{red}} \arrow[rr] \arrow[dr, "(4.14)"'] & & \prod_{\theta \in J''} \mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})}) \arrow[dl] \\ & \prod_{\theta \in J''} \mathbb{P}(P_{\theta,0}) & \end{tikzcd}$$

where the right diagonal map is the Frobenius morphism (resp. identity) on the factors indexed by $\theta = \theta_{\mathfrak{p},i,j}$ such that $j = 1$ (resp. $j > 1$). Since the schemes are all smooth over $\mathbb{F}$ and the diagonal morphisms are finite flat of the same degree (namely $p^{n'}$), it follows that the horizontal map is an isomorphism.

In order to obtain the above isomorphism, we factored out a power of Frobenius in each component of the form $\theta_{\mathfrak{p},i,1}$ by describing the corresponding morphism to $\mathbb{P}^1$ in terms of the data associated to A' (i.e., pull-backs via π_2 of vector bundles on $\widetilde{Y}_{U,\mathbb{F}}$). We can similarly describe the other components of the isomorphism in terms of the bundles associated to A', again with a shift in the index, but the relation is easier to obtain since no Frobenius factor intervenes. To that end, suppose that $\tau = \tau_{\mathfrak{p},i}$ and $\theta = \theta_{\mathfrak{p},i,j} \in J$ for some $j < e_\mathfrak{p}$, and consider the $\mathcal{O}_{Z_{\mathrm{red}}}[u]/(u^{e_\mathfrak{p}})$-linear morphism

$$\xi_\tau^* : \mathfrak{p}\mathcal{O}_{F,\mathfrak{p}} \otimes_{\mathcal{O}_{F,\mathfrak{p}}} H^1_{\mathrm{dR}}(A/Z_{\mathrm{red}})_\tau = \mathcal{H}^1_{\mathrm{dR}}((\mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A)/Z_{\mathrm{red}})_\tau \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A'/Z_{\mathrm{red}})_\tau$$

induced by the unique isogeny $\xi : A' \to \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A$ such that $\xi \circ \psi$ is the canonical isogeny $A \to \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A$. Since the image of $\mathcal{G}'^{(j)}_\tau$ under ψ_τ^* is $\mathcal{F}^{(j)}_\tau$, it follows that the image of $\mathfrak{p}\mathcal{O}_{F,\mathfrak{p}} \otimes_{\mathcal{O}_{F,\mathfrak{p}}} \mathcal{F}^{(j)}_\tau$ under ξ^*_τ is $\mathcal{F}'^{(j-1)}_\tau$. We thus obtain an $\mathcal{O}_{Z_{\mathrm{red}}}$-linear isomorphism

$$P_{\sigma\theta,0} \otimes_{\mathbb{F}} \mathcal{O}_{Z_{\mathrm{red}}} = \mathcal{P}_{\sigma\theta} \xrightarrow[\sim]{u} \mathfrak{p}\mathcal{O}_{F,\mathfrak{p}} \otimes_{\mathcal{O}_{F,\mathfrak{p}}} \mathcal{P}_{\sigma\theta} \xrightarrow[\sim]{\xi^*_\tau} \mathcal{P}'_\theta, \tag{4.17}$$

under which $\mathcal{L}'_\theta$ corresponds to the image of the morphism $\mathcal{P}'_{\sigma\theta} \to \mathcal{P}_{\sigma\theta}$ induced by ψ. Recall that if $\sigma\theta \notin J$ (i.e., $\sigma\theta \in J''$), then this is precisely the subbundle of $\mathcal{P}_{\sigma\theta} = P_{\sigma,\theta} \otimes_{\mathbb{F}} \mathcal{O}_{Z_{\mathrm{red}}}$ defining the factor indexed by $\sigma\theta$ in the morphism (4.14). However, if $\theta, \sigma\theta \in J$, then this subbundle is $L_{\sigma\theta,0} \otimes_{\mathbb{F}} \mathcal{O}_{Z_{\mathrm{red}}}$.

Theorem 4.11. *For $\theta = \theta_{\mathfrak{p},i,j} \in \Sigma$ such that $\sigma^{-1}\theta \in J$, let*

$$\delta_\theta : P_{\theta,0}^{(n_\theta^{-1})} \otimes_{\mathbb{F}} \mathcal{O}_{Z_{\mathrm{red}}} \xrightarrow{\sim} \mathcal{P}'_{\sigma^{-1}\theta}$$

denote the isomorphism defined by (4.16) or (4.17), according to whether $j = 1$ or $j > 1$ (and applied with $\sigma^{-1}\theta$ in place of θ), and let

$$\mu_\theta : Z_{\mathrm{red}} \longrightarrow \mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})})$$

denote the morphism such that the tautological line bundle pulls back to the subbundle corresponding to $\mathcal{L}'_{\sigma^{-1}\theta}$ under δ_θ. Then

1. *the resulting morphism*

$$Z_{\mathrm{red}} \xrightarrow{(\mu_\theta)_\theta} \prod_{\theta \in J''} \mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})})$$

is an isomorphism,

2. *and if $\sigma^{-1}\theta, \theta \in J$, then the morphism μ_θ is the projection to the point corresponding to $L_{\theta,0}^{(n_\theta^{-1})}$.*

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Remark 4.12. The theorem above is asserted in [ERX17a, §4.9(1)] for fibres of $\widetilde{\xi}_J$ at geometric *generic* points of $T_{J'}$. There is, however, a serious error in their argument (as already noted in [DKS23, §7]), specifically in the unjustified claim in the paragraph after [ERX17a, (4.9.3)] that one *can rearrange the choices of local parameters...* Theorem 4.11 thus fills the resulting gap in the proof of Proposition 3.19 of [ERX17a]. We note also that the latter is a cohomological vanishing result, a stronger version of which follows from the results in this paper (see Remark 5.10), and its application in [ERX17a] (as well as [DW20]) is to the construction of Hecke operators, which is ultimately achieved in greater generality in §5.4 of this paper.

4.7. Thickening

We will now give a complete description of Z, generalizing [DKS23, Thm. 7.2.4]. Our approach is based on that of [DKS23]: we will use the isomorphism of (4.16), for $\theta = \theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J$ such that $\sigma\theta \in J$, in order to extend the isomorphism Theorem 4.11(1) to one between Z and a suitable thickening of the target.

To that end, let T denote the fibre product

$$\begin{array}{ccc} T & \longrightarrow & \prod_{\sigma^{-1}\theta,\theta\in J} \mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})})) \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{F} & \longrightarrow & \prod_{\sigma^{-1}\theta,\theta\in J} \mathbb{P}(P_{\theta,0}), \end{array}$$

where the bottom arrow is defined by $L_{\theta,0}$ and the right vertical arrow is defined by the Frobenius morphism $M_\theta \mapsto M_\theta^{(p)}$ on the factors indexed by θ such that $n_\theta = p$. (Recall that these are the $\theta = \theta_{\mathfrak{p},i,j}$ such that $j = 1$, and note that we could have omitted the other factors from the definition, but we maintain them for the sake of uniformity in later formulas.)

We then let $\widetilde{Z}_{\mathrm{red}} = Z_{\mathrm{red}} \times_{\mathbb{F}} T$, and let $i : Z_{\mathrm{red}} \hookrightarrow \widetilde{Z}_{\mathrm{red}}$ denote the resulting divided power thickening (i.e., the unique section of the projection $q : \widetilde{Z}_{\mathrm{red}} \to Z_{\mathrm{red}}$). We thus have canonical isomorphisms $\mathcal{O}_{\widetilde{Z}_{\mathrm{red}}}[u]/(u^{e_\mathfrak{p}})$-linear morphisms

$$(R^1 s'_{\mathrm{crys},*}\mathcal{O}_{A',\mathrm{crys}})_{\widetilde{Z}_{\mathrm{red}},\tau} \cong q^*\mathcal{H}^1_{\mathrm{dR}}(A'/Z_{\mathrm{red}})_\tau$$

for all $\tau = \tau_{\mathfrak{p},i} \in \Sigma_0$ (where we continue to let $(\underline{A}, \underline{A}', \psi)$ denote the universal triple over Z_{red}, and $s' : A' \to Z_{\mathrm{red}}$ is the structure morphism). Furthermore, recall that to give a lift of the closed immersion $Z_{\mathrm{red}} \hookrightarrow Z$ to a morphism $\widetilde{Z}_{\mathrm{red}} \to Z$ is equivalent to giving a lift of $\mathcal{F}'^{(j)}_\tau$ (for each $\tau = \tau_{\mathfrak{p},i}$, $j = 1, \dots, e_\mathfrak{p}$) to an $\mathcal{O}_{\widetilde{Z}_{\mathrm{red}}}[u]/(u^{e_\mathfrak{p}})$-submodule $\widetilde{\mathcal{F}}'^{(j)}_\tau$ of $q^*\mathcal{H}^1_{\mathrm{dR}}(A'/Z_{\mathrm{red}})_\tau$ such that

- $\widetilde{\mathcal{F}}'^{(j-1)}_\tau \subset \widetilde{\mathcal{F}}'^{(j)}_\tau$ and $\widetilde{\mathcal{F}}'^{(j)}_\tau/\widetilde{\mathcal{F}}'^{(j-1)}_\tau$ is a line bundle on Z_{red} annihilated by u;
- the morphism
$$q^*\psi^*_\tau : q^*\mathcal{H}^1_{\mathrm{dR}}(A'/Z_{\mathrm{red}})_\tau \longrightarrow q^*\mathcal{H}^1_{\mathrm{dR}}(A/Z_{\mathrm{red}}) = H^1_{\mathrm{dR}}(A_0/\mathbb{F})_\tau \otimes_{\mathbb{F}} \mathcal{O}_{\widetilde{Z}_{\mathrm{red}}}$$
sends $\widetilde{\mathcal{F}}'^{(j)}_\tau$ to $F^{(j)}_{\tau,0} \otimes_{\mathbb{F}} \mathcal{O}_{\widetilde{Z}_{\mathrm{red}}}$ if $\tau \notin \Sigma_{\mathfrak{P},0}$, sends $\widetilde{\mathcal{G}}'^{(j)}_\tau := u^{-1}\widetilde{\mathcal{F}}'^{(j-1)}_\tau$ to $F^{(j)}_{\tau,0} \otimes_{\mathbb{F}} \mathcal{O}_{\widetilde{Z}_{\mathrm{red}}}$ if $\theta_{\mathfrak{p},i,j} \in J$, and sends $\widetilde{\mathcal{F}}'^{(j)}_\tau$ to $F^{(j-1)}_{\tau,0} \otimes_{\mathbb{F}} \mathcal{O}_{\widetilde{Z}_{\mathrm{red}}}$ otherwise.

We construct such a lift by setting $\widetilde{\mathcal{F}}'^{(j)}_\tau = q^*\mathcal{F}'^{(j)}_\tau$ unless $j = e_\mathfrak{p}$ and $\sigma^{-1}\theta, \theta \in J$, where $\theta = \theta_{\mathfrak{p},i+1,1} \in J$, in which case we define $\widetilde{\mathcal{F}}'^{(e_\mathfrak{p})}_\tau$ so that

$$\widetilde{\mathcal{L}}'_{\sigma^{-1}\theta} := \widetilde{\mathcal{F}}'^{(e_\mathfrak{p})}_\tau/\widetilde{\mathcal{F}}'^{(e_\mathfrak{p}-1)}_\tau \subset \widetilde{\mathcal{G}}'^{(e_\mathfrak{p})}_\tau/\widetilde{\mathcal{F}}'^{(e_\mathfrak{p}-1)}_\tau = q^*\mathcal{P}_{\sigma^{-1}\theta}$$

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corresponds to the pull-back of the tautological line bundle on $\mathbb{P}(P_{\theta,0}^{(p^{-1})})$ under $q^*\delta_\theta$. Note that $i^*\widetilde{\mathcal{L}}'_{\sigma^{-1}\theta} = \mathcal{L}'_{\sigma^{-1}\theta}$ since each corresponds to $L_{\theta,0}^{(p^{-1})} \otimes_{\mathbb{F}} \mathcal{O}_{Z_{\mathrm{red}}}$ under δ_θ. The fact that the resulting sheaves $\widetilde{\mathcal{F}}_\tau'^{(j)}$, including those with $j = e_{\mathfrak{p}}$, satisfy the required properties is immediate from their definition and the corresponding properties of the $\mathcal{F}_\tau'^{(j)}$. The Grothendieck–Messing Theorem thus yields a triple $(\widetilde{\underline{A}}, \widetilde{\underline{A}}', \widetilde{\psi})$ over $\widetilde{Z}_{\mathrm{red}}$ such that $i^*\widetilde{A}' = A'$ and $\widetilde{\mathcal{F}}'^{\bullet}$ corresponds to the Pappas–Rapoport filtration under the canonical isomorphism

$$\mathcal{H}^1_{\mathrm{dR}}(\widetilde{A}'/\widetilde{Z}_{\mathrm{red}}) \cong (R^1 s'_{\mathrm{crys},*}\mathcal{O}_{A',\mathrm{crys}})_{\widetilde{Z}_{\mathrm{red}}} \cong q^*\mathcal{H}^1_{\mathrm{dR}}(A'/Z_{\mathrm{red}}).$$

We claim that the resulting morphism

$$\widetilde{Z}_{\mathrm{red}} \longrightarrow Z$$

is an isomorphism. We first note that the induced morphism is injective on tangent spaces at all closed points $\widetilde{z} = (z,t)$ in $\widetilde{Z}_{\mathrm{red}}(\mathbb{F}) = Z(\mathbb{F}) \times T(\mathbb{F}) \xrightarrow{\sim} Z(\mathbb{F})$. Indeed let $R_1 = \mathbb{F}[\epsilon]$ and suppose that $\widetilde{z}_1 = (z_1, t_1) \in Z_{\mathrm{red}}(R_1) \times T(R_1) = \widetilde{Z}_{\mathrm{red}}(R_1)$ corresponds to an element of the kernel of the induced map

$$\mathrm{Tan}_{\widetilde{z}}(\widetilde{Z}_{\mathrm{red}}) = \mathrm{Tan}_z(Z_{\mathrm{red}}) \times \mathrm{Tan}_t(T) \to \mathrm{Tan}_z(Z)$$

(i.e., that the triple $(\widetilde{\underline{A}}'_1, \widetilde{\underline{A}}_1, \psi_1)$ associated to $\widetilde{z}_1$ is (isomorphic to) the pull-back of the triple $(\underline{A}_0, \underline{A}'_0, \psi_0)$ corresponding to z). The fact that $Z_{\mathrm{red}} \hookrightarrow Z$ is a closed immersion then implies that z_1 is trivial, and the fact that $\widetilde{F}'^{(e_{\mathfrak{p}})}_{\tau,1}$ corresponds to $F'^{(e_{\mathfrak{p}})}_{\tau,0} \otimes_{\mathbb{F}} R_1$ under the canonical isomorphism $H^1_{\mathrm{dR}}(\widetilde{A}'_1/R_1) \cong H^1_{\mathrm{dR}}(A'_0/\mathbb{F}) \otimes_{\mathbb{F}} R_1$ implies that $\widetilde{L}'_{\sigma^{-1}\theta,1}$ corresponds to $L^{(p^{-1})}_{\sigma^{-1}\theta,0} \otimes_{\mathbb{F}} R_1$ under the isomorphisms $\widetilde{P}'_{\sigma^{-1}\theta,1} \cong P'_{\sigma^{-1}\theta,0} \otimes_{\mathbb{F}} R_1 \cong P^{(p^{-1})}_{\theta,0} \otimes_{\mathbb{F}} R_1$, and hence that t_1 is trivial.

Since $\widetilde{Z}_{\mathrm{red}} \longrightarrow Z$ is injective on tangent spaces, it is a closed immersion, and the same argument as in the proof of [DKS23, Lemma 7.2.3] shows that it is an isomorphism. Alternatively, note that the diagram

$$\begin{array}{ccc} \widetilde{Z}_{\mathrm{red}} & \longrightarrow & Z \\ {\scriptstyle q}\downarrow & & \downarrow \\ Z_{\mathrm{red}} & \longrightarrow & \prod_{\theta \in J''} \mathbb{P}(P_{\theta,0}) \end{array}$$

commutes, where the right vertical map is the fibre of $\widetilde{\xi}_J$ over x and the bottom horizontal arrow is its restriction (i.e. (4.14)). Recall from the proof of Lemma 4.8 that these morphisms are finite flat of degree p^n and $p^{n'}$, respectively. Since q is finite flat of degree p^δ, the composite morphism $\widetilde{Z}_{\mathrm{red}} \to Z \to \prod_{\theta \in J''} \mathbb{P}(P_{\theta,0})$ is finite flat of degree p^n, and since $\widetilde{Z}_{\mathrm{red}} \to Z$ is a closed immersion, it follows that it must be an isomorphism.

Note in particular that the above construction extends the isomorphisms δ_θ of (4.16) and (4.17) to isomorphisms

$$\mathcal{P}^{(n_\theta^{-1})}_{\theta,0} \otimes_{\mathbb{F}} \mathcal{O}_{\widetilde{Z}_{\mathrm{red}}} \xrightarrow{\sim} q^*\mathcal{P}'_{\sigma^{-1}\theta} \cong \widetilde{\mathcal{P}}'_{\sigma^{-1}\theta}$$

over $\widetilde{Z}_{\mathrm{red}} \cong Z$ for all $\theta \in J$. We thus obtain the following extension of Theorem 4.11, maintaining the same notation except that now we write $\mathcal{L}'_\theta \subset \mathcal{P}'_\theta$ for the vector bundles $\mathcal{G}_\tau'^{(j)}/\mathcal{F}_\tau'^{(j-1)}$ associated to the data $\underline{A}'$, where $\tau = \tau_{\mathfrak{p},i}$, $\theta = \theta_{\mathfrak{p},i,j}$ and $(\underline{A}, \underline{A}', \psi)$ is the universal triple over Z, and we let

$$\varepsilon_\theta : \mathcal{P}'_{\sigma^{-1}\theta} \xrightarrow{\sim} \mathcal{P}^{(n_\theta^{-1})}_{\theta,0} \otimes_{\mathbb{F}} \mathcal{O}_Z \tag{4.18}$$

denote the inverse of the isomorphism just constructed over Z.

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Theorem 4.13. *The morphism*

$$Z \longrightarrow \prod_{\sigma^{-1}\theta\in J} \mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})}),$$

defined so the tautological line bundle on $\mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})})$ pulls back to $\varepsilon_\theta(\mathcal{L}'_{\sigma^{-1}\theta})$, is a closed immersion, identifying Z with the fibre over $(L_{\theta,0})_{\theta\in J-J'}$ of the morphism

$$\begin{array}{ccc} \prod_{\sigma^{-1}\theta\in J} \mathbb{P}(P_{\theta,0}^{(n_\theta^{-1})}) & \longrightarrow & \prod_{\theta\in J-J'} \mathbb{P}(P_{\theta,0}) \\ (M_\theta)_\theta & \mapsto & (M_\theta^{(n_\theta)})_\theta. \end{array}$$

We remark that the constructions in the proof of the theorem are independent of the quasi-polarizations in the data, and hence compatible with the action of $\mathcal{O}_{F,(p),+}^\times$ on the various schemes and vector bundles. One therefore obtains an identical description of the geometric fibres of the morphism $\overline{Y}_0(\mathfrak{P})_J \to \overline{Y}_{J'}$ for sufficiently small U, with $\mathcal{L}'_{\sigma^{-1}\theta} \subset \mathcal{P}'_{\sigma^{-1}\theta}$ (resp. $L_{\theta,0} \subset P_{\theta,0}$) replaced by restrictions (resp. fibres) of descents to $\overline{Y}_0(\mathfrak{P})_{J,\mathbb{F}}$ (resp. $\overline{Y}_{J',\mathbb{F}}$). Furthermore, the resulting descriptions are compatible with the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ in the obvious sense. Since we will not make use of this, we leave it to the interested reader to make precise, and simply record the following cruder, more immediate consequence of Theorem 4.13, generalizing Theorem D of [DKS23].

Corollary 4.14. *Every geometric fibre of the morphism $\overline{Y}_0(\mathfrak{P})_J \to \overline{Y}_{J'}$ is isomorphic to*

$$(\mathbb{P}^1_{\mathbb{F}})^m \times (\mathrm{Spec}\,(\mathbb{F}[T]/T^p))^\delta$$

where m and δ are as in Lemma 4.8, i.e., $m = |J'| = |J''|$ and $\delta = |\Delta| = \#\{\,\tau = \tau_{\mathfrak{p},i} \mid \theta_{\mathfrak{p},i,e_\mathfrak{p}}, \theta_{\mathfrak{p},i+1,1} \in J\,\}$.

5. Cohomological vanishing

5.1. Level $U_1(\mathfrak{P})$

One of our main results will be the vanishing of the higher direct images of both the structure and dualizing sheaves of $Y_{U_0(\mathfrak{P})}$ under the projection to Y_U. We will prove this also holds with $U_0(\mathfrak{P})$ replaced by its open compact subgroup $U_1(\mathfrak{P}) := \{\, g \in U \mid g_\mathfrak{p} \in I_1(\mathfrak{p}) \text{ for all } \mathfrak{p}|\mathfrak{P}\,\}$, where

$$I_1(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{p}}) \,\middle|\, c, d-1 \in \mathfrak{p}\mathcal{O}_{F,\mathfrak{p}} \right\}.$$

We first recall the definition of suitable integral models for Hilbert modular varieties of level $U_1(\mathfrak{P})$, following [Pap95]. We consider the functor which, to a locally Noetherian $\mathcal{O}$-scheme S, associates the set of isomorphism classes of data $(\underline{A}, \underline{A}', \psi, P)$, where $(\underline{A}, \underline{A}', \psi, P)$ corresponds to an element of $\widetilde{Y}_{U_0(\mathfrak{P})}(S)$, and $P \in A(S)$ is an $(\mathcal{O}_F/\mathfrak{P})$-generator of $\ker(\psi)$. The functor is represented by an $\mathcal{O}$-scheme which we denote $\widetilde{Y}_{U_1(\mathfrak{P})}$, and the forgetful morphism $\widetilde{f} : \widetilde{Y}_{U_1(\mathfrak{P})} \to \widetilde{Y}_{U_0(\mathfrak{P})}$ is finite flat.

The scheme $\widetilde{Y}_{U_1(\mathfrak{P})}$ can be described explicitly as follows in terms of $H = \ker(\psi)$, where $(\underline{A}, \underline{A}', \psi)$ is the universal triple over $S := \widetilde{Y}_{U_0(\mathfrak{P})}$. Firstly, the same argument as when p is unramified in F shows that H is a Raynaud $(\mathcal{O}_F/\mathfrak{P})$-module scheme (in the sense that $H[\mathfrak{p}]$ is a Raynaud $(\mathcal{O}_F/\mathfrak{p})$-vector space scheme for each $\mathfrak{p}|\mathfrak{P}$). The correspondence of [Ray74, Thm. 1.4.1] thus associates to H an invertible $\mathcal{O}_S$-module $\mathcal{R}_\tau$ for each $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$, together with morphisms

$$s_\tau : \mathcal{R}_\tau^{\otimes p} \longrightarrow \mathcal{R}_{\phi\circ\tau} \quad \text{and} \quad t_\tau : \mathcal{R}_{\phi\circ\tau} \longrightarrow \mathcal{R}_\tau^{\otimes p}$$

such that $s_\tau \circ t_\tau = w_\mathfrak{p}$ for a certain fixed element $w_\mathfrak{p} \in p\mathcal{O}^\times$. We then have

$$H = \mathbf{Spec}((\mathrm{Sym}_{\mathcal{O}_S} \mathcal{R})/\mathcal{I}),$$

where $\mathcal{R} = \bigoplus_{\tau \in \Sigma_{\mathfrak{P},0}} \mathcal{R}_\tau$, $\alpha \in \mathcal{O}_F$ acts on $\mathcal{R}_\tau$ as the Teichmüller lift of $\overline{\tau}(\alpha)$, and $\mathcal{I}$ is the sheaf of ideals generated by the $\mathcal{O}_S$-submodules $(s_\tau - 1)\mathcal{R}_\tau^{\otimes p}$ for $\tau \in \Sigma_{\mathfrak{P},0}$. The comultiplication on H is given via duality by the scheme structure on the Cartier dual

$$H^\vee = \mathbf{Spec}((\mathrm{Sym}_{\mathcal{O}_S} \mathcal{R}^\vee)/\mathcal{J}),$$

where $\mathcal{R}^\vee = \bigoplus_{\tau \in \Sigma_{\mathfrak{P},0}} \mathcal{R}_\tau^{-1}$, and $\mathcal{J}$ is generated by $(t_\tau - 1)(\mathcal{R}_\tau^{-1})^{\otimes p}$ for $\tau \in \Sigma_{\mathfrak{P},0}$, where t_τ is regarded as a morphism $(\mathcal{R}_\tau^{-1})^{\otimes p} \to \mathcal{R}_{\phi\circ\tau}^{-1}$. The same argument as in [Pap95, 5.1] then shows that $\widetilde{Y}_{U_1(\mathfrak{P})}$ can be identified with the closed subscheme of H defined by the sheaf of ideals generated by the

$$(s_\mathfrak{p} - 1)\left(\bigotimes_{\tau \in \Sigma_{\mathfrak{p},0}} \mathcal{R}_\tau^{\otimes(p-1)}\right)$$

for $\mathfrak{p}|\mathfrak{P}$, where $s_\mathfrak{p} := \bigotimes_{\tau \in \Sigma_{\mathfrak{p},0}} s_\tau$ is viewed as a morphism $\bigotimes_{\tau \in \Sigma_{\mathfrak{p},0}} \mathcal{R}_\tau^{\otimes(p-1)} \to \mathcal{O}_S$.

Note that the scheme $\widetilde{Y}_{U_1(\mathfrak{P})}$ is equipped with a natural action of $(\mathcal{O}_F/\mathfrak{P})^\times$ over $\widetilde{Y}_{U_0(\mathfrak{P})}$, defined by $\alpha \cdot (\underline{A}, \underline{A}', \psi, P) = (\underline{A}, \underline{A}', \psi, \alpha P)$. Furthermore, we may write

$$\tilde{f}_* \mathcal{O}_{\widetilde{Y}_{U_1(\mathfrak{P})}} = \bigoplus_\chi \mathcal{R}_\chi \tag{5.1}$$

where χ runs over all characters $(\mathcal{O}_F/\mathfrak{P})^\times \to \mathcal{O}^\times$, and each summand may be written as

$$\mathcal{R}_\chi = \bigotimes_{\tau \in \Sigma_{\mathfrak{P},0}} \mathcal{R}_\tau^{\otimes m_{\chi,\tau}}$$

where the integers $m_{\tau,\chi}$ are uniquely determined by the conditions

- $0 \le m_{\chi,\tau} \le p-1$ for all $\tau \in \Sigma_{\mathfrak{P},0}$;
- $m_{\chi,\tau} < p-1$ for some $\tau \in \Sigma_{\mathfrak{p},0}$ for each $\mathfrak{p}|\mathfrak{P}$;
- $\overline{\chi}(\alpha) = \prod_\tau \overline{\tau}(\alpha)^{m_{\chi,\tau}}$ for all $\alpha \in (\mathcal{O}_F/\mathfrak{P})^\times$.

Since $\tilde{f}$ is finite flat, we may similarly decompose the direct image of the dualizing sheaf as

$$\tilde{f}_* \mathcal{K}_{\widetilde{Y}_{U_1(\mathfrak{P})}/\mathcal{O}} = \mathcal{H}om_{\mathcal{O}_S}(\tilde{f}_* \mathcal{O}_{\widetilde{Y}_{U_1(\mathfrak{P})}}, \mathcal{K}_{S/\mathcal{O}}) = \bigoplus_\chi \mathcal{H}om(\mathcal{R}_\chi, \mathcal{K}_{S/\mathcal{O}}). \tag{5.2}$$

Assuming as usual that U is sufficiently small, and in particular $\alpha - 1 \in \mathfrak{P}$ for all $\alpha \in U \cap F^\times$, the action of $\mathcal{O}_{F,(p),+}^\times$ on $\widetilde{Y}_{U_1(\mathfrak{P})}$ via multiplication on quasi-polarizations factors through a free action of $\mathcal{O}_{F,(p),+}^\times/(U \cap F^\times)^2$, and we let $Y_{U_1(\mathfrak{P})}$ denote the quotient scheme. The morphism $\tilde{f}$ descends to a finite flat morphism $f : Y_{U_1(\mathfrak{P})} \to Y_{U_0(\mathfrak{P})}$, so $Y_{U_1(\mathfrak{P})}$ is Cohen–Macaulay and quasi-projective of relative dimension d over $\mathcal{O}$. Furthermore, for sufficiently small U, U' of level prime to p and $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ such that $g^{-1}Ug \subset U'$, we obtain a finite étale morphism $\tilde{\rho}_g : \widetilde{Y}_{U_1(\mathfrak{P})} \to \widetilde{Y}_{U_1'(\mathfrak{P})}$, descending to $\rho_g : Y_{U_1(\mathfrak{P})} \to Y_{U_1'(\mathfrak{P})}$ and satisfying the usual compatibilities.

Note that the assumption on U implies that the canonical isomorphism $\nu^*\underline{A} \cong \underline{A}$ over $\widetilde{Y}_{U_0(\mathfrak{P})}$ is the identity on H for all $\nu \in (U \cap F^\times)^2$, so H descends to a Raynaud $\mathcal{O}_F/\mathfrak{P}$-module scheme on $Y_{U_0(\mathfrak{P})}$, which we also denote by H. Furthermore, the line bundles $\mathcal{R}_\tau$ and morphisms s_τ, t_τ all descend to give the same descriptions of H, $H^\vee$ and $Y_{U_1(\mathfrak{P})}$ as spectra of finite flat $\mathcal{O}_{Y_{U_0(\mathfrak{P})}}$-algebras (with $(\mathcal{O}_F/\mathfrak{P})^\times$-actions) over $Y_{U_0(\mathfrak{P})}$, and we obtain analogous decompositions of $f_*\mathcal{O}_{Y_{U_1(\mathfrak{P})}}$ and $f_*\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}}$.

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5.2. Dicing

We will use the method of 'dicing', introduced in [DKS23, §6.2], to reduce the proofs of the cohomological vanishing results to consideration of line bundles on irreducible components[19] of the special fibre of $\widetilde{Y}_{U_0(\mathfrak{P})}$, or equivalently of the schemes denoted S_J.

Recall from §4.1 that for any subset $J \subset \Sigma_{\mathfrak{P}}$, the closed subscheme S_J of $\overline{S} := \widetilde{Y}_{U_0(\mathfrak{P}),k}$ is defined by the vanishing of

$$\{\psi^*_{\theta,\mathcal{M}} \,|\, \theta \in J\} \quad \cup \{\psi^*_{\theta,\mathcal{L}} \,|\, \theta \notin J\},$$

where $\psi : A \longrightarrow A'$ is the universal isogeny over $\overline{S}$ and

$$\psi_{\theta,\mathcal{M}} : \mathcal{M}'_\theta \to \mathcal{M}_\theta, \qquad \psi^*_{\theta,\mathcal{L}} : \mathcal{L}'_\theta \to \mathcal{L}_\theta$$

are the morphisms induced by ψ on the indicated subquotients of de Rham cohomology sheaves of A and A'.

Following [DKS23, §6.2], we consider the sheaves of ideals $\mathcal{I}_j \subset \mathcal{O}_{\overline{S}}$ corresponding to the closed subscheme of $\overline{S}$ defined by the vanishing of

$$\left\{ \bigotimes_{\theta \in J} \psi^*_{\mathcal{L},\theta} \,\middle|\, J \subset \Sigma_{\mathfrak{P}},\ |J| = j \right\},$$

or equivalently the image of the morphism

$$\bigoplus_{|J|=j} \left(\bigotimes_{\theta \in J} \mathcal{L}_\theta^{-1} \mathcal{L}'_\theta \right) \longrightarrow \mathcal{O}_{\overline{S}}$$

induced by the $\psi^*_{\theta,\mathcal{L}}$. We thus have

$$\mathcal{O}_{\overline{S}} = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots \mathcal{I}_{d_{\mathfrak{P}}} \supset \mathcal{I}_{1+d_{\mathfrak{P}}} = 0,$$

where $d_{\mathfrak{P}} = |\Sigma_{\mathfrak{P}}| = \sum_{\mathfrak{p}|\mathfrak{P}} e_{\mathfrak{p}} f_{\mathfrak{p}}$. Letting i_J denote the closed immersion $S_J \hookrightarrow \overline{S}$ and $\mathcal{I}_J$ the sheaf of ideals on S_J defining the vanishing locus of $\bigotimes_{\theta \in J} \psi^*_{\theta,\mathcal{L}}$, we see exactly as in [DKS23, §6.2.1] that $\mathcal{I}_J$ is invertible and the proof of [DKS23, Lemma 6.2.1] carries over *mutatis mutandis* to give the following:

Lemma 5.1. *The natural map* $\mathrm{gr}^j \mathcal{O}_{\overline{S}} \longrightarrow \bigoplus_{|J|=j} i_{J,*}\mathcal{I}_J$ *is an isomorphism for all* $j = 0, \ldots, d_{\mathfrak{P}}$.

Similarly, letting $\mathcal{J}_J$ denote the (invertible) sheaf of ideals on S_J defining the vanishing locus of[20] $\bigotimes_{\theta \notin J} \psi^*_{\theta,\mathcal{M}}$, the same argument as in [DKS23, §6.2.2] yields the following description of the graded pieces of the induced filtration $\mathrm{Fil}^j \mathcal{K}_{\overline{S}/k} := \mathcal{I}_j \mathcal{K}_{\overline{S}/k}$ on the dualizing sheaf:

Lemma 5.2. *The natural map* $\mathrm{gr}^j \mathcal{K}_{\overline{S}/k} \longrightarrow \bigoplus_{|J|=j} i_{J,*}(\mathcal{J}_J^{-1} \mathcal{K}_{S_J/k})$ *is an isomorphism for all* $j = 0, \ldots, d_{\mathfrak{P}}$.

Furthermore, letting $\overline{S}_1 = \widetilde{Y}_{U_1(\mathfrak{P}),k}$, the decompositions (5.1) and (5.2) restrict to give the decompositions

$$\tilde{f}_* \mathcal{O}_{\overline{S}_1} = \bigoplus_{\chi} \mathcal{R}_\chi \quad \text{and} \quad \tilde{f}_* \mathcal{K}_{\overline{S}_1/k} = \bigoplus_{\chi} \mathcal{R}_\chi^{-1} \mathcal{K}_{\overline{S}/k}, \tag{5.3}$$

[19]We could alternatively have used $\overline{Y}_0(\mathfrak{P})$ and $\overline{Y}_0(\mathfrak{P})_J$, as in [DKS23], but for the purpose of proving the desired vanishing results, we can replace them by étale covers.

[20]Since $\psi^*_{\theta,\mathcal{M}} : \mathcal{M}'_\theta \to \mathcal{M}_\theta$ is an isomorphism for $\theta \notin \Sigma_{\mathfrak{P}}$, we may view the tensor product as being over either $\Sigma - J$ or $\Sigma_{\mathfrak{P}} - J$, and we write simply $\theta \notin J$.

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where we again write $\widetilde{f} : \overline{S}_1 \to \overline{S}$ for the natural projection and $\mathcal{R}_\chi$ for the Raynaud bundles associated to $H = \ker(\psi)$ over $\overline{S}$. Note also that we may now view the decompositions as being over characters $\chi : (\mathcal{O}_F/\mathfrak{P})^\times \to k^\times$.

Following the method of [DKS23, §7.3] will reduce the proof of the desired cohomological vanishing to a computation involving the line bundles $\mathcal{I}_J$, $\mathcal{J}_J$, $\mathcal{K}_{S_J/k}$ and (the pull-back of) the $\mathcal{R}_\tau$ over S_J. We will now describe all of these in terms of the line bundles $\mathcal{L}_\theta$, $\mathcal{L}'_\theta$, $\mathcal{M}_\theta$ and $\mathcal{M}'_\theta$.

We start with the Raynaud bundles $\mathcal{R}_\tau$ over S_J for $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$. Suppose first that $\theta := \theta_{\mathfrak{p},i,e_\mathfrak{p}} \notin J$, so that $\psi^*_{\mathcal{L},\theta}$ vanishes on S_J, and therefore

$$\psi^*_\tau : \mathcal{H}^1_{\mathrm{dR}}(A'/S_J)_\tau \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A/S_J)$$

maps $\mathcal{F}'^{(e_\mathfrak{p})}_\tau$ onto $\mathcal{F}^{e_\mathfrak{p}-1}_\tau$ (where we are now writing $(\underline{A}, \underline{A}', \psi)$ for the universal object over S_J). It follows that the cokernel of $(s'_*\Omega^1_{A'/S_J})_\tau \to (s_*\Omega^1_{A/S_J})_\tau$ is isomorphic to the invertible sheaf $\mathcal{L}_\theta = \mathcal{F}^{(e_\mathfrak{p})}_\tau/\mathcal{F}^{(e_\mathfrak{p}-1)}_\tau$ on S_J, so

$$\mathcal{L}ie(H/S_J)_\tau = \ker(\mathcal{L}ie(A/S_J)_\tau \longrightarrow \mathcal{L}ie(A'/S_J)_\tau)$$

is isomorphic to $\mathcal{L}_\theta^{-1}$, and therefore, $i_J^*\mathcal{R}_\tau \cong \mathcal{L}_\theta$ (as in the proof of [DKS23, Lem. 5.1.1]).

Similarly, if $\theta = \theta_{\mathfrak{p},i,e_\mathfrak{p}} \in J$, then $\mathcal{G}'^{(e_\mathfrak{p})}_\tau$ is the preimage of $\mathcal{F}^{(e_\mathfrak{p})}_\tau$ under ψ^*_τ, so the kernel of $R^1s'_*\mathcal{O}_{A'} \to R^1s_*\mathcal{O}_A$ is isomorphic to $\mathcal{M}'_\theta = \mathcal{G}'^{(e_\mathfrak{p})}_\tau/\mathcal{F}'^{(e_\mathfrak{p})}_\tau$. It follows that

$$\mathcal{L}ie(H^\vee/S_J)_\tau \cong \ker(\mathcal{L}ie((A')^\vee/S_J)_\tau \longrightarrow \mathcal{L}ie(A^\vee/S_J)_\tau)$$

is isomorphic to $\mathcal{M}'_\theta$, and therefore so is $i_J^*\mathcal{R}_\tau$. Summing up, we have constructed a canonical isomorphism

$$i_J^*\mathcal{R}_\tau \;\cong\; \begin{cases} \mathcal{L}_\theta, & \text{if } \theta \notin J. \\ \mathcal{M}'_\theta & \text{if } \theta \in J, \end{cases} \tag{5.4}$$

where $\tau = \tau_{\mathfrak{p},i} \in \Sigma_{\mathfrak{P},0}$ and $\theta = \theta_{\mathfrak{p},i,e_\mathfrak{p}}$.

Next, considering $\mathcal{I}_J$, its definition (and invertibility) imply that $\bigotimes_{\theta\in J}\psi^*_{\theta,\mathcal{L}}$ induces an isomorphism

$$\bigotimes_{\theta\in J}\mathcal{L}_\theta^{-1}\mathcal{L}'_\theta \;\stackrel{\sim}{\longrightarrow}\; \mathcal{I}_J. \tag{5.5}$$

Similarly, we have that $\bigotimes_{\theta\notin J}\psi^*_{\theta,\mathcal{M}}$ induces an isomorphism

$$\bigotimes_{\theta\notin J}\mathcal{M}_\theta^{-1}\mathcal{M}'_\theta \;\stackrel{\sim}{\longrightarrow}\; \mathcal{J}_J, \tag{5.6}$$

but we will need to further relate the $\mathcal{M}'_\theta$ (for $\theta \in \Sigma_\mathfrak{P} - J$) to other line bundles on S_J.

Let $\theta = \theta_{\mathfrak{p},i,j}$ for some $\mathfrak{p}|\mathfrak{P}$, and let $\tau = \tau_{\mathfrak{p},i}$, and suppose first that $\theta, \sigma^{-1}\theta \notin J$. If $j > 1$, then consider the commutative diagram

$$\begin{tikzcd} \mathcal{P}'_\theta \arrow[r, "\psi^*_\theta"] \arrow[d, "u"'] & \mathcal{P}_\theta \arrow[d, "u"] \\ \mathcal{P}'_{\sigma^{-1}\theta} \arrow[r, "\psi^*_{\sigma^{-1}\theta}"] & \mathcal{P}_{\sigma^{-1}\theta}. \end{tikzcd}$$

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Note that the image of $u : \mathcal{P}'_\theta \to \mathcal{P}'_{\sigma^{-1}\theta}$ is $\mathcal{L}'_{\sigma^{-1}\theta} = \ker(\psi^*_{\sigma^{-1}\theta})$ (since $\sigma^{-1}\theta \notin J$), so the image of ψ^*_θ is contained in the kernel of $u : \mathcal{P}_\theta \to \mathcal{P}_{\sigma^{-1}\theta}$, which is $\mathcal{M}_{\sigma^{-1}\theta} = \mathcal{G}^{(j-1)}_\tau/\mathcal{F}^{(j-1)}_\tau$. Since each is a rank one subbundle of $\mathcal{P}_{\sigma^{-1}\theta}$, it follows that

$$\mathcal{M}_{\sigma^{-1}\theta} = \mathrm{im}(\psi^*_\theta) \cong \mathcal{P}'_\theta/\ker(\psi^*_\theta) = \mathcal{P}'_\theta/\mathcal{L}'_\theta = \mathcal{M}'_\theta.$$

Similarly, if $j = 1$, then we consider instead the commutative diagram

$$\begin{tikzcd} \mathcal{P}'_\theta \arrow[r, "\psi^*_\theta"] \arrow[d] & \mathcal{P}_\theta \arrow[d] \\ \mathcal{P}'^{(p)}_{\sigma^{-1}\theta} \arrow[r, "\psi_{\sigma^{-1}\theta}^{*(p)}"] & \mathcal{P}^{(p)}_{\sigma^{-1}\theta}, \end{tikzcd}$$

where the vertical maps are defined by the partial Hasse invariant (see (4.1) and (4.2)). The same reasoning as above then shows that $\mathcal{M}'_\theta$ is isomorphic to the kernel of

$$\widetilde{h}_\theta : \mathcal{P}_\theta \longrightarrow \mathcal{L}^{(p)}_{\sigma^{-1}\theta},$$

which in this case is isomorphic to $\mathcal{M}^{(p)}_{\sigma^{-1}\theta}$, via the morphism induced by the restriction of Frob* to

$$(\mathcal{G}^{(e_\mathfrak{p})}_{\phi^{-1}\circ\tau})^{(p)} = u^{-1}(\mathcal{F}^{(e_\mathfrak{p}-1)}_{\phi^{-1}\circ\tau})^{(p)} \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(A/S_J)_\tau[u] = \mathcal{P}_\theta.$$

Suppose, however, that $\theta \notin J$, but $\sigma^{-1}\theta \in J$, so $\theta \in J''$. Since $\underline{A}''$ corresponds to an element of $T_{J''}(S_J)$, we have that the kernel of

$$\widetilde{h}'_\theta : \mathcal{P}'_\theta \longrightarrow \mathcal{L}'^{(n_\theta)}_{\sigma^{-1}\theta}$$

is $\mathcal{L}'_\theta$, and hence that $\mathcal{M}'_\theta$ is isomorphic to $\mathcal{L}'^{(n_\theta)}_{\sigma^{-1}\theta}$.

Summing up, we have shown that if $\theta \in \Sigma_\mathfrak{P} - J$, then there is a canonical isomorphism

$$\mathcal{M}'_\theta \;\cong\; \begin{cases} \mathcal{M}^{(n_\theta)}_{\sigma^{-1}\theta}, & \text{if } \sigma^{-1}\theta \notin J \\ \mathcal{L}'^{(n_\theta)}_{\sigma^{-1}\theta}, & \text{if } \sigma^{-1}\theta \in J. \end{cases} \tag{5.7}$$

Finally, although we will not need the following description of the line bundles $\mathcal{K}_{S_J/k}$, we provide it for completeness and coherence with the theme of Kodaira–Spencer isomorphisms. Recall that the isomorphism of Theorem 3.5 arises from a canonical isomorphism

$$\mathcal{K}_{\widetilde{Y}_{U_0(\mathfrak{P})}/\mathcal{O}} \;\cong\; \bigotimes_{\theta\in\Sigma}\left(\widetilde{\mathcal{M}}'^{-1}_\theta\widetilde{\mathcal{L}}_\theta\right)$$

(using $\widetilde{\cdot}$ for universal data over $\widetilde{Y}_{U_0(\mathfrak{P})}$). However, the proof of Lemma 5.2 (see [DKS23, §6.2.2]) also gives a canonical isomorphism

$$\mathcal{K}_{S_J/k} \;\cong\; \mathcal{I}_J\mathcal{J}_J i^*_J\mathcal{K}_{\overline{S}/k}.$$

Combining these with (5.5) and (5.6), we obtain a canonical isomorphism

$$\mathcal{K}_{S_J/k} \;\cong\; \left(\bigotimes_{\theta\in\Sigma-J}\mathcal{M}^{-1}_\theta\mathcal{L}_\theta\right)\otimes\left(\bigotimes_{\theta\in J}\mathcal{M}'^{-1}_\theta\mathcal{L}'_\theta\right). \tag{5.8}$$

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(Recall that for $\theta \notin \Sigma_{\mathfrak{P}}$, we have $\mathcal{L}_\theta \cong \mathcal{L}'_\theta$ and $\mathcal{M}_\theta \cong \mathcal{M}'_\theta$, so these factors could just as well have been included with $\theta \in J$ instead of $\theta \in \Sigma_{\mathfrak{P}} - J$.)

We remark that (5.8) can also be proved more directly using a deformation-theoretic argument, which furthermore produces a filtration on $\Omega^1_{S_J/k}$ analogous to the one on $\Omega^1_{Y_U/\mathcal{O}}$ defined by Reduzzi and Xiao in [RX17, §2.8] (or more precisely [Dia23, §3.3] for the current setup).

We remark also that the various isomorphisms of line bundles established in this section should be compatible in the usual sense with descent data and Hecke action, but we will not need this for the purpose of proving the desired cohomological vanishing theorems.

5.3. Vanishing, duality and flatness results

We are now ready to prove the main result, which is a generalization of Theorem E of [DKS23]. Indeed, all the ingredients are now in place to apply the same argument as in [DKS23, §7.3]. Recall that U is any sufficiently small open compact subgroup of $\mathrm{GL}_2(\mathcal{O}_{F,p})$, $\mathfrak{P}$ is the product of any set of primes of $\mathcal{O}_F$ dividing p, and

$$Y_{U_1(\mathfrak{P})} \xrightarrow{f} Y_{U_0(\mathfrak{P})} \xrightarrow{\pi_1} Y_U$$

are the natural degeneracy maps induced by the forgetful morphisms $\widetilde{f} : \widetilde{Y}_{U_1(\mathfrak{P})} \longrightarrow \widetilde{Y}_{U_0(\mathfrak{P})}$ and $\widetilde{\pi}_1 : \widetilde{Y}_{U_0(\mathfrak{P})} \longrightarrow \widetilde{Y}_U$. We will write simply Y (resp. $Y_0(\mathfrak{P})$, $Y_1(\mathfrak{P})$) for Y_U (resp. $Y_{U_0(\mathfrak{P})}$, $Y_{U_1(\mathfrak{P})}$) when Y is fixed, and similarly abbreviate $\widetilde{Y}_U$, etc. We let $\widetilde{\varphi}$ (resp. φ) denote the composite $\widetilde{\pi}_1 \circ \widetilde{f}$ (resp. $\pi_1 \circ f$).

Theorem 5.3. *The higher direct image sheaves*

$$R^i\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}} \quad \textit{and} \quad R^i\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}$$

vanish for all $i > 0$.

Proof. First, note that since $\widetilde{Y} \longrightarrow Y$ is étale and $\widetilde{Y}_1(\mathfrak{P}) = \widetilde{Y} \times_Y Y_1(\mathfrak{P})$, we may replace Y (resp. $Y_1(\mathfrak{P})$, φ) by $\widetilde{Y}$ (resp. $\widetilde{Y}_1(\mathfrak{P})$, $\widetilde{\varphi}$). Next, note that since $\widetilde{\varphi}$ is projective and $\widetilde{\varphi}_K$ is finite, we may replace these in turn by their special fibres. Furthermore, since $\widetilde{f}$ is finite, we have

$$R^i\widetilde{\varphi}_*\mathcal{O}_{\overline{S}_1} = R^i\widetilde{\pi}_{1,*}(\widetilde{f}_*\mathcal{O}_{\overline{S}_1}) \quad \text{and} \quad R^i\widetilde{\varphi}_*\mathcal{K}_{\overline{S}_1/k} = R^i\widetilde{\pi}_{1,*}(\widetilde{f}_*\mathcal{K}_{\overline{S}_1/k})$$

(where we again let $\overline{S}_1 = \widetilde{Y}_1(\mathfrak{P})_k$ and suppress the subscript k from the notation for the morphisms). Using the decompositions given in (5.2) and the filtrations whose graded pieces are described in Lemmas 5.1 and 5.2, we are therefore reduced to proving the vanishing of

$$R^i\pi_{J,*}(\mathcal{I}_J i_J^*\mathcal{R}_\chi) \quad \text{and} \quad R^i\pi_{J,*}((\mathcal{J}_J i_J^*\mathcal{R}_\chi)^{-1}\mathcal{K}_{S_J/k})$$

for all $i > 0$, $J \subset \Sigma_{\mathfrak{P}}$ and $\chi : (\mathcal{O}_F/\mathfrak{P})^\times \longrightarrow k^\times$, where i_J is the closed immersion $S_J \hookrightarrow \overline{S}$ and $\pi_J : S_J \longrightarrow T_{J'}$ is the restriction of $\widetilde{\pi}_1$.

Let us now fix J and χ, and write $\chi = \prod \overline{\tau}^{m_\tau}$, where the product is over $\tau \in \Sigma_{\mathfrak{P},0}$, $0 \le m_\tau \le p-1$ for each τ, and $m_\tau < p-1$ for some $\tau \in \Sigma_{\mathfrak{p},0}$ for each $\mathfrak{p}|\mathfrak{P}$. We now apply the results of the preceding section to write the line bundles $\mathcal{I}_J$, $\mathcal{J}_J$ and $i_J^*\mathcal{R}_\chi$ on S_J in terms of $\mathcal{L}_\theta$, $\mathcal{M}_\theta$, $\mathcal{L}'_\theta$ and $\mathcal{M}'_\theta$.

Firstly, by formula (5.4), we have

$$i_J^*\mathcal{R}_\chi \cong \left(\bigotimes_{\theta \in \Sigma_{\mathfrak{P}}-J} \mathcal{L}_\theta^{\otimes m_\theta}\right) \bigotimes \left(\bigotimes_{\theta \in J} \mathcal{M}'^{\otimes m_\theta}_\theta\right), \tag{5.9}$$

where $m_\theta = m_\tau$ if $\theta = \theta_{\mathfrak{p},i,e_\mathfrak{p}}$ and $\tau = \tau_{\mathfrak{p},i}$, and $m_\theta = 0$ if $\theta = \theta_{\mathfrak{p},i,j}$ for some $j < e_\mathfrak{p}$. Combining (5.9) with (5.5) and the isomorphisms $\mathcal{L}'_\theta \otimes \mathcal{M}'_\theta = \mathcal{N}'_\theta \cong \mathcal{N}_\theta = \mathcal{L}_\theta \otimes \mathcal{M}_\theta$ (see Remark 3.4), we obtain an isomorphism

$$\mathcal{I}_J i_J^* \mathcal{R}_\chi \cong \left(\bigotimes_{\theta \in \Sigma_{\mathfrak{P}} - J} \mathcal{L}_\theta^{\otimes m_\theta} \right) \otimes \left(\bigotimes_{\theta \in J} (\mathcal{M}_\theta'^{\otimes (m_\theta - 1)} \otimes \mathcal{M}_\theta) \right). \tag{5.10}$$

However, combining (5.9) with (5.6) and (5.7) gives

$$\mathcal{J}_J i_J^* \mathcal{R}_\chi \cong \left(\bigotimes_{\theta, \sigma\theta \in \Sigma_{\mathfrak{P}} - J} \mathcal{M}_\theta^{\otimes (n_{\sigma\theta} - 1)} \otimes \mathcal{L}_\theta^{\otimes m_\theta} \right) \otimes \left(\bigotimes_{\theta \notin J, \sigma\theta \in J} \mathcal{M}_\theta^{-1} \mathcal{L}_\theta^{\otimes m_\theta} \right)$$
$$\otimes \left(\bigotimes_{\theta \in J, \sigma\theta \notin J} (\mathcal{M}_\theta'^{\otimes m_\theta} \otimes \mathcal{L}_\theta'^{\otimes n_{\sigma\theta}}) \right) \otimes \left(\bigotimes_{\theta, \sigma\theta \in J} \mathcal{M}_\theta'^{\otimes m_\theta} \right).$$

Combined with the isomorphisms $\mathcal{N}_\theta \cong \mathcal{N}_\theta'$ and $\mathcal{N}_\theta^{\otimes n_{\sigma\theta}} \cong \mathcal{N}_{\sigma\theta}$ (see the last paragraph of [Dia23, §4.1] for the latter), this simplifies to

$$\begin{aligned} \mathcal{J}_J i_J^* \mathcal{R}_\chi \cong & \left(\bigotimes_{\theta, \sigma\theta \in \Sigma_{\mathfrak{P}} - J} \mathcal{L}_\theta^{\otimes (m_\theta - n_{\sigma\theta} + 1)} \right) \otimes \left(\bigotimes_{\theta \notin J, \sigma\theta \in J} \mathcal{L}_\theta^{\otimes (m_\theta + 1)} \right) \\ & \otimes \left(\bigotimes_{\theta \in J, \sigma\theta \notin J} \mathcal{M}_\theta'^{\otimes (m_\theta - n_{\sigma\theta})} \right) \otimes \left(\bigotimes_{\theta, \sigma\theta \in J} \mathcal{M}_\theta'^{\otimes m_\theta} \right). \end{aligned} \tag{5.11}$$

Since π_J is flat and projective, it suffices to prove the vanishing of cohomology of fibres, that is, that

$$H^i(Z, j^*(\mathcal{I}_J i_J^* \mathcal{R}_\chi)) = 0 \quad \text{and} \quad H^i(Z, j^*((\mathcal{J}_J i_J^* \mathcal{R}_\chi)^{-1} \mathcal{K}_{S_J/k})) = 0$$

for all $x \in T_{J'}(\overline{\mathbb{F}}_p)$, where $j : Z \to S_J$ is the closed immersion of the fibre of π_J over x. Furthermore, since π_J is Cohen-Macaulay, we have $j^*\mathcal{K}_{S_J/k} = \mathcal{K}_{Z/\overline{\mathbb{F}}_p}$. Since the line bundles $j^*\mathcal{L}_\theta = L_{\theta,x} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_Z$ and $j^*\mathcal{M}_\theta = M_{\theta,x} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_Z$ are trivial on Z, the isomorphisms (5.10) and (5.11) give

$$j^*(\mathcal{I}_J i_J^* \mathcal{R}_\chi) \cong \bigotimes_{\theta \in J} j^* \mathcal{M}_\theta'^{\otimes (m_\theta - 1)} \quad \text{and} \quad j^*(\mathcal{J}_J i_J^* \mathcal{R}_\chi)^{-1} \cong \bigotimes_{\theta \in J} j^* \mathcal{M}_\theta'^{\otimes (\ell_\theta)}, \tag{5.12}$$

where $\ell_\theta = n_{\sigma\theta} - m_\theta$ if $\sigma\theta \notin J$ and $\ell_\theta = -m_\theta$ if $\sigma\theta \in J$.

We now appeal to the description of Z in Theorem 4.13, which provides an isomorphism

$$Z \xrightarrow{\sim} \operatorname{Spec} C \times \prod_{\theta \in J''} \mathbb{P}^1$$

for a finite (Gorenstein) $\overline{\mathbb{F}}_p$-algebra C, such that if $\theta \in J$, then $j^*\mathcal{M}_\theta'$ corresponds to the pull-back of $\mathcal{O}(1)_{\sigma\theta}$ (resp. an invertible C-module) if $\sigma\theta \notin J$ (resp. $\sigma\theta \in J$). It therefore follows from (5.12) that

$$H^i(Z,\, j^*(\mathcal{I}_J i_J^* \mathcal{R}_\chi)) \cong H^i\left(\prod_{\theta \in J''} \mathbb{P}^1, \bigotimes_{\theta \in J''} \mathcal{O}(m_{\sigma^{-1}\theta} - 1)_\theta \right) \otimes_{\overline{\mathbb{F}}_p} C,$$

which vanishes since $m_{\sigma^{-1}\theta} \geq 0$ for all $\theta \in J''$. Similarly, since $\mathcal{K}_{Z/\overline{\mathbb{F}}_p}$ corresponds to the dualizing sheaf of $\operatorname{Spec} C \times \prod_{\theta \in J''} \mathbb{P}^1$, which is isomorphic to $\otimes_{\theta \in J''} \mathcal{O}(-2)_\theta$, we have

$$H^i(Z,\, j^*((\mathcal{J}_J i_J^* \mathcal{R}_\chi)^{-1} \mathcal{K}_{S_J/k})) \cong H^i\left(\prod_{\theta \in J''} \mathbb{P}^1, \bigotimes_{\theta \in J''} \mathcal{O}(\ell_{\sigma^{-1}\theta} - 2)_\theta \right) \otimes_{\overline{\mathbb{F}}_p} C,$$

which vanishes since $\ell_{\sigma^{-1}\theta} = n_\theta - m_{\sigma^{-1}\theta} \geq 1$ for all $\theta \in J''$. □

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Corollary 5.4. *The higher direct image sheaves*

$$R^i\pi_{j,*}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}} \quad \textit{and} \quad R^i\pi_{j,*}\mathcal{O}_{Y_0(\mathfrak{P})}$$

vanish for all $i > 0$ *and* $j = 1, 2$.

Proof. As in the proof of the theorem, we may replace π_j by $\widetilde{\pi}_j : \widetilde{Y}_0(\mathfrak{P}) \to \widetilde{Y}$. By formula (5.1), the line bundle $\mathcal{O}_{\widetilde{Y}_0(\mathfrak{P})}$ is a direct summand of $\widetilde{f}_*\mathcal{O}_{\widetilde{Y}_1(\mathfrak{P})}$, so the vanishing of

$$R^i\widetilde{\varphi}_*\mathcal{O}_{\widetilde{Y}_1(\mathfrak{P})} = R^i\widetilde{\pi}_{1,*}(\widetilde{f}_*\mathcal{O}_{\widetilde{Y}_1(\mathfrak{P})})$$

for all $i > 0$ implies that of $R^i\widetilde{\pi}_{1,*}\mathcal{O}_{\widetilde{Y}_0(\mathfrak{P})}$. Similarly, the vanishing of $R^i\widetilde{\pi}_{1,*}\mathcal{K}_{\widetilde{Y}_0(\mathfrak{P})/\mathcal{O}}$ follows from (5.2) and the theorem.

For the analogous assertions for $\widetilde{\pi}_2$, note that there is an automorphism $\widetilde{w}_{\mathfrak{P}}$ of $\widetilde{Y}_0(\mathfrak{P})$ such that $\widetilde{\pi}_2 = \widetilde{\pi}_1 \circ \widetilde{w}_{\mathfrak{P}}$. Indeed, define $\widetilde{w}_{\mathfrak{P}}$ by the triple $(\underline{A}', \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} \underline{A}, \xi)$, where $(\underline{A}, \underline{A}', \psi)$ is the universal triple over $\widetilde{Y}_0(\mathfrak{P})$, $\mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} \underline{A}$ denotes $\mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A$ endowed with the evident auxiliary data, in particular quasi-polarization

$$\varpi_{\mathfrak{P}}^2 \otimes \lambda : \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A \to \mathfrak{P} \otimes_{\mathcal{O}_F} A^\vee = (\mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A)^\vee,$$

and $\xi : A' \to \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A$ is such that $\xi \circ \psi$ is the canonical isogeny $A \to \mathfrak{P}^{-1} \otimes_{\mathcal{O}_F} A$. The desired vanishing of higher direct images under $\widetilde{\pi}_2$ therefore follows from the corresponding assertions for $\widetilde{\pi}_1$ and the identifications $\widetilde{w}_{\mathfrak{P},*}\mathcal{O}_{\widetilde{Y}_0(\mathfrak{P})} = \mathcal{O}_{\widetilde{Y}_0(\mathfrak{P})}$ and $\widetilde{w}_{\mathfrak{P},*}\mathcal{K}_{\widetilde{Y}_0(\mathfrak{P})/\mathcal{O}} = \mathcal{K}_{\widetilde{Y}_0(\mathfrak{P})/\mathcal{O}}$. □

Remark 5.5. We caution that the automorphism $w_{\mathfrak{P}}$ of $\widetilde{Y}_0(\mathfrak{P})$ does *not* lift to an automorphism of $\widetilde{Y}_1(\mathfrak{P})$, so the argument handling the case of π_2 in in the proof of the corollary cannot be used to deduce of vanishing of higher direct images under $\pi_2 \circ f$ from the theorem.

Corollary 5.6. *The direct image sheaves* $\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}}$ *and* $\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}$ *are locally free over* $\mathcal{O}_Y$, *of rank* $\prod_{\mathfrak{p}|\mathfrak{P}}(p^{2f_\mathfrak{p}} - 1)$. *Furthermore, there is a perfect Hecke-equivariant*[21] *pairing*

$$\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}} \otimes_{\mathcal{O}_Y} \varphi_*\mathcal{O}_{Y_1(\mathfrak{P})} \longrightarrow \mathcal{K}_{Y/\mathcal{O}}.$$

Proof. We apply Grothendieck–Serre duality to the proper morphism φ. Since $R^i\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}$ vanishes for all $i > 0$, the duality isomorphism of [StaX, §0AU3(4c)] (taking K there to be $\mathcal{O}_{Y_1(\mathfrak{P})}[d]$) degenerates to

$$R\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}} \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{O}_Y}(\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}, \mathcal{K}_{Y/\mathcal{O}}).$$

In particular, we obtain an isomorphism

$$\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}, \mathcal{K}_{Y/\mathcal{O}}). \tag{5.13}$$

Furthermore, since $R^i\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}}$ vanishes for all $i > 0$, it follows that so does $\mathcal{E}xt^i_{\mathcal{O}_Y}(\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}, \mathcal{K}_{Y/\mathcal{O}})$, and hence,

$$\mathcal{E}xt^i_{\mathcal{O}_Y}(\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}, \mathcal{O}_Y) = 0$$

for all $i > 0$. Since Y is regular, it follows that $\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})}$ is locally free, with rank given by the degree of the finite morphism φ_K – namely,

$$[U : U_1(\mathfrak{P})] = \prod_{\mathfrak{p}|\mathfrak{P}}[\mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{p}}) : I_1(\mathfrak{p})] = \prod_{\mathfrak{p}|\mathfrak{P}}(p^{2f_\mathfrak{p}} - 1).$$

[21] In the sense that diagram (5.15) in the proof commutes.

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The same argument with the roles of $\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}}$ and $\mathcal{O}_{Y_1(\mathfrak{P})}$ exchanged shows that $\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}}$ is locally free of the same rank.

Furthermore, it follows from the construction of the duality isomorphism that (5.13) is induced by the trace morphism denoted $\mathrm{Tr}_{\varphi,\mathcal{K}_{Y/\mathcal{O}}}$ in [StaX, §0AWG], which may be viewed as a morphism

$$\mathrm{Tr}: \varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}} \longrightarrow \mathcal{K}_{Y/\mathcal{O}} \tag{5.14}$$

since $R^i\varphi_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}}$ vanishes for all $i > 0$.

Suppose then that U and U' are sufficiently small and that $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ is such that $g^{-1}U'g \subset U'$, giving étale morphisms $\rho_g : Y_{U'} \to Y_U$ and $\rho'_g : Y_{U'_1(\mathfrak{P})} \to Y_{U_1(\mathfrak{P})}$ such that the diagram

$$\begin{tikzcd}
Y_{U'_1(\mathfrak{P})} \arrow[r, "\rho'_g"] \arrow[d, "\varphi'"'] & Y_{U_1(\mathfrak{P})} \arrow[d, "\varphi"] \\
Y_{U'} \arrow[r, "\rho_g"] & Y_U
\end{tikzcd}$$

is Cartesian, and identifications $\rho_g^*\mathcal{K}_{Y_U/\mathcal{O}} = \mathcal{K}_{Y_{U'}/\mathcal{O}}$ and $\rho'^*_g\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}} = \mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}}$. The commutativity of the resulting diagram

$$\begin{tikzcd}
\rho_g^*\varphi_*\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}} \arrow[rr, "\rho_g^*(\mathrm{Tr})"] \arrow[d, "\wr"] & & \rho_g^*\mathcal{K}_{Y_U/\mathcal{O}} \arrow[d, equal] \\
\varphi'_*\rho'^*_g\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}} \arrow[r, equal] & \varphi'_*\mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}} \arrow[r, "\mathrm{Tr}'"] & \mathcal{K}_{Y_{U'}/\mathcal{O}}
\end{tikzcd}$$

then follows from [StaX, Lemma 0B6J] and implies that of

$$\begin{tikzcd}
\rho_g^*\varphi_*\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}} \arrow[r] \arrow[d, "\wr"] & \mathcal{H}om_{\mathcal{O}_{Y_{U'}}}(\rho_g^*\varphi_*\mathcal{O}_{Y_{U_1(\mathfrak{P})}}, \rho_g^*\varphi_*\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}}) \arrow[r] \arrow[d, "\wr"] & \mathcal{H}om_{\mathcal{O}_{Y_{U'}}}(\rho_g^*\varphi_*\mathcal{O}_{Y_{U_1(\mathfrak{P})}}, \rho_g^*\mathcal{K}_{Y_U/\mathcal{O}}) \arrow[d, equal] \\
\varphi'_*\mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}} \arrow[r] \arrow[d, equal] & \mathcal{H}om_{\mathcal{O}_{Y_{U'}}}(\rho_g^*\varphi_*\mathcal{O}_{Y_{U_1(\mathfrak{P})}}, \varphi'_*\mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}}) \arrow[r] & \mathcal{H}om_{\mathcal{O}_{Y_{U'}}}(\rho_g^*\varphi_*\mathcal{O}_{Y_{U_1(\mathfrak{P})}}, \mathcal{K}_{Y_{U'}/\mathcal{O}}) \\
\varphi'_*\mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}} \arrow[r] & \mathcal{H}om_{\mathcal{O}_{Y_{U'}}}(\varphi'_*\mathcal{O}_{Y_{U'_1(\mathfrak{P})}}, \varphi'_*\mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}}) \arrow[r] \arrow[u, "\wr"'] & \mathcal{H}om_{\mathcal{O}_{Y_{U'}}}(\varphi'_*\mathcal{O}_{Y_{U'_1(\mathfrak{P})}}, \mathcal{K}_{Y_{U'}/\mathcal{O}}), \arrow[u, "\wr"']
\end{tikzcd}$$

where the vertical maps are base-change isomorphisms, the horizontal maps on the left are the canonical morphisms, and the ones on the right are composition with $\rho_g^*(\mathrm{Tr})$ or Tr'. This in turn gives the Hecke-equivariance of the pairing – that is, the commutativity of

$$\begin{tikzcd}
\rho_g^*\varphi_*\mathcal{K}_{Y_{U_1(\mathfrak{P})}/\mathcal{O}} \otimes_{\mathcal{O}_{Y_{U'}}} \rho_g^*\varphi_*\mathcal{O}_{Y_{U_1(\mathfrak{P})}} \arrow[r] \arrow[d, "\wr"] & \rho_g^*\mathcal{K}_{Y_U/\mathcal{O}}, \arrow[d, equal] \\
\varphi'_*\mathcal{K}_{Y_{U'_1(\mathfrak{P})}/\mathcal{O}} \otimes_{\mathcal{O}_{Y_{U'}}} \varphi'_*\mathcal{O}_{Y_{U'_1(\mathfrak{P})}} \arrow[r] & \mathcal{K}_{Y_{U'}/\mathcal{O}}
\end{tikzcd} \tag{5.15}$$

where the top (resp. bottom) horizontal arrow is induced by Tr (resp. Tr′) and the vertical maps are the canonical isomorphisms. □

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Recall that for sufficiently small U, the finite flat Raynaud $(\mathcal{O}_F/\mathfrak{P})$-module scheme H over $\widetilde{Y}_0(\mathfrak{P})$ descends to $Y_0(\mathfrak{P})$, and hence so do the decompositions (5.1) and (5.2), giving

$$f_*\mathcal{O}_{Y_1(\mathfrak{P})} = \bigoplus_{\chi} \mathcal{R}_\chi \quad \text{and} \quad f_*\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}} = \bigoplus_{\chi} \mathcal{R}_\chi^{-1}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}},$$

where the line bundles $\mathcal{R}_\chi$ (now on $Y_0(\mathfrak{P})$) are indexed by the characters $\chi : (\mathcal{O}_F/\mathfrak{P})^\times \to \mathcal{O}^\times$. Theorem 5.3 thus implies the vanishing of $R^i\pi_{1,*}\mathcal{R}_\chi$ and $R^i\pi_{1,*}(\mathcal{R}_\chi^{-1}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}})$ for all $i > 0$ and characters χ, and Corollary 5.6 implies that $\pi_{1,*}\mathcal{R}_\chi$ and $\pi_{1,*}(\mathcal{R}_\chi^{-1}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}})$ are locally free, now of rank $\prod_{\mathfrak{p}|\mathfrak{P}}(p^{f_\mathfrak{p}}+1)$. Furthermore, the same argument as in the proof of its Hecke-equivariance shows that the pairing defined in the corollary is compatible with the natural action of $(\mathcal{O}_F/\mathfrak{P})^\times$ on $Y_1(\mathfrak{P})$, and hence respects the decompositions and restricts to perfect Hecke-equivariant pairings

$$\pi_{1,*}(\mathcal{R}_{\chi^{-1}}^{-1}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}}) \otimes_{\mathcal{O}_Y} \pi_{1,*}\mathcal{R}_\chi \longrightarrow \mathcal{K}_{Y/\mathcal{O}}$$

for all characters χ. We record this in the case of the trivial character:

Corollary 5.7. *The direct image sheaves $\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}}$ and $\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{P})}$ are locally free of rank $\prod_{\mathfrak{p}|\mathfrak{P}}(p^{f_\mathfrak{p}}+1)$ over $\mathcal{O}_Y$, and there is a perfect Hecke-equivariant pairing*

$$\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{P})/\mathcal{O}} \otimes_{\mathcal{O}_Y} \pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{P})} \longrightarrow \mathcal{K}_{Y/\mathcal{O}}.$$

Remark 5.8. The same argument as in the proof of Corollary 5.4 shows that we can replace π_1 by π_2 in the statement of Corollary 5.7. (In fact, the automorphism $\widetilde{w}_\mathfrak{P}$ descends to an automorphism $w_\mathfrak{P}$ of $Y_0(\mathfrak{P})$ such that $\pi_2 = \pi_1 \circ w_\mathfrak{P}$.) Note, however, that this argument does not allow one to replace π_1 by π_2 in the assertions for the twists by the bundles $\mathcal{R}_\chi$ (cf. Remark 5.5).

Suppose now that R is any (Noetherian) $\mathcal{O}$-algebra and consider the base extensions from $\mathcal{O}$ to R of the schemes Y and $Y_i(\mathfrak{P})$ for $i = 0, 1$ and morphisms φ, π_j for $j = 1, 2$, which we denote Y_R, etc. We then have the following:

Corollary 5.9. *For $\mathcal{E} = \mathcal{O}_{Y_1(\mathfrak{P})}$ and $\mathcal{K}_{Y_1(\mathfrak{P})/\mathcal{O}}$, the base-change morphisms*

$$(R^i\varphi_*\mathcal{E})_R \to R^i\varphi_{R,*}(\mathcal{E}_R)$$

are isomorphisms for all $i \geq 0$. In particular, $R^i\varphi_{R,*}(\mathcal{E}_R) = 0$ for all $i > 0$, $\varphi_{R,*}(\mathcal{E}_R)$ is locally free over $\mathcal{O}_{Y_R}$ of rank $\prod_{\mathfrak{p}|\mathfrak{P}}(p^{2f_\mathfrak{p}}-1)$, and there is a perfect Hecke-equivariant pairing

$$\varphi_{R,*}\mathcal{K}_{Y_1(\mathfrak{P})_R/R} \otimes_{\mathcal{O}_{Y_R}} \varphi_{R,*}\mathcal{O}_{Y_1(\mathfrak{P})_R} \longrightarrow \mathcal{K}_{Y_R/R}.$$

Furthermore, analogous assertions hold with $U_1(\mathfrak{P})$ replaced by $U_0(\mathfrak{P})$ and φ replaced by π_j for $j = 1$ and 2.

Proof. Since Y and $Y_1(\mathfrak{P})$ are flat over $\mathcal{O}$, the schemes Y_R and $Y_1(\mathfrak{P})$ are Tor independent over Y (in the sense of [StaX, Defn. 08IA]). The assertions in the case of $U_1(\mathfrak{P})$ are then immediate from [StaX, Lem. 08IB], Theorem 5.3, Corollary 5.6, and the compatibility with base-change of formation of dualizing sheaves for Cohen–Macaulay morphisms (see [Con00, Thm. 3.6.1] or Lemmas 0E2Y and 0E9W of [StaX]). The same argument applies for $U_0(\mathfrak{P})$, but using Corollaries 5.4 and 5.7 and Remark 5.8. □

Remark 5.10. Our method also yields an improvement on the cohomological vanishing results asserted in [ERX17a, Prop. 3.19]. More precisely, suppose that $\mathbf{k}, \mathbf{m} \in \mathbb{Z}^\Sigma$ and consider the automorphic line bundle $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}$ on $\widetilde{Y}$ (as defined in §5.4). Applying the same analysis as above to the line bundle $\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}$ on $\widetilde{Y}_{U_0(\mathfrak{P})}$ instead of $\mathcal{R}_\chi$, one obtains a filtration on $\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m},k}$ for which the graded pieces have fibres (relative to $\widetilde{\pi}_1$) described exactly as in the first formula of (5.12), but with the exponents[22] m_θ replaced

[22]Note that the m_θ in (5.12) are not the constituents of the weight vector $\mathbf{m}$ appearing in §5.4.

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by $n_{\sigma\theta}k_{\sigma\theta} - k_\theta$. (This refines a similar analysis carried out in [ERX17a, §4.9] for reduced fibres at generic points, whose description is completed by our Theorem 4.11; see Remark 4.12.) The same argument as in the proof of Theorem 5.3 therefore shows that if $n_{\sigma\theta}k_{\sigma\theta} \geq k_\theta$ for all $\theta \in \Sigma_{\mathfrak{P}}$, then $R^i\widetilde{\pi}_{1,*}(\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}})$ vanishes[23] for all $i > 0$. Furthermore, the same argument as in the proof of Corollary 5.9 shows that, for such $\mathbf{k}$, the base-change morphism $(\widetilde{\pi}_{1,*}\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}})_R \to \widetilde{\pi}_{1,*}(\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m},R})$ is an isomorphism and $R^i\widetilde{\pi}_{1,*}(\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m},R}) = 0$ for all $i > 0$. It follows also that if $\prod_\theta \theta(\mu)^{k_\theta + 2m_\theta}$ has trivial image in R for all $\mu \in U \cap \mathcal{O}_F^\times$, then $R^i\pi_{1,*}(\pi_2^*\mathcal{A}_{\mathbf{k},\mathbf{m},R}) = 0$ for all $i > 0$, and the base-change map $(\pi_{1,*}(\pi_2^*\mathcal{A}_{\mathbf{k},\mathbf{m},R}))_{R'} \to \pi_{1,*}(\pi_2^*\mathcal{A}_{\mathbf{k},\mathbf{m},R'})$ is an isomorphism for all Noetherian R-algebras R'. (Note, however, that the proof of Corollary 5.6 does not carry over, as it would require analogous results for $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}^{-1}\mathcal{K}_{\widetilde{Y}_0(\mathfrak{P})/\mathcal{O}}$, for which the corresponding inequalities do not necessary hold.)

We may also use the results above to construct integral models for Hilbert modular varieties of levels of $U_0(\mathfrak{P})$ and $U_1(\mathfrak{P})$ which are finite and flat over the smooth models of level U. Indeed, consider the Stein factorizations

$$Y_1(\mathfrak{P}) \to Y_1'(\mathfrak{P}) \to Y \quad \text{and} \quad Y_0(\mathfrak{P}) \to Y_0'(\mathfrak{P}) \to Y,$$

where $Y_1'(\mathfrak{P}) := \mathbf{Spec}(\varphi_*\mathcal{O}_{Y_1(\mathfrak{P})})$ and $Y_0'(\mathfrak{P}) := \mathbf{Spec}(\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{P})})$. Note that $Y_i(\mathfrak{P})_K = Y_i'(\mathfrak{P})_K$ for $i = 1, 2$ and that Corollaries 5.6 and 5.7 immediately imply the following:

Corollary 5.11. *The schemes $Y_0'(\mathfrak{P})$ and $Y_1'(\mathfrak{P})$ are finite and flat over Y; in particular, they are Cohen–Macaulay over $\mathcal{O}$.*

Note that the same conclusions apply to $Y_0''(\mathfrak{P}) := \mathbf{Spec}(\pi_{2,*}\mathcal{O}_{Y_0(\mathfrak{P})})$; in fact $w_{\mathfrak{P}}$ induces an isomorphism $Y_0''(\mathfrak{P}) \xrightarrow{\sim} Y_0'(\mathfrak{P})$ over Y. Note also that the natural map $Y_1'(\mathfrak{P}) \to Y$ factors through $Y_0'(\mathfrak{P})$. Furthermore, if U, U' are sufficiently small and $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ is such that $g^{-1}U'g \subset U$, then (with the obvious notation) the morphisms $\rho_g : Y_{U_i'(\mathfrak{P})} \to Y_{U_i(\mathfrak{P})}$ for $i = 0, 1$ induce morphisms $\rho_g' : Y'_{U_i'(\mathfrak{P})} \to Y'_{U_i(\mathfrak{P})}$ satisfying the usual compatibilities.

Finally, we note the following consequence of Corollary 5.11, pointed out to us by G. Pappas.

Corollary 5.12. *The normalization of Y in $Y_0(\mathfrak{P})_K$ relative to π_1 (or to π_2) is flat over Y.*

Proof. First, recall that $Y_0(\mathfrak{P})$ is normal, since, for example, it is regular in codimension one and Cohen–Macaulay. Therefore, it follows from [StaX, Lem. 035L] that $Y_0'(\mathfrak{P})$ is normal. Since $Y_0'(\mathfrak{P})$ is also finite over Y, it is the normalization of Y in $Y_0'(\mathfrak{P})_K = Y_0(\mathfrak{P})_K$ (relative to π_1), and its flatness over Y_U follows from Corollary 5.11.

The same argument applies with π_1 (resp. $Y_0'(\mathfrak{P})$) replaced by π_2 (resp. $Y_0''(\mathfrak{P})$). □

5.4. Hecke operators

We now specialize to the case of $\mathfrak{P} = \mathfrak{p}$ and consider the morphisms

$$\pi_1, \pi_2 : Y_0(\mathfrak{p}) \to Y,$$

Y (resp. $Y_0(\mathfrak{p})$) for Y_U (resp. $Y_{U_0(\mathfrak{p})}$) when U is fixed.

The pairing of Corollary 5.7 thus defines an isomorphism

$$\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}, \mathcal{K}_{Y/\mathcal{O}}), \tag{5.16}$$

under which the image of the unit section is the trace morphism

$$\mathrm{Tr} : \pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}} \longrightarrow \mathcal{K}_{Y/\mathcal{O}} \tag{5.17}$$

[23] The assertion in [ERX17a] is that the codimension of its support is at least $i + 1$; stated in these terms, the codimension is in fact infinite.

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(defined as in (5.14) since $R^i\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}} = 0$ for all $i > 0$). However, we have the morphism $\pi_1^*\mathcal{K}_{Y/\mathcal{O}} \to \mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}$ defined in (3.8), which by Proposition 3.7 extends the canonical isomorphism over $Y_0(\mathfrak{p})_K$. It follows that the composite of its direct image with Tr,

$$\pi_{1,*}\pi_1^*\mathcal{K}_{Y/\mathcal{O}} \longrightarrow \pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}} \longrightarrow \mathcal{K}_{Y/\mathcal{O}}, \tag{5.18}$$

extends the trace morphism over Y_K (associated to the finite flat morphism $\pi_{1,K}$). Thus, if $\mathcal{F}$ is any locally free sheaf over Y, we obtain an extension of the trace over Y_K to a morphism

$$\mathrm{tr}_{\mathcal{F}} : \pi_{1,*}\pi_1^*\mathcal{F} \longrightarrow \mathcal{F}$$

by tensoring (5.18) with $\mathcal{K}_{Y/\mathcal{O}}^{-1} \otimes \mathcal{F}$ and applying the projection formula.

Recall that we have the Kodaira–Spencer isomorphism $\mathcal{K}_{Y/\mathcal{O}} \cong \delta^{-1}\omega^{\otimes 2}$ on Y, and that Theorem 3.5 gives $\mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}} \cong \pi_1^*\omega \otimes \pi_2^*(\delta^{-1}\omega)$. Recall also that the universal isogeny on $\widetilde{Y}_0(\mathfrak{p})$ induces an isomorphism $\widetilde{\pi}_2^*\widetilde{\delta} \xrightarrow{\sim} p^{f_\mathfrak{p}}\widetilde{\pi}_1^*\widetilde{\delta}$ which descends to $Y_0(\mathfrak{p})$, so composing with multiplication by $p^{-f_\mathfrak{p}}$ yields an isomorphism $\pi_2^*\delta \xrightarrow{\sim} \pi_1^*\delta$, or equivalently $\pi_1^*\delta^{-1} \xrightarrow{\sim} \pi_2^*\delta^{-1}$. We may therefore view (5.16) as giving rise to an isomorphism

$$\begin{aligned}
\pi_{1,*}\mathcal{O}_{Y_0(\mathfrak{p})} &\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\pi_{1,*}(\pi_1^*\omega \otimes_{\mathcal{O}_{Y_0(\mathfrak{p})}} \pi_2^*(\delta^{-1}\omega)),\ \delta^{-1}\omega^{\otimes 2})\\
&\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\pi_{1,*}(\pi_1^*(\delta^{-1}\omega) \otimes_{\mathcal{O}_{Y_0(\mathfrak{p})}} \pi_2^*\omega),\ \delta^{-1}\omega^{\otimes 2})\\
&\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\delta^{-1}\omega \otimes_{\mathcal{O}_Y} \pi_{1,*}\pi_2^*\omega,\ \delta^{-1}\omega \otimes_{\mathcal{O}_Y} \omega)\\
&\xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\pi_{1,*}\pi_2^*\omega,\ \omega),
\end{aligned} \tag{5.19}$$

and we call the image of the unit section the *saving trace*

$$\mathrm{st} :\ \pi_{1,*}\pi_2^*\omega \longrightarrow \omega. \tag{5.20}$$

It follows from Proposition 3.7 and the definition of the saving trace that the diagram

$$\begin{tikzcd}
\pi_{1,*}\pi_2^*\omega \arrow[r] \arrow[d, "\mathrm{st}"'] & \pi_{1,*}\pi_1^*\omega \arrow[d, "\mathrm{tr}_\omega"] \\
\omega \arrow[r, "p^{f_\mathfrak{p}}"] & \omega
\end{tikzcd} \tag{5.21}$$

commutes, where the top arrow is the direct image of the descent of the morphism over $\widetilde{Y}_{U_0(\mathfrak{p})}$ induced by the universal isogeny.

More generally, consider the morphisms $\pi_1, \pi_2 : Y_0(\mathfrak{p})_R \to Y_R$ for any (Noetherian) $\mathcal{O}$-algebra R (omitting the subscript R from the notation for the degeneracy maps). It follows from Theorem 3.5 and Corollary 5.9 (for $\mathcal{E} = \mathcal{K}_{Y_0(\mathfrak{p})/\mathcal{O}}$ and $i = 0$) that $(\pi_{1,*}\pi_2^*\omega)_R \xrightarrow{\sim} \pi_{1,*}\pi_2^*(\omega_R)$ so the base-change of (5.20) defines a saving trace over R:

$$\mathrm{st}_R :\ \pi_{1,*}\pi_2^*(\omega_R) \longrightarrow \omega_R.$$

Note also that Theorem 3.5 and Corollary 5.9 imply that $R^i\pi_{1,*}(\pi_2^*\omega_R) \cong \omega_R^{-1}R^i\pi_{1,*}\mathcal{K}_{Y_0(\mathfrak{p})_R/R} = 0$ for all $i > 0$. More generally, it follows that if $\mathcal{F}$ is any locally free sheaf on Y_R, then

$$R^i\pi_{1,*}(\pi_1^*\mathcal{F} \otimes_{\mathcal{O}_{Y_0(\mathfrak{p})_R}} \pi_2^*\omega_R) \cong \mathcal{F} \otimes_{\mathcal{O}_{Y_R}} R^i\pi_{1,*}\pi_2^*\omega_R = 0$$

for all $i > 0$, and hence that

$$H^i(Y_0(\mathfrak{p})_R,\ \pi_1^*\mathcal{F} \otimes_{\mathcal{O}_{Y_0(\mathfrak{p})_R}} \pi_2^*\omega_R) \cong H^i(Y_R,\ \mathcal{F} \otimes_{\mathcal{O}_{Y_R}} \pi_{1,*}\pi_2^*\omega_R). \tag{5.22}$$

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For $(\mathbf{k},\mathbf{m}) \in \mathbb{Z}^\Sigma$, we let $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m}}$ denote the line bundle $\bigotimes_\theta(\mathcal{L}_\theta^{\otimes k_\theta} \otimes \mathcal{N}_\theta^{\otimes m_\theta})$ on $\widetilde{Y}$. Recall that if $\mu^{\mathbf{k}+2\mathbf{m}} := \prod_\theta \theta(\mu)^{k_\theta+2m_\theta}$ has trivial image in R for all $\mu \in U \cap \mathcal{O}_F^\times$, then $\widetilde{\mathcal{A}}_{\mathbf{k},\mathbf{m},R}$ descends to a line bundle on Y_R which we denote $\mathcal{A}_{\mathbf{k},\mathbf{m},R}$. We will now explain how to use the saving trace to construct a Hecke operator $T_\mathfrak{p}$ on $H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ for suitable $\mathbf{k},\mathbf{m} \in \mathbb{Z}^\Sigma$ (and in particular whenever $k_\theta \geq 1$ and $m_\theta \geq 0$ for all θ). Our main interest is in the case $i = 0$ and $F \neq \mathbb{Q}$, in which case

$$M_{\mathbf{k},\mathbf{m}}(U; R) := H^0(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$$

is the space of Hilbert modular forms of weight $(\mathbf{k},\mathbf{m})$ and level U with coefficients in R.

Recall that the universal isogeny $\psi : A_1 \to A_2$ over $\widetilde{Y}_0(\mathfrak{p})$ induces morphisms $\psi_\theta^* : \widetilde{\pi}_2^*\mathcal{P}_\theta \to \widetilde{\pi}_1^*\mathcal{P}_\theta$, and in turn $\widetilde{\pi}_2^*\mathcal{L}_\theta \to \widetilde{\pi}_1^*\mathcal{L}_\theta$ and $\wedge^2\psi_\theta^* : \widetilde{\pi}_2^*\mathcal{N}_\theta \to \widetilde{\pi}_1^*\mathcal{N}_\theta$ such that

$$\theta(\varpi_\mathfrak{p})\widetilde{\pi}_1^*\mathcal{L}_\theta \subset \psi_\theta^*(\widetilde{\pi}_2^*\mathcal{L}_\theta) \subset \widetilde{\pi}_1^*\mathcal{L}_\theta \quad \text{and} \quad \wedge^2\psi_\theta^*(\widetilde{\pi}_2^*\mathcal{N}_\theta) = \theta(\varpi_\mathfrak{p})\widetilde{\pi}_1^*\mathcal{N}_\theta.$$

Since $v_\mathcal{O}(\theta(\varpi_\mathfrak{p}))$ is independent of $\theta \in \Sigma_\mathfrak{p}$ and trivial for $\theta \notin \Sigma_\mathfrak{p}$, it follows that if

$$\sum_{\theta\in\Sigma_\mathfrak{p}} \min\{m_\theta, m_\theta + k_\theta - 1\} \geq 0, \tag{5.23}$$

then ψ_θ^* induces a morphism $\widetilde{\pi}_2^*\widetilde{\mathcal{A}}_{\mathbf{k}-\mathbf{1},\mathbf{m}} \to \widetilde{\pi}_1^*\widetilde{\mathcal{A}}_{\mathbf{k}-\mathbf{1},\mathbf{m}}$. Furthermore, its base-change to $\widetilde{Y}_0(\mathfrak{p})_R$ descends to a morphism

$$\pi_2^*\mathcal{A}_{\mathbf{k}-\mathbf{1},\mathbf{m},R} \to \pi_1^*\mathcal{A}_{\mathbf{k}-\mathbf{1},\mathbf{m},R} \tag{5.24}$$

over Y_R.

For $(\mathbf{k},\mathbf{m})$ satisfying (5.23), we define the endomorphism $T_\mathfrak{p}$ of $H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ as the composite

$$\begin{aligned} H^i(Y_R,\, \mathcal{A}_{\mathbf{k},\mathbf{m},R}) &\xrightarrow{\pi_2^*} H^i(Y_0(\mathfrak{p})_R,\, \pi_2^*\mathcal{A}_{\mathbf{k}-\mathbf{1},\mathbf{m},R} \otimes_{\mathcal{O}_{Y_0(\mathfrak{p})_R}} \pi_2^*\omega_R) \\ &\longrightarrow H^i(Y_0(\mathfrak{p})_R,\, \pi_1^*\mathcal{A}_{\mathbf{k}-\mathbf{1},\mathbf{m},R} \otimes_{\mathcal{O}_{Y_0(\mathfrak{p})_R}} \pi_2^*\omega_R) \\ &\xrightarrow{\sim} H^i(Y_R,\, \mathcal{A}_{\mathbf{k}-\mathbf{1},\mathbf{m},R} \otimes_{\mathcal{O}_{Y_R}} \pi_{1,*}\pi_2^*\omega_R) \\ &\xrightarrow{1\otimes \mathrm{st}_R} H^i(Y_R,\, \mathcal{A}_{\mathbf{k},\mathbf{m},R}), \end{aligned} \tag{5.25}$$

where the second arrow is induced by (5.24) and the third is (5.22).

It follows from the construction that the operator $T_\mathfrak{p}$ is compatible with base-change, in the sense that if $R \to R'$ is any homomorphism of $\mathcal{O}$-algebras, then the resulting diagram

$$\begin{tikzcd} H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R}) \otimes_R R' \arrow[r] \arrow[d, "T_\mathfrak{p}\otimes 1"'] & H^i(Y_{R'}, \mathcal{A}_{\mathbf{k},\mathbf{m},R'}) \arrow[d, "T_\mathfrak{p}"] \\ H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R}) \otimes_R R' \arrow[r] & H^i(Y_{R'}, \mathcal{A}_{\mathbf{k},\mathbf{m},R'}) \end{tikzcd}$$

commutes. Furthermore, it is straightforward to check that $T_\mathfrak{p}$ is compatible with the action of $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$, in the sense that if U and U' are sufficiently small and $g^{-1}Ug \subset U'$, then the resulting diagram

$$\begin{tikzcd} H^i(Y'_R, \mathcal{A}'_{\mathbf{k},\mathbf{m},R}) \arrow[r] \arrow[d, "T_\mathfrak{p}"'] & H^i(Y_R, \rho_g^*\mathcal{A}'_{\mathbf{k},\mathbf{m},R}) \arrow[r, "\sim"] & H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R}) \arrow[d, "T_\mathfrak{p}"] \\ H^i(Y'_R, \mathcal{A}'_{\mathbf{k},\mathbf{m},R}) \arrow[r] & H^i(Y_R, \rho_g^*\mathcal{A}'_{\mathbf{k},\mathbf{m},R}) \arrow[r, "\sim"] & H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R}) \end{tikzcd}$$

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commutes, where $\rho_g : Y_R \to Y'_R := Y_{U',R}$ and $\rho_g^*\mathcal{A}'_{\mathbf{k},\mathbf{m},R} \xrightarrow{\sim} \mathcal{A}_{\mathbf{k},\mathbf{m},R}$ are defined in §2.3. Finally, the commutativity of (5.21) implies that $T_\mathfrak{p}$ coincides with the classical Hecke operator so denoted on the space $H^0(Y_K, \mathcal{A}_{\mathbf{k},\mathbf{m},K})$ of Hilbert modular forms of level U and weight $(\mathbf{k},\mathbf{m})$ over K.

We record the result as follows:

Theorem 5.13. *For sufficiently small open compact subgroups $U \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$, Noetherian $\mathcal{O}$-algebras R, and $\mathbf{k},\mathbf{m} \in \mathbb{Z}^\Sigma$ such that $\mu^{\mathbf{k}+2\mathbf{m}}$ has trivial image in R for all $\mu \in U \cap \mathcal{O}_F^\times$ and*

$$\sum_{\theta\in\Sigma_\mathfrak{p}} \min\{m_\theta, m_\theta + k_\theta - 1\} \geq 0,$$

the operators $T_\mathfrak{p}$ defined on $H^i(Y_{U,R}, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ by (5.25) are compatible with base-change and the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$, and coincide with the classical Hecke operator $T_\mathfrak{p}$ if R is a K-algebra and $i = 0$.

Remark 5.14. The inequality (5.23) is needed to ensure integrality, but results for more general $\mathbf{k}$, $\mathbf{m}$ follow from twisting arguments. Indeed, for fixed $\mathbf{k}$ and varying $\mathbf{m}$, the modules $H^i(Y_{U,R}, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ are isomorphic for sufficiently small U. The isomorphisms are not canonical, but one can keep track of its twisting effect on the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ (see, for example, [DS23, Lemma 4.6.1] and [Dia23, Prop. 3.2.2]), and the same considerations apply to its effect on $T_\mathfrak{p}$. In particular, if R has characteristic zero (so $\mathbf{k} + 2\mathbf{m}$ is parallel), then we recover the same integrality conditions as in [FP23] if p is unramified in F.

However, if R has finite characteristic, then there is no restriction on $\mathbf{k}$ and $\mathbf{m}$, and the critical case for $i = 0$ becomes $\mathbf{m} = \mathbf{0}$. Note that the hypothesis (5.23) is then equivalent to $k_\theta \geq 1$ for all $\theta \in \Sigma_\mathfrak{p}$, which is known to hold in the cases of primary interest thanks to the main result of [DK23] and [DDW24, Prop. 1.13] (see also [Dia23, Thm. D]).

Note also that if $\mathbf{m} = \mathbf{0}$, then the inequality (5.23) becomes an equality. By contrast, if the inequality is strict one finds that $T_\mathfrak{p} = 0$ if R is a k-algebra, and more generally that $T_\mathfrak{p}$ is nilpotent if $p^n R = 0$.

Remark 5.15. Interchanging the roles of π_1 and π_2, one can similarly define a saving trace $\mathrm{st}' : \pi_{2,*}\pi_1^*\omega \to \omega$ (without using the isomorphism $\pi_2^*\delta \cong \pi_1^*\delta$). A construction analogous to the one above then defines an operator $T'_\mathfrak{p}$ on $H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ whenever $\sum_{\theta\in\Sigma_\mathfrak{p}} \max\{m_\theta, m_\theta + k_\theta - 1\} \leq 0$, and the analogue of the commutativity of (5.21) is then that of the diagram

$$\begin{tikzcd} \pi_{2,*}\pi_2^*\omega \arrow[rr] \arrow[dr, "\mathrm{tr}'_\omega"'] & & \pi_{2,*}\pi_1^*\omega \arrow[dl, "\mathrm{st}'"] \\ & \omega, & \end{tikzcd} \tag{5.26}$$

where tr'_ω extends the trace relative to $\pi_{2,K}$.

Note that if $\mathbf{k} = \mathbf{1}$ and $\mathbf{m} = \mathbf{0}$ (or more generally $k_\theta = 1$ for all $\theta \in \Sigma_\mathfrak{p}$ and $\sum_{\theta\in\Sigma_\mathfrak{p}} m_\theta = 0$), then both $T_\mathfrak{p}$ and $T'_\mathfrak{p}$ are defined. Using the commutativity of (5.21) and (5.26), it is straightforward to check that $T_\mathfrak{p}$ is the composite of $T'_\mathfrak{p}$ with the automorphism of $H^i(Y_R, \omega_R)$ induced by the map $\sigma_\mathfrak{p} : Y \xrightarrow{\sim} Y$ obtained by descent from $\underline{A} \mapsto \underline{A} \otimes_{\mathcal{O}_F} \mathfrak{p}^{-1}$, together with the isomorphism $\sigma_\mathfrak{p}^*\omega = \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{p}) \otimes_\mathbb{Z} \omega \xrightarrow{\sim} \omega$ defined by $p^{-f_\mathfrak{p}}$.

Remark 5.16. The main results of this paper, in particular Theorem 3.2.1 and Theorem 5.3.1 (and its corollaries), are extended in [Dia24, §5] to toroidal compactifications. Furthermore, the effect of the saving trace on q-expansions is given by Proposition 5.3.1 of [Dia24], leading to that of $T_\mathfrak{p}$ in Proposition 6.8.1. In particular, this implies the commutativity of the Hecke operators $T_\mathfrak{p}$ (whenever defined) on $H^0(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ for varying $\mathfrak{p}$ in S_p.

By contrast, checking this directly from their construction leads to a formidable diagram whose commutativity ultimately seems to require analogues of some of the results in §5.3 for direct images under the projections $Y_0(\mathfrak{p}\mathfrak{p}') \to Y_0(\mathfrak{p}')$. While these might follow from arguments along similar lines to those for direct images under $Y_0(\mathfrak{p}) \to Y$ or from their compatibility with more general base-changes

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than we considered (more precisely, with respect to $Y_0(\mathfrak{p})$ instead of $\mathcal{O}$), we have not carried these out. Consequently, we have not shown the commutativity of the operators $T_\mathfrak{p}$ on $H^i(Y_R, \mathcal{A}_{\mathbf{k},\mathbf{m},R})$ for $i > 0$.

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