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Letter to the Editor

Weibel instabilities in dense quantum plasmas

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Abstract. The quantum effect on the Weibel instability in an unmagnetized plasma is presented. Our analysis shows that the quantum effect tends to stabilize the Weibel instability in the hydrodynamic regime, whereas it produces a new oscillatory instability in the kinetic regime. A novel effect called the quantum damping, which is associated with the Landau damping, is disclosed. The new quantum Weibel instability may be responsible for the generation of non-stationary magnetic fields in compact astrophysical objects as well as in the forthcoming intense laser–solid density plasma interaction experiments.

The Weibel instability [1] arises in a variety of plasmas including fusion plasmas, both magnetic and inertial confinement, space/astrophysical plasmas, as well as in plasmas created by high-intensity free-electron X-ray laser pulses. The Weibel instability is of significant interest since it generates quasi-stationary magnetic fields, which can account for seed magnetic fields in laboratory [2] and astrophysical plasmas [3]. The purely growing Weibel instability in a non-Maxwellian plasma is excited by the anisotropy of the electron distribution function. The linear and nonlinear aspects of the Weibel instability in classical electron–ion plasmas are fully understood [4].

However, in dense plasmas, such as those in compact astrophysical objects (e.g. the interior of the white dwarfs, neutron stars/magnetars, supernovae), as well as in the next-generation intense laser–solid density plasma experiments [5], in nanowires and in micromechanical systems, one notices the importance of quantum electron tunneling effects [6] at nanoscales. In dense quantum plasmas, the de Broglie wavelength associated with the plasma particles is comparable to the inter-particle spacing, and one uses either the Wigner–Maxwell equations [7] or quantum hydrodynamical models [8] to investigate numerous collective interactions [5]. To study quantum effects in plasmas, Klimontovich and Silin [9] derived a general kinetic equation for the quantum plasma, and linearizing that equation they obtained linear dispersion relations for transverse electromagnetic (EM) as well as longitudinal waves. The latter have also been studied by Pines [10], who reported the dispersion of electron plasma oscillations involving the Bohm potential [6] that causes electron tunneling.

In this letter, we present new aspects of the Weibel instability in an unmagnetized quantum plasma. For our purposes, we use the dispersion relation $k^2c^2/\omega^2 = \varepsilon^{\mathrm{tr}}$

for the EM waves, with the following transverse dielectric permeability [9, 11]

$$\varepsilon^{\mathrm{tr}} = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} + \sum_\alpha \frac{2\pi q_\alpha^2}{\hbar\omega^2} \cdot \int \frac{d^3p}{\omega - \mathbf{k}\cdot\mathbf{v}} v_\perp^2 \left[f_{0\alpha}\left(\mathbf{p} + \frac{\hbar\mathbf{k}}{2}\right) - f_{0\alpha}\left(\mathbf{p} - \frac{\hbar\mathbf{k}}{2}\right)\right], \quad (1)$$

where $\mathbf{k}$ is the wavevector, c is the speed of light in vacuum, ω is the wave frequency, $\omega_{\mathrm{p}\alpha}$ is the plasma frequency of the particle species α, q_α is the charge, $\hbar$ is the Planck constant divided by 2π, $f_{0\alpha}$ is the equilibrium distribution function, $\mathbf{p}_\alpha$ is the momentum. In the non-relativistic limit, we have $\mathbf{p}_\alpha = m_{0\alpha}\mathbf{v}$, where $m_{0\alpha}$ is the rest mass and $\mathbf{v}$ is the velocity vector. Using the notation $\mathbf{v} + \hbar\mathbf{k}/2m_{0\alpha} \to \mathbf{v}$ in the first integral and $\mathbf{v} - \hbar\mathbf{k}/2m_{0\alpha} \to \mathbf{v}$ in the second, (1) is rewritten in the form

$$\varepsilon^{\mathrm{tr}} = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}_\alpha}^2}{\omega^2} + \sum_\alpha \frac{2\pi q_\alpha^2}{\hbar\omega^2} \times \int d^3v\, v_\perp^2 f_{0\alpha}(v)\left(\frac{1}{\omega + \hbar k^2/2m_{0\alpha} - \mathbf{k}\cdot\mathbf{v}} - \frac{1}{\omega - \hbar k^2/2m_{0\alpha} - \mathbf{k}\cdot\mathbf{v}}\right). \quad (2)$$

Let us choose an anisotropic distribution function

$$f_{0\alpha} = n_{0\alpha} A_\alpha \exp\left(-\frac{m_{0\alpha}v_\perp^2}{2T_{\alpha\perp}} - \frac{m_{0\alpha}v_\parallel^2}{2T_{\alpha\parallel}}\right), \quad (3)$$

where $n_{0\alpha}$ is the equilibrium density, $T_{\alpha\perp}$ ($T_{\alpha\parallel}$) is the temperature transverse (parallel) to $\mathbf{k}$. The above distribution function can also be expressed as

$$f_{0\alpha} = n_{0\alpha} f_\alpha(v_\perp^2)\delta(v_z) \quad \text{or} \quad f_{0\alpha} = n_{0\alpha} f_\alpha(v_\perp^2)\delta(v_z - u_{0z}), \quad (4)$$

where u_{0z} is the equilibrium drift along the z-axis in a Cartesian coordinate system.

Focusing on transverse EM waves propagating along the z-axis, we can take $\mathbf{k} = (0, 0, k)$ and $\mathbf{k}\cdot\mathbf{v} = kv_z$, and introduce

$$\frac{1}{2}m_{0\alpha}\int d^3v\, v_\perp^2 f_\alpha(v_\perp, v_z) = m_{0\alpha}n_{0\alpha}\left\langle \frac{v_\perp^2}{2}\right\rangle \int dv_z\, f_{0\alpha}(v_z), \quad (5)$$

to rewrite (2) as

$$\varepsilon^{tr} = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}_\alpha}^2}{\omega^2} + \sum_\alpha \frac{\omega_{\mathrm{p}_\alpha}^2}{\omega^2\hbar}\frac{m_{0\alpha}\langle v_\perp^2\rangle}{2} \int dv_z\, f_{0\alpha}(v_z)\left(\frac{1}{\omega_+ - kv_z} - \frac{1}{\omega_- - kv_z}\right), \quad (6)$$

where $\omega_\pm = \omega \pm \hbar k^2/2m_{\alpha 0}$. We note that in the case of the Maxwellian distribution we would have $(1/2)m_{0\alpha}n_{0\alpha}\langle v_\perp^2\rangle = n_{0\alpha}T_{\alpha\perp}$.

We now consider some special cases for an electron plasma, with fixed ion background. First, choosing $f_{0\alpha} = \delta(v_z)$, we obtain

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\left(1 + \frac{k^2\langle v_\perp^2\rangle}{2(\omega^2 - \eta^2)}\right), \quad (7)$$

where $\omega_{\mathrm{pe}} = (4\pi n_{0\mathrm{e}}e^2/m_{0\mathrm{e}})^{1/2}$ is the electron plasma frequency and $\eta = \hbar k^2/2m_{0\mathrm{e}}$. Supposing that $\omega^2 \ll \eta^2$, we obtain from (7)

$$\omega^2 = k^2c^2 + \omega_{\mathrm{p}}^2\left(1 - \frac{k^2\langle v_\perp^2\rangle}{2\eta^2}\right). \quad (8)$$

Equation (8) predicts a purely growing quantum instability if $2\langle v_\perp^2\rangle m_{0\mathrm{e}}^2 > \hbar^2k^2(1 + k^2c^2/\omega_{\mathrm{pe}}^2)$. It should be stressed that if in the expression (6) $f_0(v_z)$ is the Maxwellian

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distribution function of the form $f_0(v_z) = (m_{0\mathrm{e}}/2\pi T_{\mathrm{e}\|})^{1/2} \exp(-m_{0\mathrm{e}} v_z^2/2T_{\mathrm{e}\|})$, then by assuming $|\omega - kv_z| \ll \hbar k^2/2m_{0\mathrm{e}}$ in the integral of (6), one would obtain the dispersion relation (8), which shows that this expression does not depend on the parallel electron temperature.

Equation (7) indicates that the quantum effect can stabilize the Weibel instability for short wavelengths. We observe that (7) has four roots, two of which are the low frequencies ($\omega \ll \omega_{\mathrm{pe}}$). From (7) we obtain

$$\omega^2 = \eta^2 - \frac{\omega_{\mathrm{pe}}^2 k^2 \langle v_\perp^2 \rangle}{2(\omega_{\mathrm{pe}}^2 + k^2 c^2 - \eta^2)}. \tag{9}$$

In order to estimate the wavelengths for which the quantum effect can stabilize the Weibel instability, we suppose that $\omega_{\mathrm{pe}}^2 \sim k^2 c^2$. This leads to a condition of stabilization from (9)

$$\frac{\hbar^2 k^2}{m_{0\mathrm{e}}^2} > \langle v_\perp^2 \rangle \quad \text{or} \quad \frac{T_{\mathrm{e}\perp}}{2m_{0\mathrm{e}}}.$$

Next, we study the kinetic quantum effect in plasmas. In the following, we assume that the distribution function $f_0(v_z)$ is of the form

$$f_0(v_z) = \frac{1}{v_\| \sqrt{\pi}} \exp\left(-\frac{v_z^2}{v_\|^2}\right), \tag{10}$$

where $v_\| = (2T_{\mathrm{e}\|}/m_{0\mathrm{e}})^{1/2}$. Introducing the dimensionless quantities

$$u = \frac{v_z}{v_\|}, \quad z_\pm = \frac{\omega \pm \eta}{kv_\|},$$

we can express (6) as

$$\varepsilon^{\mathrm{tr}} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} + \frac{\omega_{\mathrm{p}}^2}{\omega^2} \frac{\langle m_{0\mathrm{e}} v_\perp^2/2 \rangle}{\hbar k v_\|} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} du\, e^{-u^2} \left(\frac{1}{z_+ - u} - \frac{1}{z_- - u} \right). \tag{11}$$

Here the integral $(1/\sqrt{\pi}) \int (z-u)^{-1}\, du \exp(-u^2) = -i\sqrt{\pi} w(z)$, where

$$w(z) = \exp(-z^2) \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z \exp(t^2)\, dt \right). \tag{12}$$

The function $w(z)$ is related with the function $I_+(z)$ through

$$\frac{I_+(z)}{z} = -i\sqrt{\pi} w(z), \tag{13}$$

and the asymptotes of $I_+(z)$ are

$$I_+(z) = \begin{cases} 1 + \dfrac{1}{2z^2} + \dfrac{3}{z^4} + \cdots - i\sqrt{\pi} z \exp(-z^2) & \text{for } |z| \gg 1 \text{ and } |\mathrm{Im}\, z| \ll 1, \\ -i\sqrt{\pi} z (1 - z^2) + 2z^2 & \text{for } |z| \ll 1. \end{cases} \tag{14}$$

We now rewrite the expression (11) as

$$\varepsilon^{\mathrm{tr}} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} + \frac{\omega_{\mathrm{pe}}^2}{2\omega^2} \frac{m_{0\mathrm{e}} \langle v_\perp^2 \rangle}{\hbar k v_\|} \left(\frac{I_+(z_+)}{z_+} - \frac{I_-(z_-)}{z_-} \right). \tag{15}$$

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Consider the case $z_{\pm} \gg 1$, so that (15) can be written as

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{\rm pe}^2}{\omega^2}\left(1 + \frac{k^2\langle v_\perp^2\rangle}{2(\omega^2-\eta^2)}\right) + 2i\frac{\sqrt{\pi}\omega_{\rm pe}^2}{\omega^2}\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}\sinh\left(\frac{\hbar\omega}{2T_{\rm e\parallel}}\right)\exp\left(-\frac{\omega^2+\eta^2}{k^2v_\parallel^2}\right). \tag{16}$$

We specifically note here that if $\sinh(\hbar\omega/2T_{\rm e\parallel}) \gtrsim 1$, then we obtain the result which we call the quantum damping. In the opposite case, that is, $\hbar\omega/T_{\rm e\parallel} \ll 1$, we obtain the classical damping.

In the low-frequency limit, namely $\omega^2 \ll \omega_{\rm pe}^2$, (16) admits solutions of the form $\omega = \omega_{\rm r} + i\omega_{\rm i}$, where the real and imaginary parts of the frequency are given by, respectively,

$$\omega_{\rm r} = \left(\eta^2 - \frac{\omega_{\rm pe}^2}{2(\omega_{\rm pe}^2+k^2c^2)}\langle v_\perp^2\rangle k^2\right)^{1/2}, \tag{17}$$

and for the quantum Landau damping (QLD)

$$\omega_{\rm i} = -\sqrt{\pi}\left(\frac{\omega_{\rm pe}^2}{\omega_{\rm pe}^2+k^2c^2}\right)^2\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}\frac{k^2\langle v_\perp^2\rangle}{2\omega_r}\sinh\left(\frac{\hbar\omega_r}{2T_{\rm e\parallel}}\right)\exp\left(-\frac{\omega^2+\eta^2}{k^2v_\parallel^2}\right). \tag{18}$$

Next, for $z_{\pm} \ll 1$, we have $I_+(z) = -i\sqrt{\pi}z(1-z^2) + 2z^2$, and for the last term in (15) we obtain $z_+^{-1}I_+(z_+) - z_-^{-1}I_+(z_-) = (2\eta/kv_\parallel) + 2i\sqrt{\pi}(\eta/kv_\parallel)\omega/kv_\parallel$. In such an approximation, we obtain

$$\varepsilon^{\rm tr} = 1 - \frac{\omega_{\rm pe}^2}{\omega^2}\left(1 - \frac{m_{0\rm e}\langle v_\perp^2\rangle}{2T_{\rm e\parallel}}\right) + i\sqrt{\pi}\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2T_{\rm e\parallel}}\frac{\omega_{\rm pe}^2}{\omega k v_\parallel},$$

or

$$k^2c^2 + \omega_{\rm pe}^2\left(1 - \frac{m_{0\rm e}\langle v_\perp^2\rangle}{2T_{\rm e\parallel}}\right) - i\sqrt{\pi}\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2T_{\rm e\parallel}}\frac{\omega_{\rm pe}^2\omega}{kv_\parallel} = 0, \tag{19}$$

which admits the solution

$$\omega = -i\frac{2}{\sqrt{\pi}}\frac{T_{\rm e\parallel}}{m_{0\rm e}\langle v_\perp^2\rangle}\frac{kv_\parallel}{\omega_{\rm pe}^2}\left[k^2c^2 + \omega_{\rm pe}^2\left(1 - \frac{m_{0\rm e}\langle v_\perp^2\rangle}{2T_{\rm e\parallel}}\right)\right]. \tag{20}$$

Equation (20) admits a purely growing instability if

$$\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2T_{\rm e\parallel}} > \frac{k^2c^2+\omega_{\rm pe}^2}{\omega_{\rm pe}^2}. \tag{21}$$

Finally, we consider the range of frequencies $\omega_+ \gg kv_z \gg \omega_-$ (Fig. 1), or $|z_+| \gg 1$ and $|z_-| \ll 1$. Clearly such situation can be realized in a quantum case alone. In this case, (15) reduces to

$$\varepsilon^{\rm tr} = 1 - \frac{\omega_{\rm pe}^2}{\omega^2} + \frac{\omega_{\rm pe}^2}{\omega^2}\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}\left(\frac{1}{z_+} - 2z_- + i\sqrt{\pi}\right) \tag{22}$$

which yields, in the first approximation,

$$\varepsilon^{\rm tr} = 1 - \frac{\omega_{\rm pe}^2}{\omega^2}\left(1 - i\sqrt{\pi}\frac{m_{0\rm e}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}\right). \tag{23}$$

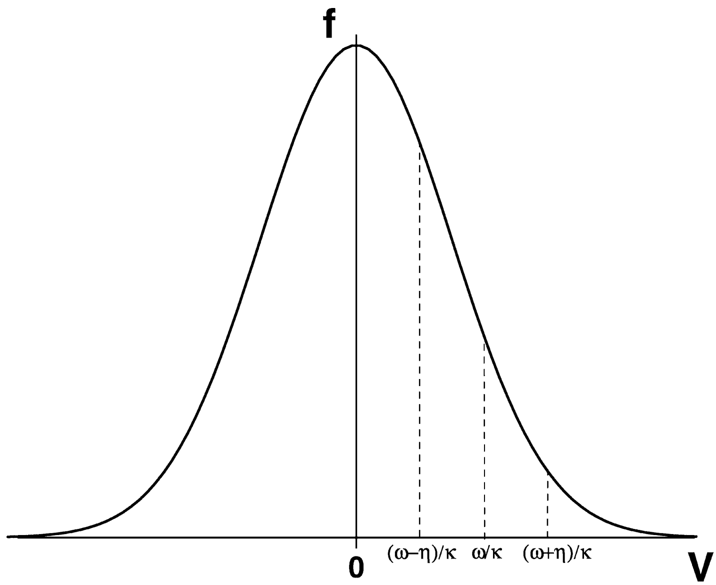

Figure 1. Ranges of interactions of the electrons with a wave for a distribution over velocities.

Accordingly, in this case, the dispersion relation reads

$$\omega^2 = \omega_{\mathrm{pe}}^2\left(1 - i\sqrt{\pi}\frac{m_{0\mathrm{e}}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}\right) + k^2c^2. \tag{24}$$

As is well known, the classical Weibel instability is a purely growing instability. We now show that the quantum effect leads to a new type of Weibel instability, which we refer to as the Weibel oscillatory instability. To this end, we rewrite (24) as

$$\omega = \pm\sqrt{\omega_{\mathrm{pe}}^2 + k^2c^2}(1+Q^2)^{1/4}\left(\cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}\right), \tag{25}$$

where $\varphi = \mathrm{arctg}\, Q$ and

$$Q = \frac{\sqrt{\pi}\omega_{\mathrm{pe}}^2}{(\omega_{\mathrm{pe}}^2 + k^2c^2)}\frac{m_{0\mathrm{e}}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}.$$

Let us consider two cases. First, for $Q \ll 1$ and $\varphi \sim Q$ the real and imaginary parts of the frequencies, deduced from (25), are

$$\omega_{\mathrm{r}} \approx \pm\sqrt{\omega_{\mathrm{pe}}^2 + k^2c^2}, \tag{26}$$

$$\omega_{\mathrm{i}} = \pm\frac{\sqrt{\pi}\omega_{\mathrm{pe}}^2}{\sqrt{\omega_{\mathrm{pe}}^2 + k^2c^2}}\frac{m_{0\mathrm{e}}\langle v_\perp^2\rangle}{2\hbar k v_\parallel}. \tag{27}$$

More vigorous effect is obtained when $Q \gg 1$. Namely, in this case $\varphi \sim \pi/2$, and we have

$$\omega_{\mathrm{r}} = \omega_{\mathrm{i}} = 0.7\sqrt{\omega_{\mathrm{pe}}^2 + k^2c^2}\,Q^{1/2}. \tag{28}$$

To summarize, we have investigated the quantum mechanical effects on the Weibel instability in an unmagnetized plasma containing electron energy anisotropy. It is shown that the quantum effect stabilizes the Weibel instabilities, but a new type of Weibel instability, the quantum Weibel instabilities, are found. These instabilities describe the quantum wave excitation with slow damping by the quantum Landau mechanism. We have demonstrated the possibility of a novel oscillatory Weibel instability, which is not found in [12, 13]. The Weibel instability reported here may be responsible for the generation of non-stationary magnetic fields in dense astrophysical objects, as well as in the next-generation intense laser–solid density plasma experiments. The random walk of the electrons in non-stationary magnetic fields can produce anomalous electron transport at quantum scales in dense plasmas.

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