An infinite integral formula

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§ 1. The object of this note is to discuss the formula

$$f(x+z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s)(-z)^s \{D^s f(x)\}\, ds, \tag{1}$$

the integral being supposed convergent for certain ranges of values of x and z. The contour is such that the poles of $\Gamma(-s)$ lie to its right and the other poles of the integrand to its left. It will be seen that all the Pincherle-Mellin-Barnes integrals are particular cases of this formula.

$D^s f(x)$ denotes the sth differential coefficient with respect to x of $f(x)$, and this must be defined for all values of s. A summary and comparison of the various methods of generalising $D^s f(x)$ for all values of s has been given by Ferrar.[1] For the purposes of this formula, any definition may be adopted which is consistent with the Nth differential coefficient when $s = N$, an integer. Hence, it is usually sufficient to write down an expression for the Nth differential coefficient, and merely remove the condition that N shall be an integer.

To establish the formula, we complete the contour with a large semi-circle to the right, and equate the integral to the sum of the residues of the integrand at the poles of $\Gamma(-s)$. The residue of $\Gamma(-s)$ at $s = N$ is

$$\frac{\cos N\pi}{\Gamma(1+N)}.$$

Hence the integral is equal to

$$\sum_{N=0}^{\infty} \frac{\cos N\pi}{\Gamma(1+N)} (-z)^N D^N f(x) = f(x+z). \tag{2}$$

It is supposed that the integral round the semi-circle tends to zero as the radius tends to infinity. This is in general equivalent to assuming

[1] *Proc. Roy. Soc., Edin*, 48 (1927-28), 92.

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the convergence of the series (2), which may be assumed for certain values of z.

§ 2. As an illustration let

$$f(x) = F(\alpha, \beta; \gamma; x).$$

If N is an integer we have

$$D^N f(x) = \frac{\Gamma(\alpha + N)\,\Gamma(\beta + N)\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma + N)} F(\alpha + N, \beta + N; \gamma + N; x).$$

Retaining this definition when N is not an integer, we have

$$F(\alpha, \beta; \gamma; x + z) =$$
$$\frac{1}{2\pi i}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s F(\alpha+s, \beta+s; \gamma+s; x)\,ds. \qquad (3)$$

This integral was given by Whittaker.[1] The ordinary Pincherle-Mellin-Barnes formula[2] is obtained by putting $x = 0$, or, in slightly different form, by putting $x = 1$ and writing

$$F(\alpha + s, \beta + s; \gamma + s; 1) = \frac{\Gamma(\gamma + s)\,\Gamma(\gamma - \alpha - \beta - s)}{\Gamma(\gamma - \alpha)\,\Gamma(\gamma - \beta)}.$$

§ 3. If $f(x)$ be taken as the Legendre function $P_n(x)$, we may obtain the Nth differential coefficient from the formula—

$$\frac{d^N}{dx^N} P_n(x) = (x^2 - 1)^{-N/2} P_n^N(x),$$

and hence (1) gives

$$P_n(x + z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i}(x^2 - 1)^{-s/2}\,\Gamma(-s)\,(-z)^s P_n^s(x)\,ds. \qquad (4)$$

§ 4. For Bessel functions we may use the formula

$$\frac{d^N}{d(z^2/2)^N}\{z^{-n} J_n(z)\} = (-)^N z^{-N-n} J_{N+n}(z),$$

that is

$$\frac{d^N}{dx^N}\{(2x)^{-n/2} J_n([2x]^{\frac{1}{2}})\} = (-)^N (2x)^{-\frac{1}{2}(N+n)} J_{N+n}([2x]^{\frac{1}{2}}),$$

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[1] *Proc. Edin. Math. Soc.* (2), 3 (1931), 189.

[2] *Cf.* Whittaker and Watson, *Modern Analysis* (Cambridge, 1927), §14·5.

and hence we have

$$2^{-n/2}(x+z)^{-n/2} J_n([2x+2z]^{\frac{1}{2}}) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s)\, z^s (2x)^{-\frac{1}{2}(n+s)} J_{n+s}([2x]^{\frac{1}{2}})\, ds. \qquad (5)$$

Putting $x = 0$, and writing $z = \frac{1}{2}u^2$, we have

$$J_n(u) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\,(u/2)^{n+2s}}{\Gamma(n+s+1)}\, ds.$$

§ 5. From the integral representation of the Confluent Hypergeometric function

$$W_{k,m}(z) = \frac{e^{-z/2} z^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t/z)^{k+m-\frac{1}{2}} e^{-t}\, dt,$$

it can be deduced that

$$D^N\{e^{x/2}x^{m-\frac{1}{2}}W_{k,m}(x)\} = \frac{\Gamma(k+m+\frac{1}{2})}{\Gamma(k+m+\frac{1}{2}-N)}\, e^{x/2}x^{m-\frac{1}{2}-N/2}W_{k-N/2,\,m-N/2}(x).$$

Hence (1) gives

$$W_{k,m}(x+z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(-s)\,\Gamma(k+m+\frac{1}{2})}{\Gamma(k+m+\frac{1}{2}-s)}\, e^{-z/2}(-z)^s x^{m-\frac{1}{2}-s/2}(x+z)^{\frac{1}{2}-m} W_{k-s/2,\,m-s/2}(x)\, ds. \qquad (6)$$

§ 6. The formula (1) may be applied to various types of generalised Hypergeometric functions. As an example we have,

$$3F_2\left[\begin{matrix}\alpha, \beta, \gamma, \\ \delta, \epsilon,\end{matrix}\; x+z\right] = \frac{1}{2\pi i}\frac{\Gamma(\delta)\,\Gamma(\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_{-\infty i}^{\infty i}\frac{\Gamma(-s)\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma+s)}{\Gamma(\delta+s)\,\Gamma(\epsilon+s)}(-z)^s {}_3F_2\left[\begin{matrix}\alpha+s, \beta+s, \gamma+s, \\ \delta+s, \epsilon+s,\end{matrix}\; x\right] ds. \qquad (7)$$

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