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A MARKOV-MODULATED DIFFUSION MODEL FOR ENERGY HARVESTING SENSOR NODES

Omer H. Abdelrahman
Department of Electrical and Electronic Engineering,
Intelligent Systems and Networks,Imperial College,
London SW7 2BT, UK
E-mail: o.abd06@imperial.ac.uk

This paper presents a probability model of an energy-harvesting wireless sensor node, with the objective of linking quality of sensed data to energy consumption and self-sustainability. The model departs from the common energy discretization framework used in the literature, and instead uses a diffusion process modulated by discrete packet arrival and transmission processes for the detailed representation of renewable energy supply, consumption and storage. An analytical–numerical method is developed to compute the average time until the node experiences an outage, due to lack of energy, for a given workload and ambient energy characteristics, battery capacity and initial charge. The results are illustrated with numerical examples.

Keywords: energy depletion, energy harvesting, diffusion process, first-passage time, internet of things, self-sustainability

1. INTRODUCTION

Energy harvesting is a promising solution for powering the Internet of things (IoT). It allows wireless sensing devices to operate autonomously for an extended period of time without the need to frequently recharge batteries. However, this seemingly free energy alternative creates new network design challenges (Gelenbe et al. [11], Gubbi et al. [20]), which require careful balancing of the power budget during different phases of operation, with the amount of energy available from the environment, along with the capacity and efficiency of energy storage. Ideally, such sensing devices need to operate in an energy neutral manner (Kansal et al. [24]), by adapting data gathering and transmission processes to the availability of renewable and intermittent sources of energy.

In this paper, we study the evolution of the energy stored in a wireless sensor node that harvests energy and uses the energy to collect and forward data. The arrival of energy and of data packets to the node are both assumed to be random processes: energy is harvested from random sources such as light, motion, and temperature, and data accumulates into the node, also at random, through sensing or reception from other nodes. However, while data are measured in terms of discrete packets, energy is quantified or measured in continuous units using diffusion processes. Our aim is to provide a fine-grained mathematical framework for optimizing the performance of these systems, which complements recent coarse-grained

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approaches based on queueing theory (Gelenbe and Kadioglu [12], Gelenbe [17], Gelenbe and Marin [19], Seyedi and Sikdar [29], Tandon and Motani [31]), or based on fluid flow approximations (Galinina et al. [8], Gautam and Mohapatra [9], Jones et al. [22], Tunc and Akar [32]) that may not capture adequately the stochastic nature of energy availability and consumption.

Much work has been devoted in recent years to the fundamental analysis and optimization methods that can be used in energy-harvesting wireless communications. A common approach in the literature is to represent the energy stored by a node in discrete units, called *energy packets* in Gelenbe and Ceran [10], Gelenbe and Kadioglu [12], Gelenbe [15–17], Gelenbe and Marin [19], where an energy packet is defined as the minimum amount of energy needed to transmit a single data packet. Intuitively, one would expect that the energy flow into the node should balance the flow of data packets. However, Gelenbe [17] has shown that if the flows of energy and of data packets are exactly balanced, then the system exhibits an unstable behavior, such that the variance of the imbalance between data and energy packets increases indefinitely with time. This is a frequently encountered problem in queueing theory (Abdelrahman and Gelenbe [1], Kendall [26]) concerning the stability of the synchronization of two *independent* streams, when the synchronization time is negligibly small. Smart scheduling policies have been suggested (Sharma et al. [30]) to adapt sensing and transmission power to the energy and channel conditions, thus providing stability and good performance. The approach presented herein can be extended to analyze the performance of such energy-aware policies.

Cai et al. [6] modeled the state of battery for a renewable-operated wireless access point as a $G/G/1$ queue with arbitrary arrivals and departures of unit energy. Diffusion approximation is then used to analyze the time-dependent behavior of the buffer. We developed this idea further in our previous work (Abdelrahman and Gelenbe [4]) by representing more explicitly the interactions between the energy and data buffers, as well as the energy costs of sensing, processing and communication, which are usually neglected (Cai et al. [6]) or combined (Jones et al. [22], Tunc and Akar [32]) in the literature. In the current paper, we extend (Abdelrahman and Gelenbe [4]) by developing an analytical–numerical method for computing the average time until the battery is depleted, which is a useful metric for dimensioning the node and evaluating its performance. Our modeling framework is based on Markov-modulated diffusion processes whose steady-state properties are well-understood (Asmussen [5], Karandikar and Kulkarni [25]). While we do not consider temporal variations in the environment (Jornet and Akyildiz [23], Naderi, Basagni, and Chowdhury [27]) or adaptive communication policies based on instantaneous battery level (Jones et al. [22], Tunc and Akar [32]), both aspects can be incorporated without much difficulty via multi-regime models and piecewise constant approximations (Abdelrahman and Gelenbe [2]).

The rest of the paper is organized as follows. Section 2 introduces the mathematical model and derives the stationary solution of the system along with the main performance metric of interest. Numerical examples are presented in Section 2.3. The paper concludes in Section 3 with a summary of results and directions for future work.

2. THE MODEL

Consider a wireless harvesting sensor node with energy storage of size $B > 0$, and let $0 \leq X_t \leq B$ be the amount of energy stored at time t. We denote by $Y_t \in \{0, 1, 2, 3, 4\}$ the state of the node at t where:

$$Y_t = \begin{cases} 0, & \text{if node is non-operational due to lack of energy,} \\ 1, & \text{if node is idle with no data buffered,} \\ 2, & \text{if node is collecting data,} \\ 3, & \text{if node is idle with data buffered,} \\ 4, & \text{if node is transmitting data.} \end{cases}$$

At the beginning of operation, we assume that the energy stored is $X_0 = x_0 > 0$ and the data buffer is empty. The time when the node first runs out of energy is then:

$$T = \inf\{t : X_t = 0 | X_0 = x_0\}. \tag{1}$$

When the state of the node is $Y_t = i \geq 1$ (i.e. the battery is non-empty), we model $\{X_t : t \geq 0\}$ as a diffusion process (Abdelrahman and Gelenbe [2,3], Gelenbe [13,14], Gelenbe and Abdelrahman [18]) in which the mean change in the amount of energy stored in a small time interval $[t, t + \Delta t)$ is $b_i \Delta t$, while the variance of the energy over the same time interval is $c_i \Delta t$:

$$b_i = \lim_{\Delta t \to 0} \frac{E[X_{t+\Delta t} - X_t | Y_t = i]}{\Delta t} \quad \text{and}$$
$$c_i = \lim_{\Delta t \to 0} \frac{E[(X_{t+\Delta t} - X_t)^2 - E[X_{t+\Delta t} - X_t]^2 | Y_t = i]}{\Delta t} > 0, \quad \text{for } 1 \leq i \leq 4.$$

The variance c_i is assumed to be strictly positive, so that there is no mass at the boundary B.

A schematic representation of the model is presented in Figure 1, showing the evolution of the data and energy buffers. If there is energy in the battery, the state of the node is governed by the following Markov process: data are accumulated at rate λ whenever the node is idle, and each sensing process takes on average τ_c^{-1} time; the data buffer is emptied with rate μ and the average transmission time is τ_s^{-1}.

When $Y_t = 1$, the sensor node is idle, does not have any data buffered and is harvesting energy. Accordingly, the parameters b_1 and c_1 capture the following energy aspects of the node's operation:

- Energy extracted from the environment, which typically varies with time.
- Leakage from the energy storage.
- Baseline power consumption of the node, excluding data processing and communication.

Thus, the drift b_1 can sometimes be positive when the amount of energy being harvested exceeds on average the node's idle consumption, while in other times it can be negative, for example, during the night in the case of solar harvesting.

The node will leave the idle state 1 at time t with rate λ and move to state 2, where it will collect data by either: (a) receiving a transmission from a neighboring node, or (b) sampling its own sensor and processing the sampled value. This data collection process is assumed to last for an exponentially distributed time of average τ_c^{-1}, during which the energy level varies according to a diffusion process with parameters b_2 and c_2, which is the result of both energy harvesting and data collection. Note, however, τ_c^{-1} will typically be very small, while $b_2 \ll 0$ to represent the negative *jump* in the energy level due to data reception or acquisition.

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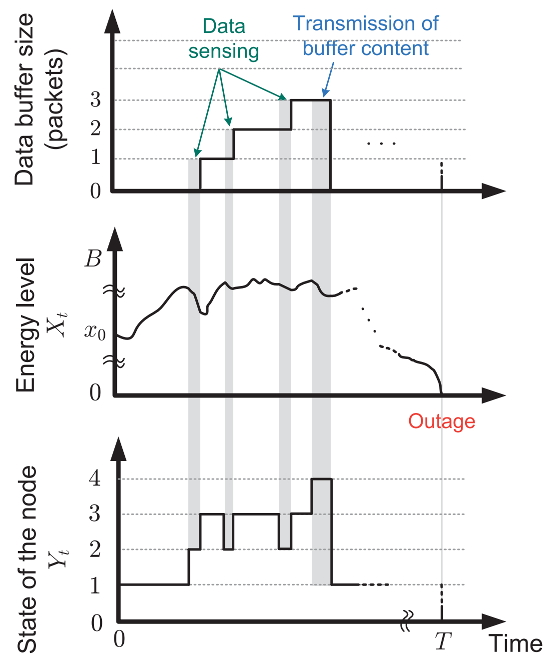

FIGURE 1. The evolution of the data buffer (top), battery charge $0 \leq X_t \leq B$ (middle) and state of the node $Y_t \in \{0, 1, 2, 3, 4\}$ (bottom). In each state $T_t \geq 1$, battery level varies following a distinct diffusion process that represents the combined effect of harvesting and any processing performed within the state (e.g. sensing in state 2 and transmission in state 3). The node experiences an outage due to lack of energy at time T.

The data collected by the node are subsequently stored in a data buffer, and the state of the node changes to 3 indicating that the node has some data to transmit. Since the node may have a different duty cycle when its data buffer is occupied than when it is empty, the diffusion parameters in states 3 and 1 can be different. From state 3, the node may again move to state 2, with rate λ, in order to gather and process more samples (e.g. to improve quality of information) or it may move to state 4 with rate μ signifying the beginning of a transmission. The sending of data occurs over an exponentially distributed time with mean τ_s^{-1} during which energy level diffuses with parameters b_4 and c_4, independently of the number of data collection operations performed since the previous transmission. Thus, we assume that the node may gather and process a number of samples (i.e. visiting states 2 and 3 multiple times), before finally sending all data in *one* packet in state 4. After that the buffer becomes empty and the node returns to state 1. Again, the negative jump in energy level due to a transmission can be represented by a small duration τ_s^{-1} and a negative drift $b_4 \ll 0$.

Note that the energy costs of sensing and of wireless transmission vary according to the application domain, the sensing environment, the type of raw input used by the sensor and so on. Indeed, transmission consumes most of the energy for low-quality, low-rate sensor nodes, and for simple sensing modalities such as temperature and light. On the other hand, sensing can be the most energy demanding process for some acoustic and seismic sensing applications that require high-rate and high-resolution analog-to-digital conversion (Raghunathan, Ganeriwal, and Srivastava [28]). Therefore, we do not make any assumptions regarding the relative magnitude of the diffusion parameters in states 1 and 3, except that $c_i > 0$.

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The above normal operation of the node is interrupted if the battery is depleted, taking the state of the node to $Y_t = 0$, where the node becomes temporarily non-operational until energy is replenished. However, as an artifact to construct a recurrent random process in order to simplify the computation of $E[T]$ – the average battery depletion time – it is assumed that after one time unit the energy level is restored to x_0 and the diffusion process proceeds as before.

Denote by $f_i(x, t)$ the probability density that the sensor node has energy level x and in occupancy state $i = 1, 2, 3, 4$ at time t, and by $p(t)$ the probability that the node runs out of energy at time t. Mathematically, these quantities are defined as:

$$p(t) = \Pr[X_t = 0],$$
$$f_i(x,t) = \lim_{\Delta x \to 0} \frac{\Pr[x \le X_t \le x + \Delta x,\ Y_t = i]}{\Delta x}, \quad 1 \le i \le 4.$$

Thus, $Y_t = 0$ is a fictitious state that we use to make the diffusion process repeats itself indefinitely, and $E[T]$ is the average time that it takes from any successive start of the process until the first instance when state 0 is reached again. Let $\lim_{t\to\infty} p(t) = p$, then:

$$p = \frac{1}{1 + E[T]}, \quad E[T] = p^{-1} - 1. \tag{2}$$

2.1. System of Differential Equations

The evolution of $f_i(x, t), x > 0$ is governed by the coupled diffusion or Fokker–Planck equations:

$$\begin{aligned}
\partial_t f_1(x,t) &= \frac{c_1}{2}\partial_{xx} f_1(x,t) - b_1 \partial_x f_1(x,t) - \lambda f_1(x,t) + \tau_s f_4(x,t) + p(t)\delta(x - x_0),\\
\partial_t f_2(x,t) &= \frac{c_2}{2}\partial_{xx} f_2(x,t) - b_2 \partial_x f_2(x,t) - \tau_c f_2(x,t) + \lambda[f_1(x,t) + f_3(x,t)],\\
\partial_t f_3(x,t) &= \frac{c_3}{2}\partial_{xx} f_3(x,t) - b_3 \partial_x f_3(x,t) - [\lambda + \mu] f_3(x,t) + \tau_c f_2(x,t),\\
\partial_t f_4(x,t) &= \frac{c_4}{2}\partial_{xx} f_4(x,t) - b_4 \partial_x f_4(x,t) - \tau_s f_4(x,t) + \mu f_3(x,t),
\end{aligned} \tag{3}$$

where $\delta(x - x_0)$ is the Dirac delta function at x_0. Furthermore, the probability mass $p(t)$ satisfies the equation:

$$\partial_t p(t) = -p(t) + \sum_{i=1}^{4} \lim_{x\to 0} \left[\frac{c_i}{2}\partial_x f_i(x,t) - b_i f_i(x,t)\right]. \tag{4}$$

The initial conditions can be written as:

$$p(0) = 0, \tag{5}$$

$$f_i(x,0) = \begin{cases} \delta(x - x_0), & \text{for } i = 1,\\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Since energy level cannot exceed B, the probability current across the upper boundary of the energy storage must be zero:

$$\lim_{x\to B}\left[-\frac{c_i}{2}\partial_x f_i(x,t) + b_i f_i(x,t)\right] = 0, \tag{7}$$

while the probability densities will vanish at the lower boundary:

$$f_i(0,t) = 0. \tag{8}$$

We also have that the sum of the probabilities at any time t must be one:

$$p(t) + \sum_{i=1}^{4} \int_0^B f_i(x,t)\ dx = 1. \tag{9}$$

2.2. Stationary Analysis

To derive the stationary solution of the system, we set $\lim_{t\to\infty} \partial_t f_i(x,t) = 0$, $\lim_{t\to\infty} f_i(x,t) = f_i(x)$ and $\lim_{t\to\infty} p(t) = p$ so that (3) and (4) become:

$$\begin{aligned} -p\delta(x-x_0) &= \frac{c_1}{2} f_1''(x) - b_1 \partial_x f_1(x) - \lambda f_1(x) + \tau_s f_4(x), \\ 0 &= \frac{c_2}{2} f_2''(x) - b_2 f_2'(x) - \tau_c f_2(x) + \lambda[f_1(x) + f_3(x)], \\ 0 &= \frac{c_3}{2} f_3''(x) - b_3 f_3'(x) - [\lambda+\mu] f_3(x) + \tau_c f_2(x), \\ 0 &= \frac{c_4}{2} f_4''(x) - b_4 f_4'(x) - \tau_s f_4(x) + \mu f_3(x) \end{aligned} \tag{10}$$

and

$$p = \sum_{i=1}^{4} \lim_{x\to 0} \left[\frac{c_i}{2} \partial_x f_i(x)\right], \tag{11}$$

with the boundary conditions:

$$f_i(0) = 0, \quad \text{and} \quad \lim_{x\to B} \left[\frac{c_i}{2} \partial_x f_i(x) - b_i f_i(x)\right] = 0, \quad 1 \le i \le 4 \tag{12}$$

and the normalization condition:

$$p + \sum_{i=1}^{4} \int_0^B f_i(x)\ dx = 1. \tag{13}$$

The solution to the above set of equations can be obtained by considering two regimes for the energy level: (i) $0 \le x < x_0$ and (ii) $x_0 < x \le B$. In each regime, the set of equations (10) can be written in matrix form as:

$$f''(x) - \mathbf{C}^{-1}\mathbf{B} f'(x) - \mathbf{C}^{-1}\mathbf{A} f(x) = 0, \tag{14}$$

where $f = \big(f_1(x),\ f_2(x),\ f_3(x),\ f_4(x)\big)^T$ and

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 & 0 & -\tau_s \\ -\lambda & \tau_c & -\lambda & 0 \\ 0 & -\tau_c & \lambda+\mu & 0 \\ 0 & 0 & -\mu & \tau_s \end{pmatrix}, \quad \mathbf{B} = \operatorname{diag}\Big(b_1,\dots,b_4\Big), \quad \mathbf{C} = \frac{1}{2}\operatorname{diag}\Big(c_1,\dots,c_4\Big). \tag{15}$$

The linear system of four second-order ordinary differential equations (14) can be transformed into a set of 8 first-order equations as follows:

$$\frac{d}{dx}\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C}^{-1}\mathbf{A} & \mathbf{C}^{-1}\mathbf{B} \end{pmatrix} \begin{pmatrix} f \\ f' \end{pmatrix} \equiv \mathbf{M} \begin{pmatrix} f \\ f' \end{pmatrix}, \tag{16}$$

where $\mathbf{0}$ and $\mathbf{I}$ are the zero and identity matrices, respectively.

Solving (16) involves computing the eigen-decomposition of the matrix $\mathbf{M}$. Since the characteristic equation $|\mathbf{M} - \xi \mathbf{I}| = 0$ has real coefficients, any complex roots must occur in conjugate pairs, each of which will give, up to sign, the same solutions and therefore only one of the conjugate eigenvalue–vector pair is required.

Result 1. Let ξ^R be the vector of real eigenvalues of $\mathbf{M}$ and ξ^C the vector of complex eigenvalues where only one of each conjugate pairs is included, and denote by $\mathbf{\Phi}^R, \mathbf{\Phi}^C$ the matrices formed by taking the first four elements of the corresponding eigenvectors. Also, define the following quantities:

$$\begin{aligned}
& \mathbf{Q} = \mathrm{diag}(\xi^R), && \mathbf{Q}_{\exp}(x) = \mathrm{diag}(e^{\xi^R x}), \\
r = \Re(\xi^C), \quad & \mathbf{R} = \mathrm{diag}(r), && \mathbf{R}_{\exp}(x) = \mathrm{diag}(e^{rx}), \\
s = \Im(\xi^C), \quad & \mathbf{S} = \mathrm{diag}(s), && \mathbf{S}_{\sin}(x) = \mathrm{diag}(\sin(sx)), \quad \mathbf{S}_{\cos}(x) = \mathrm{diag}(\cos(sx)), \\
& \mathbf{\Psi} = \Re(\mathbf{\Phi}^C), && \mathbf{\Omega} = \Im(\mathbf{\Phi}^C),
\end{aligned} \tag{17}$$

where $\Re, \Im$ denote the real and imaginary parts of a complex number. Then, the average time until the node runs out of energy with an initial charge of x_0 is:

$$E[T] = \left[\frac{c_1}{2} G_1'(x_0)(k^- - k^+)\right]^{-1} - 1, \tag{18}$$

where k^- and k^+ are vectors of size 8 such that $k = \begin{pmatrix} k^- \\ k^+ \end{pmatrix}$ is the last column of the inverse of the 16×16 matrix:

$$\begin{pmatrix}
\mathbf{G}(0) & \mathbf{0} \\
\mathbf{0} & \mathbf{C}\ \mathbf{G}'(B) - \mathbf{B}\ \mathbf{G}(B) \\
\mathbf{G}(x_0) & -\mathbf{G}(x_0) \\
\mathbf{G}'_{-1}(x_0) & -\mathbf{G}'_{-1}(x_0) \\
1_{1\times 4} \left(\mathbf{C}\ \mathbf{G}'(0) + \int_0^{x_0} \mathbf{G}(x)dx\right) & 1_{1\times 4} \int_{x_0}^{B} \mathbf{G}(x)dx
\end{pmatrix}. \tag{19}$$

Here $1_{1\times 4}$ denotes a row vector of ones; $\mathbf{B}$ and $\mathbf{C}$ are the energy diffusion matrices as specified in (15); and $\mathbf{G}(x)$ is the 4×8 matrix:

$$\mathbf{G}(x) \equiv \begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix} = \begin{pmatrix} \mathbf{U}(x) & \mathbf{V}(x) & \mathbf{W}(x) \end{pmatrix}, \quad \mathbf{G}_{-1}(x) = \begin{pmatrix} G_2(x) \\ G_3(x) \\ G_4(x) \end{pmatrix}$$

whose components are constructed as follows:

$$\begin{aligned}
\mathbf{U}(x) &= \mathbf{\Phi}^R \mathbf{Q}_{\exp}(x), \\
\mathbf{V}(x) &= \left[\mathbf{\Psi}\mathbf{S}_{\cos}(x) - \mathbf{\Omega}\mathbf{S}_{\sin}(x)\right]\mathbf{R}_{\exp}(x), \\
\mathbf{W}(x) &= \left[\mathbf{\Psi}\mathbf{S}_{\sin}(x) + \mathbf{\Omega}\mathbf{S}_{\cos}(x)\right]\mathbf{R}_{\exp}(x),
\end{aligned} \tag{20}$$

with first derivative and integral given by:

$$\mathbf{U}'(x) = \mathbf{U}(x)\mathbf{Q}, \qquad \int \mathbf{U}(x)dx = \mathbf{U}(x)\mathbf{Q}^{-1},$$
$$\mathbf{V}'(x) = \mathbf{V}(x)\mathbf{R} - \mathbf{W}(x)\mathbf{S}, \qquad \int \mathbf{V}(x)dx = [\mathbf{V}(x)\mathbf{R} + \mathbf{W}(x)\mathbf{S}](\mathbf{R}^2 + \mathbf{S}^2)^{-1},$$
$$\mathbf{W}'(x) = \mathbf{W}(x)\mathbf{R} + \mathbf{V}(x)\mathbf{S}, \qquad \int \mathbf{W}(x)dx = [\mathbf{W}(x)\mathbf{R} - \mathbf{V}(x)\mathbf{S}](\mathbf{R}^2 + \mathbf{S}^2)^{-1}.$$

PROOF: The solution to (16) has the form:

$$f(x) = \begin{cases} \sum_j \left[\alpha_j^- \Re(e^{\xi_j x}\phi_j) + \beta_j^- \Im(e^{\xi_j x}\phi_j)\right], & \text{for } 0 \le x \le x_0, \\ \sum_j \left[\alpha_j^+ \Re(e^{\xi_j x}\phi_j) + \beta_j^+ \Im(e^{\xi_j x}\phi_j)\right], & \text{for } x_0 \le x \le B, \end{cases}$$

where ξ_j is an eigenvalue of $\mathbf{M}$, ϕ_j is the corresponding eigenvector (its first four elements since we are only interested in solving for f and not f'), and $\alpha_j^\mp, \beta_j^\mp$ are constants to be determined from (a) the boundary conditions (12) at 0 and B, (b) the continuity condition of the probability density function and the probability current at x_0, and (c) the normalization condition (13). Note that $f(x)$ takes a slightly different form than above if there are repeated eigenvalues, but we do not go into the details of this special case.

Using the definitions in (17) we can write the jth complex eigenvector and eigenvalue more explicitly as $\phi_j^C = \psi_j + \omega_j\sqrt{-1}$ and $\xi_j^C = r_j + s_j\sqrt{-1}$ so that $e^{\xi_j^C x} = e^{r_j x}\left[\cos(s_j x) + \sin(s_j x)\sqrt{-1}\right]$ and:

$$\Re(e^{\xi_j^C x}\phi_j^C) = e^{r_j x}[\psi_j \cos(s_j x) - \omega_j \sin(s_j x)],$$
$$\Im(e^{\xi_j^C x}\phi_j^C) = e^{r_j x}[\omega_j \cos(s_j x) + \psi_j \sin(s_j x)],$$

which represent the jth columns of the matrices $\mathbf{V}(x)$ and $\mathbf{W}(x)$ defined in (20). It is then straightforward to express $f(x)$ in matrix form as:

$$f(x) = \begin{pmatrix}\mathbf{U}(x) & \mathbf{V}(x) & \mathbf{W}(x)\end{pmatrix}\begin{pmatrix}k^- \\ k^+\end{pmatrix}$$

where k^- is the vector of constants for $x \le x_0$, and k^+ the vector for $x \ge x_0$; the entries of the matrix $\mathbf{U}(x)$ are associated with the real eigenvalues, while those of $\mathbf{V}(x)$ and $\mathbf{W}(x)$ correspond, respectively, to the real and imaginary parts of the complex eigenvalues.

Now if we apply the boundary conditions (12) at $x = 0$ and $x = B$ we obtain:

$$\mathbf{G}(0)k^- = 0, \quad \left[\mathbf{C}\ \mathbf{G}'(B) - \mathbf{B}\ \mathbf{G}(B)\right]k^+ = 0,$$

which form the first two rows of (19). Furthermore, to ensure continuity of the probability density function at $x = x_0$, we have $f(x_0^-) = f(x_0^+)$ or equivalently:

$$\mathbf{G}(x_0)k^- - \mathbf{G}(x_0)k^+ = 0$$

yielding the third row of the solution matrix. Moreover, integrating the differential equations (10) from $x = x - \epsilon$ to $x = x + \epsilon$ and taking the limit as $\epsilon \to 0$ give $p = \frac{c_1}{2}[f_1'(x_0^-) - f_1'(x_0^+)]$

and $f_i'(x_0^-) - f_i'(x_0^+) = 0,\ i > 1$ or:

$$p = \frac{c_1}{2} G_1'(x_0)[k^- - k^+], \quad G_i'(x_0)k^- - G_i'(x_0)k^+ = 0, \quad 2 \le i \le 4$$

from which the expression for $E[T]$ in (18) and the fourth row of (19) follow. We can also obtain another equivalent expression for p using (11):

$$p = 1_{1\times 4}\ \mathbf{C}\ \mathbf{G}'(0)k^-$$

which can be substituted in the normalization equation (13) to arrive at the last row of (19). Thus, k is the solution of a system of linear equations and is given by the inverse of (19) multiplied by a vector with zeros everywhere except for a one in the last position. ■

2.3. Numerical Examples

We illustrate in Figure 2 the effect of increasing the node's power budget and workload on the average time until battery is first depleted. Specifically, we plot $E[T]$ as a function of the data gathering rate λ, when the net energy cost during sensing (calculated as the product of average power consumption and sensing duration, i.e. b_2/τ_c) and during transmission (which is b_4/τ_s) is increased in relation to the battery size $B = 1$. The parameters of the model are set as follows: the mean and variance of energy in the inactive states are $(b_1, c_1) = (b_3, c_3) = (0.01, 0.01)$, while for sensing and transmission we have $b_2 = b_4 \in [-0.01, -0.05, -0.1]$ and $c_2 = c_4 = 0.001$; the average sensing and transmission durations are $\tau_c^{-1} = \tau_s^{-1} = 0.2s$; the data buffer is flushed at rate $\mu = 0.2$; and the initial charge x_0 is 60% of the total battery capacity. As λ is increased, the node spends more time gathering data, which improves the quality of information but reduces $E[T]$ significantly when the associated energy expenditure is high. However, if sensing and communication incur negligible costs, we see that the average battery depletion time is unaffected by λ and is largely determined by energy availability, battery leakage and idle consumption whose combined effects are captured by the diffusion parameters.

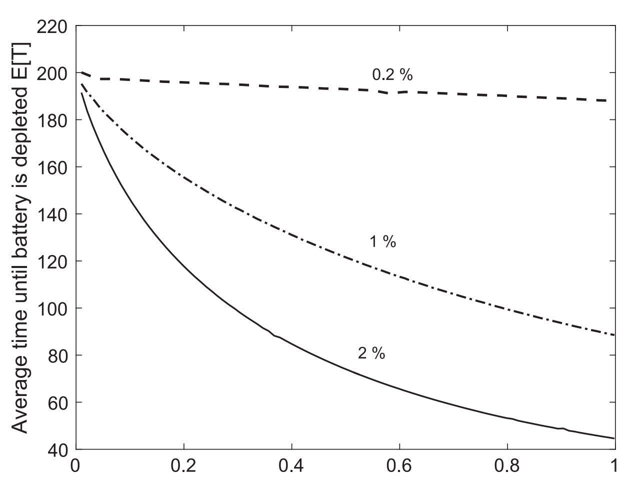

FIGURE 2. Average battery depletion time $E[T]$ versus the data sensing rate λ when the energy cost of sensing or transmission is z% of the battery size, with $z \in [0.2, 1, 2]$.

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3. CONCLUSIONS

In this paper, we presented a diffusion model of a battery-operated wireless sensor node that uses renewable energy sources such as wind and solar. We obtained an expression for the average time before the node's battery is depleted, by solving a set of diffusion equations with specific boundary conditions. In future work, we will extend the model to incorporate temporal variations in the energy source as well as battery-level-dependent behaviors. We also plan to investigate variants of the model that capture the same dynamics of the node's operation but with fewer parameters and states. This could be achieved by combining discrete Poisson jumps, to represent the sudden drops in energy level due to sensing or communication, with background stochastic diffusion for the energy harvesting and leakage processes. Such *jump-diffusion* models have been applied successfully in other domains such as financial engineering, insurance, mathematical biology and medicine (Dufresne and Gerber [7], Hanson [21]). Finally, the model of a single node will be integrated into a network setting so as to predict the quality of service and amount of energy available across multiple interconnected sensor nodes.

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