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BROOKS' THEOREM FOR MEASURABLE COLORINGS

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Abstract

We generalize Brooks' theorem to show that if G is a Borel graph on a standard Borel space X of degree bounded by $d \geqslant 3$ which contains no $(d+1)$-cliques, then G admits a μ-measurable d-coloring with respect to any Borel probability measure μ on X, and a Baire measurable d-coloring with respect to any compatible Polish topology on X. The proof of this theorem uses a new technique for constructing one-ended spanning subforests of Borel graphs, as well as ideas from the study of list colorings. We apply the theorem to graphs arising from group actions to obtain factor of IID d-colorings of Cayley graphs of degree d, except in two exceptional cases.

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1. Introduction

We begin by recalling a classical theorem of Brooks from finite combinatorics.

THEOREM 1.1 (Brooks' theorem [9, Theorem 5.2.4]). *Suppose G is a finite graph with vertex degree bounded by d. Suppose further that G contains no complete graph on $d+1$ vertices, and if $d=2$ that G contains no odd cycles. Then G has a (proper) vertex d-coloring.*

It is easy to see that if a graph G has vertex degree bounded by d, then G has a $(d+1)$ coloring: greedily color the vertices one by one, using the least color not already assigned to a neighboring vertex. One way of regarding Brooks' theorem is that it is a complete characterization of the graphs for which this obvious upper bound cannot be improved: odd cycles and complete graphs.

Brooks' theorem is a fundamental result of graph coloring which has been generalized in a variety of different settings. See [8] for a recent survey. This paper examines measurable generalizations of Brooks' theorem. While a straightforward compactness argument extends Brooks' theorem to infinite graphs, such an argument cannot in general produce a coloring with desirable measurability properties such as being μ-measurable with respect to some probability measure, or being Baire measurable with respect to some *Polish* (separable, completely metrizable) topology. Recall that a *standard Borel space* is a set X equipped with a σ-algebra generated by a Polish topology. The graphs we will consider are *Borel graphs* where the vertices of the graph are the elements of some standard Borel space X, and whose edge relation is Borel as a subset of $X \times X$.

In studying Borel graphs, Kechris *et al.* [16, Proposition 4.6] have shown that every Borel graph of vertex degree bounded by a finite d admits a Borel $(d+1)$-coloring, in analogy to the fact above. Hence, we can ask again for which exceptional cases this obvious bound cannot be improved. Marks [19, Theorem 1.3] has shown that for every finite d, there is an acyclic Borel graph of degree d with no Borel d-coloring. Hence, to obtain a reasonable measurable analogue of Brooks' theorem as in Theorem 1.2, we must consider measurability constraints weaker than Borel measurability. In this paper, we focus on Baire measurability with respect to some compatible Polish topology and μ-measurability with respect to some Borel probability measure μ.

Still, the obvious analogue of the $d = 2$ case of Brooks' theorem does not hold for either of these measurability notions. Let $S : \mathbb{T} \to \mathbb{T}$ be an irrational rotation of the unit circle $\mathbb{T}$, and let G_S be the graph on $\mathbb{T}$ rendering adjacent each point $x \in \mathbb{T}$ and its image $S(x)$ under S so G_S is acyclic and each vertex has degree 2. Now an easy ergodicity argument shows that G_S has no Lebesgue measurable 2-coloring: since S is measure-preserving, the two color sets would have to have equal measure, but since S^2 is ergodic, the color sets would each have to be null or conull. Similarly, G_S has no Baire measurable 2-coloring (see Section 6). This example may be considered an infinite analogue of the odd cycle exemption in 1.1.

Our main result is the following measurable analogue of Brooks' theorem for the case $d \geqslant 3$.

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THEOREM 1.2. *Suppose that G is a Borel graph on a standard Borel space X with vertex degree bounded by a finite $d \geqslant 3$. Suppose further that G contains no complete graph on $d+1$ vertices.*

(1) *Let μ be any Borel probability measure on X. Then G admits a μ-measurable d-coloring.*

(2) *Let τ be any Polish topology compatible with the Borel structure on X. Then G admits a Baire measurable d-coloring.*

This improves a prior result of Conley and Kechris who proved an analogous theorem for approximate colorings where one is allowed to discard a set of arbitrarily small measure [5, Theorems 2.19, 2.20].

Our proof of Theorem 1.2 uses ideas from the study of list colorings in graph theory. Recall that if G is a graph on X and L is a function mapping each $x \in X$ to a set $L(x)$, then a *coloring of G from the lists L* is a coloring c of X such that for every x, $c(x) \in L(x)$. Given a function $f: X \to \mathbb{N}$, we say X is f-list-colorable if for every function L on X with $|L(x)| = f(x)$, there is a coloring of G from the lists L. We say G is degree-list-colorable if it is f-list-colorable for the function $f(x) = \deg_G(x)$, where $\deg_G(x)$ is the degree of x. Borodin [4] and Erdős *et al.* [10] independently generalized Brooks' theorem to list colorings by classifying the finite graphs G which are not degree-list-colorable. Of course, if a graph G has degree bounded by d, then being degree-list-colorable implies that G is d-colorable since we can color from the lists $L(x) = \{1, \ldots, \deg_G(x)\}$ for every x. For degree-list-coloring, the exceptional graphs are the finite Gallai trees:

THEOREM 1.3 (Borodin [4], Erdős *et al.* [10]). *A finite connected graph is degree-list-colorable if and only if it is not a Gallai tree.*

Recall here that a set S of vertices from a graph G is *biconnected* if the induced subgraph $G \restriction S$ remains connected after removing any single vertex from S. A *block* of a graph is a maximal biconnected set and a *Gallai tree* is a connected graph whose blocks are complete graphs or odd cycles.

In addition to making key use of this result, we also generalize it to Borel graph colorings. Recall that if Y is a set, we use $[Y]^{<\infty}$ to denote the collection of finite subsets of Y, and if Y is a standard Borel space then so is $[Y]^{<\infty}$ with the Borel structure induced as a quotient of $\bigsqcup_{n \in \mathbb{N}} Y^n$. Say that a locally finite Borel graph G on X is *Borel degree-list-colorable* if for every Polish space and every Borel function $L: X \to [Y]^{<\infty}$ such that $|L(x)| = \deg_G(x)$, there is a Borel coloring $c: X \to Y$ of G from the lists L.

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THEOREM 1.4. *Suppose that G is a locally finite Borel graph on a standard Borel space X, and that no connected component of G is a Gallai tree. Then G is Borel degree-list-colorable.*

Recall that if G is a locally finite graph on X, then two rays $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ in G are *end-equivalent* if for every finite set $S \subseteq X$, the rays eventually lie in the same connected component of $G \restriction (X \setminus S)$. If G is an acyclic graph, this is equivalent to (x_i) and (y_i) being tail equivalent. An *end* of a graph is an end-equivalence class of rays. A *one-ended spanning subforest* of G is a acyclic graph $T \subseteq G$ on X such that every $x \in X$ is incident on some edge in T, and every connected component of T has exactly one end.

The other tool we use to prove Theorem 1.2 is a new technique for constructing μ-measurable and Baire measurable one-ended spanning subforests of acyclic Borel graphs. In the case where the connected components of G are Gallai trees and we cannot apply Theorem 1.4, we use a one-ended subforest of the Gallai tree to give a skeleton along which we may color the graph to prove Theorem 1.2.

THEOREM 1.5. *Suppose that G is a locally finite acyclic Borel graph on a standard Borel space X such that no connected component of G has 0 or 2 ends.*

(1) *Let μ be a Borel probability measure on X. Then there is a μ-conull Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.*

(2) *Let τ be a compatible Polish topology on X. Then there is a τ-comeager Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.*

We also discuss a version of Theorem 1.5.1 for locally countable graphs in Section 3.

In Section 5, we apply Theorem 1.2 to graphs arising from group actions, and we apply the methods of Section 3 along with results from probability to obtain factor of IID d-colorings of Cayley graphs of degree d, apart from two exceptional cases.

Finally, in the case $d = 2$ we show that the ergodic-theoretic obstruction discussed above is in a sense the only counterexample to Brooks' theorem.

THEOREM 1.6. *Suppose G is a Borel graph on a standard Borel space X with vertex degree bounded by $d = 2$ such that G contains no odd cycles. Let $E_{2,G}$ be the equivalence relation on X where $x \; E_{2,G} \; y$ if x and y are connected by a path of even length in G.*

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(1) *Let μ be a G-quasiinvariant Borel probability measure on X. Then G admits a μ-measurable 2-coloring if and only if there does not exist a nonnull G-invariant Borel set A such that every $E_{2,G}$-invariant Borel subset of A differs from a G-invariant set by a null set.*

(2) *Let τ be a G-quasiinvariant Polish topology compatible with the Borel structure on X. Then G admits a Baire measurable 2-coloring if and only if there does not exist a nonmeager G-invariant Borel set A so that every $E_{2,G}$-invariant Borel subset of A differs from a G-invariant set by a meager set.*

2. Preliminaries

A (simple, undirected) *graph* on a set of vertices X is a symmetric irreflexive relation on X. Given a graph G on X, we say that two points $x, y \in X$ are *neighbors* or are *adjacent* in G if $x\ G\ y$. The (G-)*degree* of a vertex x, denoted $\deg_G(x)$, is the number of neighbors of x and a graph G has bounded degree d if every vertex has degree at most d. We say that G is *locally finite* (respectively *locally countable*) if the degree of every vertex in G is finite (respectively countable). A set $A \subseteq X$ of vertices of G is (G-)*independent* if for every x, $y \in A$ it is not the case that $x\ G\ y$. If $f : X \to X$ is a function, then the graph G_f generated by f is defined by $x\ G_f\ y$ if $x \neq y$ and either $f(x) = y$ or $f(y) = x$.

A (simple) *path* in a graph G is a finite sequence $x_0, \ldots, x_n$ of distinct vertices such that $x_0\ G\ x_1\ G\ \ldots\ G\ x_n$. We say that such a path has *length* n. A (simple) *ray* is an infinite sequence $(x_i)_{i\in\mathbb{N}}$ of distinct vertices such that $x_i\ G\ x_{i+1}$ for every $i \in \mathbb{N}$, and a *line* is a bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ such that $x_i\ G\ x_{i+1}$ for every $i \in \mathbb{Z}$. If G is a graph on X, then the graph metric $d_G : X^2 \to \mathbb{N} \cup \{\infty\}$ on G maps $(x, y) \in X^2$ to the length of the shortest path connecting x and y, if such a path exists; otherwise we set $d_G(x, y) = \infty$. A *cycle* in a graph G is a sequence of vertices $x_0\ G\ x_1\ G\ x_2\ G\ \ldots\ G\ x_n = x_0$ such that $n > 2$, and $x_i \neq x_j$ for all $i < j < n$. We say the length of such a cycle is n. We say that a graph is *acyclic* if it does not contain any cycles. If G is a graph on X, then we let E_G be the connectedness relation of G, where $x\ E_G\ y$ if there is a path in G from x to y. We say that a set $A \subseteq X$ is G-*invariant* if it is E_G-invariant, that is, a union of connected components of G.

A *Borel graph* is a graph whose vertices are the elements of a standard Borel space X, and whose edge relation is Borel as a subset of $X \times X$. The restriction $G \restriction A$ of G to a set $A \subseteq X$ is the graph on A is the induced subgraph obtained by restricting the relation G to A. If G is a Borel graph, and A is a Borel set, then since A inherits the standard Borel structure of X, we see that $G \restriction A$ is also a Borel graph.

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Let X and Y be standard Borel spaces. Let μ be a Borel probability measure on X. We say that a function $f : X \to Y$ is *μ-measurable* if it is measurable for the completion of μ. Let τ be a compatible Polish topology on X (by *compatible* we mean that the σ-algebra generated by the τ-open sets coincides with the given Borel σ-algebra on X). We say that a function $f : X \to Y$ is *Baire measurable (with respect to τ)* if it is measurable for the σ-algebra of sets which have the Baire property with respect to the completion of τ; the smallest σ-algebra containing the Borel sets and all τ-meager sets.

There is an equivalence between admitting a μ-measurable or Baire measurable coloring, and admitting a Borel coloring modulo an invariant null or meager set:

PROPOSITION 2.1. *Suppose G is a locally countable Borel graph on a standard Borel space X. Suppose that G admits some n-coloring.*

(1) *Let μ be any Borel probability measure on X. Then G admits a μ-measurable n-coloring if and only if there is a μ-conull G-invariant Borel set $A \subseteq X$ such that $G \restriction A$ has a Borel n-coloring.*

(2) *Let τ be any Polish topology compatible with the Borel structure on X. Then G admits a Baire measurable n-coloring if and only if there is a comeager G-invariant Borel set $A \subseteq X$ such that $G \restriction A$ has a Borel n-coloring.*

Proof. We begin with the direction $\Rightarrow$ of (1). Suppose c is a μ-measurable n-coloring of G. By the Feldman–Moore theorem [15, Theorem 1.3], let $\{T_i\}_{i\in\mathbb{N}}$ be a set of Borel automorphisms of X so that $E_G = \bigcup_{i\in\mathbb{N}} T_i$. Now for each i, since $c \circ T_i$ is μ-measurable, there is a μ-conull set A_i such that $c \circ T_i \restriction A_i$ is Borel. Thus, if $A = [\bigcap_{i\in\mathbb{N}} A_i]_{E_G}$, then A is μ-conull, and $c \restriction A$ is Borel. This is because for all $x \in A$, if i is least such that $T_i^{-1}(x) \in A_i$, then $c(x) = (c \circ T_i)(T_i^{-1}(x))$.

The direction $\Leftarrow$ of (1) is straightforward. Given a Borel n-coloring c of $G \restriction A$ where A is a μ-conull G-invariant set, let c' be an arbitrary n-coloring of $G \restriction (X \setminus A)$. Then $c \cup c'$ is a μ-measurable coloring of G.

The proof of part (2) is identical to the above. Simply replace the phrase μ-conull with comeager (with respect to τ), and μ-measurable with Baire measurable (with respect to τ). □

While we have stated our main results in terms of the existence of μ-measurable and Baire measurable colorings, throughout the paper we will mostly work with the equivalent formulations given by Proposition 2.1 above. Note here that the classical Brooks' theorem shows the existence of the requisite d-coloring that we will need to apply the above proposition. An analogous fact is also true for list-coloring.

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Suppose G is a locally countable Borel graph on a standard Borel space X, and μ is a Borel probability measure on X. Then we say that μ is *G-quasiinvariant* if every μ-null set is contained in a G-invariant μ-null set. Now for every Borel probability measure μ on X, there exists a G-quasiinvariant Borel probability measure μ' on X such every μ'-null set is μ-null (that is, μ' dominates μ). This follows from the Feldman–Moore theorem [15, Theorem 1.3] by letting $\{T_i\}_{i\in\mathbb{N}}$ be a set of Borel automorphisms of X such that $E_G = \bigcup_i \operatorname{graph}(T_i)$, and then setting $\mu'(A) = \sum_{i\geqslant 1} 2^{-n}\nu(T_i(A))$ (see [15, Section 8]). A key property of a quasiinvariant measure is that if A is μ'-conull, then it contains a G-invariant μ'-conull set. This is because the set $\{x : x \notin A \wedge x \in [A]_{E_G}\}$ is null since it is contained in the complement of A, and hence is saturation is null.

Similarly, suppose G is a Borel graph on X, and τ is a compatible Polish topology for X. Then we say that τ is *G-quasiinvariant* if every τ-meager set is contained in a G-invariant τ-meager set. It follows from a result of Zakrzewski [23] that if G is a locally countable Borel graph, then for every compatible Polish topology τ on X there is a G-quasiinvariant compatible Polish topology τ' such that every τ'-meager set is τ-meager.

The combination of the above discussion and Proposition 2.1 justifies our assumption from now on that our measures and topologies are quasiinvariant with respect to the graphs we consider. This is because Proposition 2.1 allows us to reformulate Theorems 1.2 and 1.6 to state the existence of a Borel d-coloring of $G \upharpoonright A$ for some G-invariant Borel A which is conull or comeager. Thus, the assumption of quasiinvariance is harmless since we may always pass to a quasiinvariant measure or topology without adding any new conull or comeager sets. Our assumption of quasiinvariance is helpful because it frees us from talking constantly about invariant sets when we discard null or meager set, since in this case a null set or meager set of vertices is always contained in a null set or meager G-invariant set, respectively.

3. One-ended subforests

This section focuses on definably isolating one-ended subforests of various classes of locally finite acyclic graphs. These subforests will subsequently provide a skeleton along which to construct a coloring. A few of our results in this section generalize to locally countable graphs and in these cases we indicate what changes need to be made in our arguments to work in this greater generality. It is worth noting that since the graphs under investigation are acyclic, the two natural definitions of end-equivalence in terms of deleting vertices and deleting edges in fact coincide.

Suppose G is an acyclic graph on X and every connected component of G has one end. We can form a function $f : X \to X$ as follows. For each $x \in X$, there

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is a unique infinite G-ray $(x_i)_{i\in\mathbb{N}}$ such that $x = x_0$. Then define $f(x) = x_1$, so f points 'towards' the unique end in the graph. Note that f generates G, and since G is everywhere one-ended, there is no infinite *descending* sequence $x_0, x_1, \ldots \in X$ with $f(x_{i+1}) = x_i$ for every i. That is, as a relation, f is well founded. Conversely, suppose $f: X \to X$ is a function containing no infinite descending sequence (in particular, no fixed points). Then the graph G_f generated by f has one end in every connected component. Thus, if H is a graph on X, finding a one-ended spanning subforest of H is equivalent to finding a function $f: X \to X$ contained in H admitting no infinite descending sequences.

Now suppose $f: X \to X$ is a partial function. We say that f is *one-ended* if there is no infinite descending sequence $x_0, x_1, \ldots \in X$ in f. Note that if f is a one-ended partial function, then any connected component of G_f containing a point y not in the domain of f will contain 0 ends (and not one end!). Our terminology here is inspired by regarding such a $y \notin \mathrm{dom}(f)$ as a 'point at infinity'. In the case where f is finite-to-one, by König's lemma, f is one-ended if and only if all of its backward orbits $f^{-\mathbb{N}}(x) = \bigcup_{n\in\mathbb{N}} f^{-n}(x)$ are finite.

PROPOSITION 3.1. *Suppose that G is a locally finite Borel graph on a standard Borel space X, and $A \subseteq X$ is Borel. Then there is a one-ended Borel function $f: [A]_{E_G} \setminus A \to [A]_{E_G}$ whose graph is contained in G.*

Proof. Without loss of generality we may assume that $[A]_{E_G} = X$ since $G \restriction [A]_{E_G}$ is a Borel graph. Let B be the set of $x \in X \setminus A$ such that there exists a G-ray $(x_i)_{i\in\mathbb{N}}$ with $x_0 = x$ and $d(x_{i+1}, A) > d(x_i, A)$ for all $i \in \mathbb{N}$. Note that B is Borel by König's lemma. By Lusin–Novikov uniformization [14, Theorem 18.10], there is a Borel function $f: X \setminus A \to X$ such that

(1) if $x \in B$, then $f(x)$ is a neighbor of x such that $f(x) \in B$ and $d(f(x), A) > d(x, A)$.

(2) if $x \notin B$, then $f(x)$ is a neighbor of x such that $d(f(x), A) < d(x, A)$.

To see that f is as desired, suppose first that $x \notin B$. Then $f^{-\mathbb{N}}(x) \subseteq X \setminus B$, and if $f^{-\mathbb{N}}(x)$ were infinite an application of König's lemma would allow the construction of an injective G-ray as in the definition of B, contradicting the fact that $x \notin B$. On the other hand, if $x \in B$ then $f^{-\mathbb{N}}(x) \cap B$ is finite and is in fact contained in $\bigcup_{i<d(x,A)} f^{-i}(x)$. Consequently $f^{-\mathbb{N}}(x)$ is the union of this finite set with $\bigcup_{i\leqslant d(x,A)}\{f^{-\mathbb{N}}(y) : y \in f^{-i}(x) \setminus B\}$, which by the previous case is a finite union of finite sets. □

We note that in the case where G is a locally countable Borel graph, the same function f is one-ended. However, the function f will not be Borel in general

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since while B is analytic it may not be Borel. Nevertheless, f will be $\sigma(\Sigma^1_1)$-measurable, where $\sigma(\Sigma^1_1)$ is the σ-algebra generated by the analytic sets. Hence, the function f is Borel after discarding an appropriate null or meager set.

Iteratively applying Proposition 3.1 can be used to find one-ended Borel functions whose domains are G-invariant.

LEMMA 3.2. *Suppose G is a locally finite Borel graph on X, and there is a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots$ of Borel sets with empty intersection such that A_{i+1} meets each connected component of $G \restriction A_i$. Then there is a one-ended Borel function $f : [A_0]_{E_G} \to [A_0]_{E_G}$ whose graph is contained in G.*

Proof. We may assume A_0 is G-invariant by replacing A_0 with $[A_0]_{E_G}$. Apply Proposition 3.1 to find a one-ended Borel function $f_i : (A_i \setminus A_{i+1}) \to A_i$ whose graph is contained in G. Then let $f = \bigcup_i f_i$. The function f will be one-ended since any decreasing sequence in f would contain a subsequence which is decreasing in some f_i. □

Indeed, the hypothesis of Lemma 3.2 is equivalent to the existence of a Borel one-ended function whose graph is contained in G; if f is a one-ended Borel function, then let $A_n = f^n[X]$.

Next, we use Lemma 3.2 to construct one-ended Borel functions with conull domain in bounded-degree acyclic Borel graphs with an additional property that we call ampleness.

DEFINITION 3.3. We say that a graph G on X is *ample* if every vertex has degree at least 2, and for all $x \in X$ every connected component of $G \restriction (X \setminus \{x\})$ contains a vertex of degree at least 3.

Geometrically, an acyclic locally finite graph is ample if it contains no isolated ends. Equivalently, an acyclic locally finite graph G is ample if it can be obtained from an acyclic graph with each vertex of degree at least 3 by 'subdividing' each edge by adding some vertices of degree 2.

LEMMA 3.4. *Suppose that G is a bounded-degree acyclic Borel graph on a standard Borel space X. Suppose moreover that G is ample. Let μ be a Borel probability measure on X. Then there is a μ-conull Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.*

Proof. Fix d bounding the degree of vertices of G, so $d \geqslant 3$. The heart of the construction rests in the following claim.

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Claim. There is a Borel subset $A \subseteq X$ meeting each connected component of G and with $\mu(A) \leqslant 1 - d^{-3}$, such that $G \restriction A$ is ample.

Proof of the Claim. Let $X' = \{x \in X : \deg_G(x) \geqslant 3\}$ and define an auxiliary graph G' on X' by putting $x\ G'\ y$ if $x\ E_G\ y$ and the unique G-path from x to y contains no other points of X'. Define a Borel map $\pi : X \to X'$ selecting for each x a closest element of X' with respect to the graph metric on G. Let $\nu = \pi_* \mu$ be the pushforward measure of μ on X' under π, so $\nu(B) = \mu(\pi^{-1}(B))$ for all Borel B.

Finally, let H be the distance $\leqslant 3$ graph associated with G', so two distinct points of X' are H related if they are connected by a G' path of length at most 3. Now H has degree bounded by $d^3 - 1$, and hence by [16, Proposition 4.6] a Borel coloring in d^3 colors. Consequently, there is an H-independent Borel set $C' \subseteq X'$ with $\nu(C') \geqslant d^{-3}$.

Define $C \subseteq X$ by $x \in C$ if $x \in C'$ or $x \in X \setminus X'$ and can be connected to a point in C without using any other points of X'. Note that $\pi^{-1}(C') \subseteq C$, so in particular $\mu(C) \geqslant d^{-3}$. We then set $A = X \setminus C$, and check that A satisfies the conclusion of the claim.

The G'-independence of C (in conjunction with the ampleness of G) implies that A meets each G-component. The only thing remaining to check is that $G \restriction A$ is ample. Note that the only way a vertex x in X' can have $(G \restriction A)$-degree less than three is if it is G'-adjacent to an element of C. So the fact that distinct points of C have G' distance at least four implies that x has two G' neighbors in X' whose $(G \restriction A)$-degree remains 3. In particular, the degree of x is 2. Moreover, if x were used to witness the ampleness condition of one of its neighbors, the condition can be witnessed instead by the other neighbor. So $G \restriction A$ is ample and the claim is proved. □

By iterating the claim, we may build a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots$ of Borel sets so that $A_0 = X$, A_{i+1} meets each component of $G \restriction A_i$, and $\mu\left(\bigcap_i A_i\right) = 0$. Since we may assume that G is μ-quasiinvariant, after discarding the μ-null saturation of $\bigcap_i A_i$, we can then apply Lemma 3.2. □

The same idea works in the context of Baire category, even in the more general context where G is not bounded degree.

LEMMA 3.5. *Suppose that G is a locally finite acyclic Borel graph on a standard Borel space X. Suppose moreover that Y is a Borel set and $G \restriction Y$ is ample. Let τ be a Polish topology compatible with X. Then there is a G-invariant Borel set $B \subseteq Y$ such that $Y \setminus B$ is τ-meager and a one-ended Borel function $f : B \to X$ whose graph is contained in G.*

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Proof. The proof is very similar to Lemma 3.4 above. The statement to prove in place of the above claim is the following: $(*)$ For any nonempty τ-open set U there is a Borel subset $A \subseteq Y$ meeting each connected component of $G \restriction Y$ and with $U \setminus A$ nonmeager, such that $G \restriction A$ is ample. The proof of $(*)$ is the same as the proof of the claim except that we choose the H-independent Borel set $C \subseteq X_3$ with $U \cap \pi^{-1}(C)$ nonmeager. Then we fix a countable base $\{U_k\}_{k\in\mathbb{N}}$ of open sets for τ and, as in part (1), we iteratively apply $(*)$ to build a decreasing sequence $(A_i)_{i\in\mathbb{N}}$ of Borel sets so that $A_0 = X$, A_{i+1} meets each component of $G \restriction A_i$, and with $U_i \setminus A_i$ nonmeager. It follows that $U_k \setminus \bigcap_i A_i$ is nonmeager for all $k \in \mathbb{N}$, and therefore $\bigcap_i A_i$ is meager. The rest of the proof is as before. $\square$

Next, we show that we can reduce the problem of proving Theorem 1.5 to the case of ample graphs.

LEMMA 3.6. *Suppose G is a locally finite acyclic Borel graph on a standard Borel space X and no connected component of G has 0 or 2 ends. Then there is a Borel set B such that $G \restriction B$ is ample, and there is a one-ended Borel function $f: (X \setminus [B]_{E_G}) \to (X \setminus [B]_{E_G})$ contained in the graph of $G \restriction (X \setminus [B]_{E_G})$.*

Proof. Let A be the set of $x \in X$ such that there are disjoint rays $(y_i)_{i\in\mathbb{N}}$ and $(z_i)_{i\in\mathbb{N}}$ such that y_0 and z_0 are neighbors of x. Note that A is Borel by König's lemma. Now every vertex in the induced subgraph $G \restriction A$ has degree at least 2. Furthermore, every connected component of $G \restriction A$ has at least 3 ends, since A does not meet any connected component of G with 1 end, G contains no connected components with 0 or 2 ends.

Consider the set $X \setminus [A]_{E_G}$ of connected components that do not contain any element of A. This is the set of connected components of G that each have 1 end. Clearly $G \restriction X \setminus [A]_{E_G}$ has a Borel one-ended subforest on this set: map each x to the unique neighbor y such that there is an injective G-ray (z_i) with $z_0 = x$ and $z_1 = y$.

For each $i \in \mathbb{N}$, let A_n be the set of $x \in A$ such that x has distance at least n from every vertex y with $\deg_{G\restriction A}(y) \geqslant 3$, and x is contained in an isolated end of $G \restriction A$. Here by x being contained in an isolated end, we mean that there is an injective $(G \restriction A)$-ray $(x_i)_{i\in\mathbb{N}}$ such that $x = x_0$, and every x_i has degree 2. Note that each A_n is a Borel set. Now applying Lemma 3.2 to the sequence $A_0 \supseteq A_1 \supseteq \cdots$ we can find a one-ended Borel function of $G \restriction [A_0]_{E_G}$.

Let $B = A \setminus [A_0]_{E_G}$. Then clearly $G \restriction (X \setminus [B]_{E_G})$ has a Borel one-ended spanning subforest (since $G \restriction (X \setminus [A]_{E_G})$ and $G \restriction [A_0]_{E_G}$ both do). Furthermore, $G \restriction B$ is ample, since every vertex of $G \restriction A$ has degree at least 2, and $G \restriction B$ has no isolated ends by the definition of A_0. $\square$

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We note that Lemma 3.6 generalizes to locally countable graphs, but where B is analytic, and the one-ended subforest of $G \restriction (X \setminus [B]_{E_G})$ will be $\sigma(\Sigma^1_1)$-measurable.

We are now ready to prove Theorem 1.5.

THEOREM 3.7. *Suppose that G is a locally finite acyclic Borel graph on a standard Borel space X, such that no connected component of G has* 0 *or* 2 *ends.*

(1) *Let μ be a Borel probability measure on X. Then there is a μ-conull Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.*

(2) *Let τ be a compatible Polish topology on X. Then there is a τ-comeager Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.*

Proof. (2) follows directly from Lemma 3.6 and then Lemma 3.5 and Proposition 3.1.

We prove (1). By Lemma 3.6 we may assume that G is ample. By Lemma 3.4 and Proposition 3.1 the theorem is true when G has bounded degree.

By [**16**, Proposition 4.10], we may find a Borel edge coloring of G with $\mathbb{N}$ colors. Let G_n be the subgraph of G consisting of all edges assigned a color $\leqslant n$ so that G_n is a bounded-degree Borel graph. Let $A_n = \{x \in X : G_n \restriction [x]_{E_{G_n}}$ does not have 0 or 2 ends$\}$. Then the graph $G_n \restriction A_n$ is bounded degree, and so we can find a Borel one-ended subforest of each $G_n \restriction A_n$ modulo a null set, by our observation about bounded-degree graphs above.

Thus, modulo a null set, we can find a Borel one-ended subforest of each $G \restriction [A_n]_{E_G}$ via Proposition 3.1, and hence a Borel one-ended subforest of $G \restriction [\bigcup_n A_n]_{E_G}$.

So we need only need to construct our one-ended subforest on the graph $G \restriction (A \setminus [\bigcup_n A_n]_{E_G})$. Now each $G_n \restriction (X \setminus A_n)$ is either 0 or 2-ended. Thus, each $G_n \restriction (X \setminus A_n)$ is μ-hyperfinite since connected components with 0 ends are finite, and for those with 2 ends we can apply [**13**, Lemma 3.20]. Hence $G \restriction (A \setminus [\bigcup_n A_n]_{E_G})$ is an increasing union of μ-hyperfinite graphs and is hence also μ-hyperfinite by [**15**, Theorem 6.11]. Thus, by a result of Adams, [**13**, Lemma 3.21], there is a Borel assignment of one or two ends to each equivalence class of the graph $G \restriction (A \setminus [\bigcup_n A_n]_{E_G})$, modulo a null set. Let C_1 be set where there is a Borel assignment of one end, and C_2 be the subset where there is a Borel assignment of two ends.

Since G is acyclic, any two ends in a connected component of G are jointed by a unique line. Let $B_n \subseteq C_2$ be the set of points of distance at least n from this distinguished line. Since $G \restriction C_2$ is acyclic, each point has degree at least 2, and each connected component does not have 2 ends, B_{n+1} meets each connected component of $G \restriction B_n$ and hence we can apply Lemma 3.2 to the sequence $B_0 \supseteq B_1$ to find a one-ended function contained in the graph of $G \restriction C_2$.

On the set C_1, there is a Borel function $g \colon C_1 \to C_1$ generating $G \restriction C_1$ corresponding to the unique choice of end. In particular, for every $x, y \in C$ there are $n, m \in N$ such that $f^n(x) = f^m(y)$. By Lusin–Novikov uniformization [14, Theorem 18.10], we can find a Borel function $h \colon C_1 \to C_1$ such that for every $x \in X$, $g(h(x)) = x$. Consider the set $C^* \subseteq C_1$ of connected components in $G \restriction C_1$ on which h is not one-ended. Then the set of x such that $h(g(x)) = x$ forms a unique bi-infinite line in C^*, and hence we can find a one-ended Borel function on C^* as we did above on the set C_2. On $C_1 \setminus C^*$, the function h is one-ended, so we are done. □

We note here that (1) in Theorem 1.5 can be generalized to locally countable graphs. Several of the functions and sets that we have used in our argument will be analytic and $\sigma(\Sigma^1_1)$-measurable, but these will become Borel after discarding a null set. The only other modification we need to make is that in the proof above, locally countable acyclic graphs with 0 ends are not necessarily finite, but they are smooth by [20, Theorem A], and hence hyperfinite.

In recent work joint with Damien Gaboriau, the authors have extended Theorem 1.5 to characterize exactly when a (not necessarily acyclic) measure-preserving locally finite graph G has a one-ended spanning subforest.

Now we use the ability to find one-ended functions inside a graph to help definably color the graph.

The following proposition is a trivial modification of [16, Proposition 4.6]. It is proved by partitioning the set B into countably many G-independent Borel sets $A_0, A_1, \ldots$ and then coloring each vertex the least color in $L(x)$ (with respect to a Borel linear ordering of Y) not already used by one of its neighbors.

PROPOSITION 3.8. *Suppose G is a locally finite Borel graph on a standard Borel space X and $B \subseteq X$ is Borel. Then if Y is a Polish space and $L \colon B \to [Y]^{<\infty}$ is a Borel function such that for every $x \in B$, $d_{G \restriction B}(x) < |L(x)|$. Then $G \restriction B$ has a Borel coloring from the lists L.*

We now have the following lemma which is essentially identical to [5, Lemma 2.18].

LEMMA 3.9. *Suppose that G is a locally finite Borel graph on a standard Borel space X, B is a Borel subset of X and $f \colon B \to X$ is a one-ended Borel function*

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whose graph is contained in G. If Y is a Polish space and $L\colon B \to [Y]^{<\infty}$ is a Borel function such that $L(x) \geqslant \deg_G(x)$ for every $x \in B$, then $G \restriction B$ has a Borel coloring from the lists L.

Proof. Let $B_i = B \cap (f^i[B] \setminus f^{i+1}[B])$ so that B_i consists of the points in B that have rank i in the graph generated by f. Note that B is the disjoint union of B_0, $B_1, \ldots$. We will define a coloring c of $G \restriction B$ from the lists L. Let $L_0 = L$. Now iteratively apply Proposition 3.8 to color $G \restriction B_i$ from the lists L_i, and then define $L_{i+1}(x) = L_i(x) \setminus \{c(y) : y \in N(x) \wedge (\exists j < i) y \in B_j\}$. Note that since each $x \in B_i$ has at least one neighbor $f(x)$ not in $B_0, \ldots, B_{i-1}$, that $d_{G \restriction B_i}(x) < L_i(x)$. □

COROLLARY 3.10. *Suppose that G is a locally finite Borel graph on a standard Borel space X and there exists a one-ended Borel function $f\colon X \to X$ whose graph is contained in G. Then G is Borel degree-list-colorable.*

We note that this implies that for every finite d there is an acyclic Borel graph G on X of degree d such that there is no one-ended Borel function $f\colon X \to X$ whose graph is contained in G. This is because by [**19**, Theorem 1.3] for every finite d, there is an acyclic Borel graph of degree d with no Borel d-coloring.

4. A proof of the measurable Brooks' theorem

We are now ready to prove Theorem 1.4 from the introduction.

THEOREM 4.1. *Suppose that G is a locally finite Borel graph on a standard Borel space X and let B be the set of vertices contained in connected components of G that are not Gallai trees. Then $G \restriction B$ is Borel degree-list-colorable.*

Proof. Let $[E_G]^{<\infty} \subseteq [X]^{<\infty}$ be the finite subsets S of X that are contained in a single connected component of G. Let G_I be the intersection graph on $[E_G]^{<\infty}$ so $R\ G_I\ S$ if $R \cap S \neq \emptyset$. Then G_I has a Borel $\mathbb{N}$-coloring c_I (see [**15**, Lemma 7.3] and [**7**, Proposition 2]).

If $S \subseteq X$ is a finite set, its boundary in G is $\partial S = \{y \notin S : \exists x \in S(x\ G\ y)\}$. Let $A \subseteq [E_G]^{<\infty}$ be collection of finite sets S such that each connected component of $G \restriction S$ is not a Gallai tree. Let $A' \subseteq A$ be the set of $S \in A$ such that $c_I(S \cup \partial S) \leqslant c_I(R \cup \partial R)$ for all $R \in A$ in the same G-component as S. Let $B' = \bigcup A'$, so $B' \subseteq B$, and each connected component of $G \restriction B'$ is finite and not a Gallai tree, and B' meets each connected component of $G \restriction B$.

Let $L\colon X \to [Y]^{<\infty}$ be an assignment of lists to each element of x so that $\deg_G(x) = |L(x)|$. By Proposition 3.1, we can find a one-ended function $f\colon B \setminus$

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$B' \to B$, and by Lemma 3.9 we can find a coloring c of $G \restriction (B \setminus B')$ from the lists L. Now let $L' \colon B' \to [Y]^{<\infty}$ be defined by $L'(x) = L(x) \setminus \{c(y) : y \mathrel{G} x \text{ and } y \in B \setminus B'\}$. To finish, there is at least one coloring of each connected component of $G \restriction B'$ from the lists L'. Hence, by Lusin–Novikov uniformization [**14**, Theorem 8.10] we can extend c to a coloring of $G \restriction B$ from the lists L. □

We are now ready to prove a version of Theorem 1.2 for list colorings.

THEOREM 4.2. *Suppose that G is a locally finite Borel graph on a standard Borel space X and G contains no connected components that are finite Gallai trees, and no infinite connected components that are 2-ended Gallai trees.*

(1) *Let μ be any Borel probability measure on X. Then there is a μ-conull G-invariant Borel set B so that $G \restriction B$ is Borel degree-list-colorable.*

(2) *Let τ be any Polish topology compatible with the Borel structure on X. Then there is a G-invariant comeager Borel set B so that $G \restriction B$ is Borel degree-list-colorable.*

Proof. The theorem follows by combining Theorems 4.1, 1.5, and Corollary 3.10.

Let A be the set of vertices contained in connected components of G that are not Gallai trees. Then $G \restriction A$ is Borel degree-list-colorable by Theorem 4.1. Hence, we may as well assume that every connected component of G is an infinite Gallai tree.

Now let $Y \subseteq [X]^{<\infty}$ be the Borel set of blocks of G, and consider the intersection graph $G_I \restriction Y$ on blocks so that two distinct blocks $R, S \in Y$ are adjacent if $R \cap S \neq \emptyset$. Since these blocks are maximal biconnected components of G, there cannot be any cycles in G_I, since such a cycle would imply its constituent blocks were not maximal biconnected components of G. Similarly, any two blocks intersect at a unique vertex. Finally, since no connected components of G are 2-ended Gallai trees, no connected components of $G_I \restriction Y$ have 0 or 2 ends.

Thus, by Theorem 1.5 we can find a Borel one-ended subforest of $G_I \restriction Y$ modulo a null or meager set induced by a function f. The function f then 'lifts' to a one-ended function $\hat{f}$ contained in G as follows. Fix a Borel linear orderings $<_X$ of X and $<_Y$ of Y. Now given a vertex x, since each connected component of G is a locally finite infinite Gallai tree, x is contained at least one and at most finitely many blocks of G. Let $g(x)$ be the $<_Y$-least block containing x. Now define a Borel function $g'(x)$ by letting $g'(x) = g(x)$ if x is not contained in $f(g(x))$, and $g'(x) = f(g(x))$ otherwise. Hence, $g'(x)$ maps each vertex x to a block containing x so that x is not in $f(g'(x))$. Now let the function $\hat{f}$ map each x to the next vertex along the $<_X$-lex least path from x to an element of $f(g'(x))$.

We then finish the proof of the theorem by applying Corollary 3.10 to $\hat{f}$. □

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We note that one application of the above theorem is a new way of constructing antimatchings (see [19]). Recall that an *antimatching* of a graph G on a set X is a Borel function $f: X \to X$ contained in the graph of G such that for every x, we have $f(f(x)) \neq x$. If G is a locally finite graph, then we can map each x to the set $L(x)$ of edges in G incident to x. Then a coloring c of G from the lists L can be used to define an antimatching, by letting $f(x)$ be the unique neighbor y of x in the edge $c(x) = \{x, y\}$.

We can now prove Theorem 1.2 from the introduction:

Proof of Theorem 1.2. Suppose G is a Borel graph of bounded degree at most d, and G does not contain a complete graph on $d + 1$ vertices. Let A be the set of vertices of degree strictly less than d. We begin by d-coloring the connected components $[A]_{E_G}$ contain an element of A. Our idea is to color some element of A' 'last'.

To begin, let $A' \subseteq A$ be G-independent Borel set that meets every connected component of $G \restriction [A]_{E_G}$. Such an A' exist by taking a Borel $(d + 1)$-coloring of G by [16, Proposition 4.6] and then letting A' be the elements of A assigned the least color among all elements of A' in the same connected component. Now apply Proposition 3.1, and Lemma 3.9 to obtain a Borel d-coloring of $G \restriction [A]_{E_G} \setminus A'$, and then color each element of A' the least color not already used by one of its neighbors.

To finish, we need to color the remainder $G \restriction (X \setminus A)$ and so it suffices to show that Theorem 1.2 is true for d-regular graph. But this follows from Theorem 4.2 since the only finite Gallai trees that are d-regular are complete graphs on $d + 1$ vertices, and the only infinite regular two-ended Gallai trees are bi-infinite lines. □

We briefly discuss an alternate way of proving Theorem 1.2. The case $d = 3$ of the theorem is fairly easy to analyze directly. One can then reduce to the case $d = 3$ by iteratively removing maximal independent sets meeting every d-clique using the following proposition.

PROPOSITION 4.3. *Suppose G is a Borel graph on a standard Borel space X of finite bounded degree $\leqslant d$, where $d \geqslant 3$. Suppose further that G contains no cliques on $d + 1$ vertices.*

(1) *Let μ be any Borel probability measure on X. Then there is a μ-measurable maximal independent set $A \subseteq X$ that meets every d-clique contained in G.*

(2) *Let τ be any Polish topology compatible with the Borel structure on X. Then there is a Baire measurable maximal independent set $A \subseteq X$ that meets every d-clique contained in G.*

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Of course, this proposition follows from Theorem 1.2 by extending one of the colors in a d-coloring (which must meet every d-clique) to a maximal independent set. However, it is also simple to prove this proposition directly. Let $Y \subseteq X$ be the vertices that are contained in a unique d-clique. Then let E and F be the relations on Y where $x \ E \ y$ if the unique d-cliques containing x and y are equal and $x \ F \ y$ if $x = y$ or x and y are adjacent in G and are not E-related. One can then use [19, Lemma 4.4.1] to find a μ-measurable or Baire measurable set A meeting every E-class in exactly one point and every F-class in at most one point. From here, extending A to the desired set is straightforward.

5. Applications to group actions

We consider now (almost everywhere) free, measure-preserving actions of a finitely generated group Γ on a standard probability space (X, μ). Denote by $\mathrm{FR}(\Gamma, X, \mu)$ the set of such actions. With each $a \in \mathrm{FR}(\Gamma, X, \mu)$ and finite, symmetric generating set S of Γ not containing the identity we may associate a graph $G(S, a)$ on X by declaring x and y adjacent if there exists $s \in S$ with $s \cdot x = y$. Freeness of the action implies that almost every connected component of $G(S, a)$ is isomorphic to the Cayley graph $\mathrm{Cay}(\Gamma, S)$.

In [6, Theorem 6.1] it is shown that for finitely generated infinite groups Γ, any $a \in \mathrm{FR}(\Gamma, X, \mu)$ is weakly equivalent to some $b \in \mathrm{FR}(\Gamma, X, \mu)$ whose associated graph $G(S, b)$ is measure-theoretically $|S|$-colorable. Theorem 1.2 eliminates the need to pass to a weakly equivalent action for almost all groups.

COROLLARY 5.1. *Suppose that Γ is an infinite group with finite, symmetric generating set S such that $|S| \geqslant 3$. Then for any $a \in \mathrm{FR}(\Gamma, X, \mu)$ the graph $G(S, a)$ admits a Borel $|S|$-coloring on a conull set.*

REMARK 5.2. The only infinite groups with symmetric generating sets S satisfying $|S| < 3$ are $\mathbb{Z}$ with $S = \{\pm 1\}$ and $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^2 = b^2 = \mathrm{id} \rangle$ with $S = \{a, b\}$. Indeed, no graph associated with a free mixing action of either group admits a Borel 2-coloring on a conull set.

Finally, the methods of Section 3 may be used in conjunction with some techniques from probability to improve known bounds on the colorings of Cayley graphs attainable by factors of IID. We consider the *Bernoulli shift* action of a countable group Γ on the space $[0, 1]^\Gamma$ equipped with product Lebesgue measure μ, where $\gamma \cdot x(\delta) = x(\gamma^{-1}\delta)$. Denote by $G(\Gamma, S)$ the graph associated with the Bernoulli shift and generating set S. For convenience we sometimes work instead with the shift action of Γ on $[0, 1]^E$, where E is the edge set of the (right) Cayley

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graph $\mathrm{Cay}(\Gamma, S)$ (and as usual Γ acts by left translation on the Cayley graph). We denote the corresponding graph on $[0, 1]^E$ by $G'(\Gamma, S)$. Since the shift action on $[0, 1]^E$ is measure-theoretically isomorphic to the Bernoulli shift on $[0, 1]^\Gamma$, we lose nothing by working with $G'(\Gamma, S)$ rather than $G(\Gamma, S)$.

We may use each $x \in [0, 1]^E$ to label the edges of its connected component in $G'(\Gamma, S)$, assigning $(\gamma \cdot x, s\gamma \cdot x)$ the label $x(\gamma^{-1}, \gamma^{-1}s^{-1})$. The structure of the action ensures that this labeling is independent of the particular choice of x, and in particular this labeling is a Borel function from $G'(\Gamma, S)$ to $[0, 1]$. Following [18] we obtain the *wired minimal spanning forest*, $\mathrm{WMSF}(G'(\Gamma, S))$, by deleting those edges from $G'(\Gamma, S)$ which receive a label which is maximal in some simple cycle or bi-infinite path. By construction, $\mathrm{WMSF}(G'(\Gamma, S))$ is acyclic.

THEOREM 5.3 (Lyons–Peres–Schramm). *Suppose that Γ is a nonamenable group with finite symmetric generating set S, and consider the graph $G'(\Gamma, S)$ defined above. There is a conull, $G'(\Gamma, S)$-invariant Borel set $B \subseteq [0, 1]^E$ on which each connected component of* $\mathrm{WMSF}(G'(\Gamma, S))$ *has one end.*

Proof. See [18, Theorem 3.12], which says $\mathrm{WMSF}(G'(\Gamma, S))$ is almost surely one-ended provided the Cayley graph of Γ has no infinite clusters at critical percolation. This holds for nonamenable Cayley graphs by [2, Theorem 1.1]. □

Let $\mathrm{Aut}_{\Gamma,S}$ be the automorphism group of the Cayley graph $\mathrm{Cay}(\Gamma, S)$. Given a group Γ with generating set S and a natural number k, we may view the space $\mathrm{Col}(\Gamma, S, k)$ of k-colorings of the (right) Cayley graph $\mathrm{Cay}(\Gamma, S)$ as a closed (thus Polish) subset of k^Γ. The action of Γ by left translations on $\mathrm{Cay}(\Gamma, S)$ induces an action on $\mathrm{Col}(\Gamma, S, k)$. An *automorphism-invariant random k-coloring* of $\mathrm{Cay}(\Gamma, S)$ is a Borel probability measure on $\mathrm{Col}(\Gamma, S, k)$ invariant under this $\mathrm{Aut}_{\Gamma,S}$ action. Such a random k-coloring is a *factor of IID* if it is a factor of the Bernoulli shift of $\mathrm{Aut}_{\Gamma,S}$ on $[0, 1]^\Gamma$. That is, letting λ be Lebesgue measure on $[0, 1]$, a random k-coloring ν is a factor of IID if there is a μ-measurable equivariant function $f : [0, 1]^\Gamma \to k^\Gamma$ such that ν is the pushforward of the product measure μ^Γ under f.

In [17, Section 5] it is asked for which k can automorphism-invariant random k-colorings of Cayley graphs be attained as IID factors (see also [1, Question 10.5]). In [6, Corollary 6.4] translation-invariant random d-colorings of Cayley graphs are constructed, where as usual d is the degree of the graph, but this involves passing to actions weakly equivalent to the Bernoulli shift (or alternatively taking weak limits of IID factors). We can now strengthen this result, giving d-colorings as IID factors except in the cases $\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ where there is the usual ergodic-theoretic obstruction.

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COROLLARY 5.4. *Suppose that Γ is a countable group not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2Z)$, and suppose that S is a finite symmetric generating set for Γ with $|S| = d$. Then there is an automorphism-invariant random d-coloring of* $\mathrm{Cay}(\Gamma, S)$ *which is an IID factor.*

Proof. In the case that Γ is amenable, it has finitely many ends, and so we can apply [6, Theorem 6.7]. Otherwise, Γ is nonamenable and we can apply Corollary 3.10 to obtain from $\mathrm{WMSF}(G'(\Gamma, S))$ a Borel d-coloring $c\colon B \to d$ of the restriction of $G(\Gamma, S)$ to the conull set $B \subseteq [0, 1]^{\Gamma}$ on which $\mathrm{WMSF}(G'(\Gamma, S))$ has one end. Define $\pi : B \to \mathrm{Col}(\Gamma, S, d)$ by $(\pi(x))(\gamma) = c(\gamma^{-1} \cdot x)$. Then $\pi_*\mu$ is a automorphism-invariant random d-coloring which is a factor of IID by construction, where as usual $\pi_*\mu(A) = \mu(\pi^{-1}(A))$. □

REMARK 5.5. Russ Lyons (private communication) points out that this method of proof using spanning forests works for finitely generated groups of more than linear growth by using instead the wired *uniform* spanning forest (WUSF); see [3, Section 10]. The realization of the WUSF as a factor of IID follows from Wilson's algorithm rooted at infinity (see [11, Proof of Proposition 9]) in the case transient case and Pemantle's strong Følner independence [22] in the amenable case.

6. The case $d = 2$

In this section, we prove Theorem 1.6, giving a measurable analogue of Brooks' theorem for the case $d = 2$.

Given a graph G on X, let the equivalence relation $E_{2,G}$ be the equivalence relation on X where $x\ E_{2,G}\ y$ if x and y are connected by a path of even length in G. Then in the case where X is finite, we can rephrase the existence of an odd cycle in the following way: there is a nonempty G-invariant subset A of X such that every nonempty $E_{2,G}$-invariant subset of A is G-invariant.

Now, in the measurable context, even without the presence of odd cycles, there are Borel graphs G and measures μ for which every $E_{2,G}$-invariant Borel set differs by a null set from a Borel G-invariant set. For example, the Borel graph $G_S = \{(x, y) \in \mathbb{T}^2 : S(x) = y \text{ or } S(y) = x\}$ induced by an irrational rotation $S : \mathbb{T} \to \mathbb{T}$ of the unit circle is 2-regular and acyclic, and since S^2 is ergodic with respect to Lebesgue measure, every nonnull $E_{2,G}$-invariant Borel set is Lebesgue conull. It follows that G_S does not admit a $\mu_{\mathbb{T}}$-a.e. Borel 2-coloring, as the color sets in a measurable 2-coloring would have to be disjoint, $E_{2,G}$-invariant, and nonnull since G_S is induced by a measure-preserving transformation and is hence quasiinvariant. Likewise, there is no Baire measurable 2-coloring of G_S with respect to the usual topology on $\mathbb{T}$ since every nonmeager $E_{2,G}$-invariant Borel set of vertices in G_S is comeager.

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If we regard the phenomenon described above as generalization of possessing on odd cycle, then we have the following generalization of Brooks' theorem in the case $d = 2$:

THEOREM 6.1. *Suppose G is a Borel graph on a standard Borel space X with vertex degree bounded by $d = 2$ such that G contains no odd cycles. Let $E_{2,G}$ be the equivalence relation on X where $x \ E_{2,G} \ y$ if x and y are connected by a path of even length in G.*

(1) *Let μ be a G-quasiinvariant Borel probability measure on X. Then G admits a μ-measurable 2-coloring if and only if there does not exist a nonnull G-invariant Borel set A such that every $E_{2,G}$-invariant Borel subset of A differs from a G-invariant set by a null set.*
(2) *Let τ be a G-quasiinvariant Polish topology compatible with the Borel structure on X. Then G admits a Baire measurable 2-coloring if and only if there does not exist a nonmeager G-invariant Borel set A so that every $E_{2,G}$-invariant Borel subset of A differs from a G-invariant set by a meager set.*

Proof. We prove just part (1), since the proof of (2) is similar. Assume first that G admits a μ-measurable 2-coloring with colors sets C_0 and C_1. Now C_0 and C_1 must both be nonnull, since μ is G-quasiinvariant. However, if A is a nonnull G-invariant Borel set, then $A \cap C_0$ is $E_{2,G}$-invariant, however $A \cap C_0$ cannot differ from a G-invariant Borel set by a null set since μ is G-quasiinvariant.

For the converse, assume that for every μ-measurable nonnull G-invariant set A we can find a μ-measurable $E_{2,G}$-invariant $C \subseteq A$ which is not within a null set of being G-invariant. Then the sets $C_0 = \{x \in C : [x]_{E_G} \setminus C \neq \emptyset\}$ and $C_1 = [C_0]_{E_G} \setminus C_0$ are nonnull, and they determine a μ-measurable 2-coloring of $G \restriction [C_0]_{E_G}$. We may continue this process on $X \setminus [C_0]_{E_G}$, and by measure theoretic exhaustion we can obtain a μ-measurable 2-coloring $c : Y \to \{0, 1\}$ of $G \restriction Y$ for some G-invariant conull $Y \subseteq X$. □

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Appendix A. Borel versus measurable colorings

Let $(\mathbb{Z}/2\mathbb{Z})^{*d}$ be the d-fold free product of the group $\mathbb{Z}/2Z$. This group acts via the left shift action on the standard Borel space $\mathbb{N}^{(\mathbb{Z}/2\mathbb{Z})^{*d}}$. Let $X = \{x \in \mathbb{N}^{(\mathbb{Z}/2\mathbb{Z})^{*d}} :$

$\gamma \cdot x \neq x$ for all nonidentity $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{*d}\}$ be the free part of this action. Let $G((\mathbb{Z}/2\mathbb{Z})^{*d}, \mathbb{N})$ be the Borel graph on X where x is adjacent to y if there is a generator γ of $(\mathbb{Z}/2\mathbb{Z})^{*d}$ such that $\gamma \cdot x = y$ or $\gamma \cdot y = x$. Note this graph is acyclic and d-regular. As discussed in the introduction, Marks [19, Theorem 1.2] has shown that this graph has no Borel d-coloring. However, our Theorem 1.2 shows that for every $d \geqslant 3$, there is a μ-measurable and Baire measurable d-coloring of $G((\mathbb{Z}/2\mathbb{Z})^{*d}, \mathbb{N})$ with respect to any Borel probability measure or compatible Polish topology on X.

Hence, for all finite $d \geqslant 3$, for Borel graphs G, admitting μ-measurable d-coloring with respect to every Borel probability measure is a strictly weaker notion than admitting a Borel coloring, as is admitting a Baire measurable d-coloring with respect to every compatible Polish topology, as witnessed by these explicit graphs given above. In this appendix, we show that for 2-colorings these notions are the same, even without any degree assumptions on G.

PROPOSITION A.1. *Let G be a locally countable Borel graph on a standard Borel space X. Then the following are equivalent:*

(1) *G admits a Borel 2-coloring;*

(2) *for every Borel probability measure μ on X, G admits a μ-measurable 2-coloring;*

(3) *for every compatible Polish topology τ on X, G admits a Baire measurable 2-coloring.*

Proof. This is actually a corollary of a more general unpublished result of Louveau (see [21, Theorem 15]). We sketch Louveau's argument in this special case, which uses the G_0-dichotomy [16, Theorem 6.6]. We will prove the equivalence of (1) and (2), since the proof of the equivalence of (1) and (3) is similar. It suffices to show that if G admits no Borel 2-coloring then G admits no μ-measurable 2-coloring for some Borel probability measure μ on X. If G contains an odd cycle then it cannot be 2-colored at all, so we may assume that G contains no odd cycles. Let $G^{\text{odd}} = \{(x, y) \in X^2 : \mathrm{d}_G(x, y) \text{ is odd}\}$, where $\mathrm{d}_G : X^2 \to \mathbb{N} \cup \{\infty\}$ denotes the graph distance in G. Then G^{odd} admits no Borel $\mathbb{N}$-coloring. (Otherwise, by [16, Proposition 4.2] there is a maximal G^{odd}-independent set $A \subseteq X$ which is Borel, and since G contains no odd cycles the set $X \setminus A$ is G^{odd}-independent as well, which contradicts that G admits no Borel 2-coloring.) It follows from [16, Theorem 6.6] that there is an injective Borel homomorphism $f : 2^{\mathbb{N}} \to X$ from the graph G_0 to G^{odd}. Then f is a homomorphism from $G_0^{\text{odd}} = \{(u, v) \in (2^{\mathbb{N}})^2 : \mathrm{d}_{G_0}(u, v) \text{ is odd}\} = \{(u, v) \in 2^{\mathbb{N}} : u \text{ and } v \text{ differ on an odd number of coordinates}\}$ to G^{odd}. Let ν denote the

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uniform product measure on $2^{\mathbb{N}}$. Then every Borel G_0^{odd}-independent set is ν-null (see [5, Example 3.7]), hence every Borel G^{odd}-independent set is $f_*\nu$-null. Fix by the Feldman–Moore theorem [15, Theorem 1.3] a sequence $(T_i)_{i\in\mathbb{N}}$ of Borel automorphisms such that $E_G = \bigcup_i \mathrm{graph}(T_i)$ and let $\mu = \sum_i 2^{-i}(T_i)_* f_*\nu$, so that μ is a G-quasiinvariant probability measure with the same G-invariant null sets as $f_*\nu$. Suppose towards a contradiction that there is a μ-measurable 2-coloring of G. Then there is a Borel 2-coloring $c : B \to \{0, 1\}$ of $G \upharpoonright B$ for some G-invariant μ-conull Borel subset $B \subseteq X$. Since B is invariant, c is a 2-coloring of $G^{\mathrm{odd}} \upharpoonright B$, and since B is μ-conull it is $f_*\nu$-conull, so either $c^{-1}(0)$ or $c^{-1}(1)$ is a Borel G^{odd}-independent set with positive $f_*\nu$-measure, a contradiction. □

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