ON DOMINATED CONVERGENCE

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Introduction

The great theorem on convergence of integrals is due in its usual form to Lebesgue [**2**] though its origins go back to Arzela [**1**]. It says that the integral of the limit of a sequence of functions is the limit of the integrals if the sequence is dominated by an integrable function. This paper investigates the converse problem — if we know that we may take limits under the integral sign, then what can we say about the convergence? The answer is found for functions of a real variable, but it is easily extended to any space with a countably additive measure. Finally the result is illustrated by an application to Fourier series.

The Banach norm in the space $L(-\infty, \infty)$ is denoted by:

$$||f|| = \int_{-\infty}^{\infty} |f(x)| dx$$

LEMMA. Suppose that the real or complex functions $f_1(x), f_2(x), \cdots$ are integrable and finite on $(-\infty, \infty)$ and $f_n(x) \to 0$ for each x, and for any bounded measurable function $g(x)$:

$$\int_{-\infty}^{\infty} g(x) f_n(x) dx \to 0$$

then $||f_n|| \to 0$.

PROOF. Let $h(x) = \max_n |f_n(x)|$, it is measurable and finite for all x because $f_n(x) \to 0$.

There is an increasing sequence Σ of measurable sets such that the union of all of them is the whole line, but such that $\int_S h(x)dx$ is finite for any S in Σ.

Now suppose that the conclusion of the theorem is untrue, so that the upper limit of $||f_n||$ is not zero. There is $\beta > 0$ such that $||f_n|| > 5\beta$ for arbitrarily large n, let N be the set of all n for which this holds.

Take any $m(1)$ in N. There is $A(1)$ in Σ such that:

$$\int_{CA(1)} |f_{m(1)}(x)| dx < \beta$$

(the notation CE is used to denote the complement of any set E). Since $h(x)$ is integrable on $A(1)$ it follows from the theorem on dominated conver-

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gence that:

$$\int_{A(1)} |f_n(x)|dx \to 0.$$

Therefore in N there is $m(2) > m(1)$ such that:

$$\int_{A(1)} |f_{m(2)}(x)|dx < \beta.$$

Now take $A(2) \supset A(1)$ and also in Σ, such that:

$$\int_{CA(2)} |f_{m(2)}(x)|dx < \beta.$$

Proceeding in this way by induction we obtain:

$$m(1) < m(2) < \cdots \text{ (all in } N) \quad \text{and}$$
$$A(1) \subset A(2) \subset \cdots \text{ (all in } \Sigma)$$

such that

$$\int_{CA(i)} |f_{m(i)}(x)|dx < \beta \text{ and} \int_{A(i)} |f_{m(i+1)}(x)|dx < \beta,$$

for when $m(1), \cdots m(i)$ and $A(1), \cdots A(i-1)$ have been fixed then $A(i)$ can be chosen to satisfy the first inequality (because Σ is an increasing sequence of sets covering the line) and then $m(i+1)$ can be chosen (using the theorem on dominated convergence, the dominating function $h(x)$ being integrable on each set of Σ) to satisfy the second inequality.

Now take disjoint sets $B(1)$, $B(2), \cdots$ such that:

$$B(1) = A(1), \quad \text{and} \quad A(i+1) = A(i) \cup B(i+1) \ (i = 1, 2, \cdots).$$

Then for each i we have:

$$5\beta < \int_{-\infty}^{\infty} |f_{m(i)}(x)|dx = \int_{A(i-1)} + \int_{B(i)} + \int_{CA(i)} < 2\beta + \int_{B(i)}.$$

Now define a measurable function $g(x)$ of modulus one as follows. For each i, $g(x)$ on the set $B(i)$ is defined so that $g(x)f_{m(i)}(x) = |f_{m(i)}(x)|$ and therefore:

$$\int_{B(i)} g(x)f_{m(i)}(x)dx = \int_{B(i)} |f_{m(i)}(x)|dx > 3\beta.$$

For any $j = 1, 2, \cdots$ we have:

$$\int_{-\infty}^{\infty} g(x)f_{m(j)}(x)dx = \int_{A(j-1)} + \int_{B(j)} + \int_{CA(j)}.$$

Of the three terms on the right hand side of this equation the first and third are both of modulus less than β, and the second is real and greater than 3β.

Therefore $|\int_{-\infty}^{\infty} g(x)f_{m(j)}(x)dx| > \beta$ for all j. This is a contradiction, so that the lemma is proved.

It is trivial that if $||f_n|| \to 0$ then any sub-sequence contains a sub-sub-sequence $f_{m(n)} (n = 1, 2, \cdots)$ such that $||f_{m(n)}|| < 2^{-n}$, so that the sub-sub-sequence is dominated by the integrable function $\sum_1^\infty |f_{m(n)}(x)|$. The lemma above therefore enables us to assert the following:

THEOREM 1. If the integrable functions $f_n(x)$ tend to $f(x)$p.p. on $(-\infty, \infty)$ and if for any bounded measurable function $g(x)$:

$$\int_{-\infty}^{\infty} f_n(x)g(x)dx \to \int_{-\infty}^{\infty} f(x)g(x)dx$$

then $\|f_n - f\| \to 0$ and also every infinite sub-sequence of $f_1, f_2, \cdots$ contains an infinite sub-sub-sequence that is dominated by an integrable function.

The result can also be expressed in the following way. If $f_n \to f$ pointwise then each of the following three conditions is equivalent to the other two:

(α) The convergence is weak.

(β) The convergence is strong (or metric).

(γ) Every sub-sequence contains a dominated sub-sub-sequence.

The conditions of the theorem do not imply that the sequence itself is dominated, this is shown by taking $f_n(x)$ as the characteristic function of the interval $(\log n, \log(n+1))$.

A result in Titchmarsh [3] (paragraph 13.53, page 421) suggests the following application of Theorem 1.

THEOREM 3. Let $f(x)$ be a Lebesgue integrable function on $(0, 2\pi)$. Let its Fourier series be $\sum_{-\infty}^{\infty} a_n e^{inx}$, and let the Cesaro sums be $C_n(x) = a_0 + \sum_1^n (1 - r/n)(a_r e^{irx} + a_{-r}e^{-irx})$. Then $\|C_n - f\| \to 0$ and every sub-sequence of the Cesaro sums contains a sub-sub-sequence that is dominated by an integrable function.

PROOF. Take any bounded measurable $g(x)$ on $(0, 2\pi)$ and let its Fourier series be $\sum_{-\infty}^{\infty} b_n e^{inx}$, and let its Cesaro sums be:

$$\begin{aligned} H_n(x) &= b_0 + \sum_1^n \frac{n-r}{n}(b_r e^{irx} + b_{-r}e^{-irx}) \\ &= \frac{1}{2n\pi}\int_{-x}^{2\pi-x} \frac{\sin^2 \frac{1}{2}n\theta}{\sin^2\frac{1}{2}\theta} g(x+\theta)d\theta \end{aligned}$$

By the Fejer-Lebesgue theorem $H_n(x) \to g(x)$ p.p., and from its expression as Fejer's integral above it is clear that if $|g(x)| < M$ for all x then $|H_n(x)| < M$ for all x and all n.

Therefore by the theorem on dominated convergence:

$$\begin{aligned} \int_0^{2\pi} f(x)g(x)dx &= \lim \int_0^{2\pi} f(x)H_n(x)dx \\ &= \lim \int_0^{2\pi} b_0 f(x) + \sum^n \frac{n-r}{n} f(x)(b_r e^{irx} + b_{-r}e^{-irx})dx \\ &= 2\pi a_0 b_0 + \lim \sum_1^n \frac{n-r}{n} 2\pi(a_{-r}b_r + a_r b_{-r}) \\ &= \lim \int g(x)C_n(x)dx. \end{aligned}$$

The result now follows by theorem 1 above.

The fact that the Cesaro means are not themselves dominated is shown by the example of:

$$f(x) = x^{-1}(\log x)^{-2} \quad \text{in} \quad (0, 1/4), \quad \text{and} \quad = 0 \quad \text{in} \quad (1/4, 2\pi).$$

For any x in $(0, 1/4)$ there is n such that $1 < 2nx < \frac{1}{2}\pi$, and then $C_n(x) > > 1/(-20\, x \log x)$, and the integral of $\max_n C_n(x)$ therefore diverges at the origin.

References

[1] Arzela, C., Sulla integrazione per serie, *Rom. Acc. L. Rend.*, Vol 1, (1885) 532—537, 566—569.
[2] Lebesgue, H., Sur l'intégration des fonctions discontinues, *Ann. Ecole Norm.* (3) 27, (1910) 361—450.
[3] Titchmarsh, E. C., The theory of functions, Oxford University Press, (1932)

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