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LAPLACE TRANSFORM IDENTITIES FOR THE VOLUME OF STOPPING SETS BASED ON POISSON POINT PROCESSES

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Abstract

We derive Laplace transform identities for the volume content of random stopping sets based on Poisson point processes. Our results are based on anticipating Girsanov identities for Poisson point processes under a cyclic vanishing condition for a finite difference gradient. This approach does not require classical assumptions based on set-indexed martingales and the (partial) ordering of index sets. The examples treated focus on stopping sets in finite volume, and include the random missed volume of Poisson convex hulls.

Keywords: Poisson point process; stopping set; gamma-type distribution; Girsanov identity; anticipating stochastic calculus

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1. Introduction

Gamma-type results for the area of random domains constructed from a finite number of 'typical' Poisson distributed points, and more generally known as complementary theorems, have been obtained in [7]; see also, for example, [14, Theorem 10.4.8].

Stopping sets are random sets that carry over the notion of stopping time to set-indexed processes (cf. [8, Definition 2.27]) based on stochastic calculus for set-indexed martingales; see, for example, [6]. Gamma-type results for the probability law of the volume content of random sets have been obtained in the framework of stopping sets in [15], via Laplace transforms, using the martingale property of set-indexed stochastic exponentials, see [16] for the strong Markov property for point processes; see also [2] for extensions to Poisson processes of k-flats in $\mathbb{R}^d$.

The above mentioned approaches make use of changes of measures by modifying the intensity of the underlying Poisson point process; see also [4, Section 6]. In this paper we further develop and extend the change of measure approach to the derivation of the probability distribution of random sets, based on anticipating Girsanov identities under a measure with density; see Proposition 1 and Corollary 1. Instead of relying on set-indexed adaptedness, we use a cyclic vanishing condition of quasi-nilpotence type for the finite difference gradient of stochastic processes. This approach does not require any (partial) ordering of index sets in the spirit of anticipating stochastic calculus on the Poisson and Wiener spaces.

As a consequence of Girsanov identities we derive Laplace transform identities for the volume of stopping sets in finite volume; see Proposition 4 and Corollaries 3 and 4 below. This approach also recovers classical gamma-type identities [9], [15], for the Laplace transform of the volume of stopping sets; see Corollary 2.

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This paper is organized as follows. In Sections 2 and 3 we state the definitions and preliminary results needed on stopping sets. In Section 4 we state an extension of the Girsanov identities of [10] to measures under a density. In Section 5 we derive equations for the conditional Laplace transform of the random volume content of stopping sets. Examples in finite volume are given, including the convex hull of a Poisson point process. Appendix A contains the technical proof of the anticipating Girsanov identities of Section 4.

2. Poisson point processes

We work with a Poisson point process having a sigma-finite diffuse intensity measure σ on a sigma-compact metric space X with Borel sigma-algebra $\mathcal{B}(X)$. The underlying probability space

$$\Omega^X = \{\omega := (x_i)_{i=1,\ldots,N} \subset X,\ x_i \neq x_j,\ \text{ for all } i \neq j,\ N \in \mathbb{N} \cup \{\infty\}\}$$

is the space of configurations whose elements $\omega \in \Omega^X$ are at most countable and locally finite subsets of X, which are identified with the Radon point measure

$$\omega = \sum_{x \in \omega} \delta_x,$$

where δ_x denotes the Dirac measure at $x \in X$ and $\omega(K) \in \mathbb{N} \cup \{\infty\}$ represents the cardinality of $K \cap \omega$.

Given K in the collection $\mathcal{K}(X)$ of compact subsets of X, we let

$$\mathcal{F}_K = \sigma(\omega(U) \colon U \subset K,\ \sigma(U) < \infty)$$

denote the sigma-algebra generated by $\omega \mapsto \omega(U)$, $U \subset K$, $\sigma(U) < \infty$.

Letting $\mathcal{F} = \bigvee_{K \in \mathcal{K}(X)} \mathcal{F}_K$, the space $(\Omega^X, \mathcal{F})$ is endowed with the probability π_σ on X such that for all compact disjoint subsets $K_1, \ldots, K_n$ of X, $n \geq 1$, the mapping $\omega \mapsto (\omega(K_1), \ldots, \omega(K_n))$ is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective parameters $\sigma(K_1), \ldots, \sigma(K_n)$.

We will use the finite difference operator D_x defined as

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad x \in X,$$

and the iterated difference operator $D_{\mathfrak{s}_k}$ defined by

$$D_{\mathfrak{s}_k} F = D_{s_1} \cdots D_{s_k} F,$$

where $\mathfrak{s}_k = (s_1, \ldots, s_k) \in X^k$, $k \geq 1$, and $D_\varnothing F = F$. Recall that the standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ on $X = \mathbb{R}_+$ is defined by $N_t(\omega) = \omega([0, t])$, $t \in \mathbb{R}_+$. In particular, we have the relation

$$D_{\mathfrak{s}_k} F(\omega) = \sum_{\eta \subset \{s_1, \ldots, s_k\}} (-1)^{k-|\eta|} F(\omega \cup \eta),$$

where the above summation is taken over all (possibly empty) subsets η of Θ.

3. Stopping sets

We recall the definition of a *stopping set*; see [15] and [8, Definition 2.27, p. 335].

Definition 1. A random compact set $A(\omega)$ is called a stopping set if

$$\{\omega\colon A(\omega) \subset K\} \in \mathcal{F}_K \quad \text{for all } K \in \mathcal{K}(X).$$

When $X = \mathbb{R}_+$ and $d = 1$, the interval $[0, \tau]$ is a stopping set when τ is a stopping time in the usual sense with respect to the forward filtration generated by $(N_t)_{t\in\mathbb{R}_+}$. In particular, any interval $[0, T_n]$, where T_n is the nth Poisson jump time is a stopping set. In finite volume with $X = [0, T]$ we can also consider any interval $[0, T_n \wedge T]$ as well as the interval $[T_{N_T}, T]$, where T_{N_T} is the last Poisson jump time before time T with $T_{N_T} = 0$ if $N_T = 0$ (note that the process $\mathbf{1}_{[0,T_{N_T})}(t)$ is predictable with respect to the *backward* Poisson process filtration generated by $(N_t)_{t\in\mathbb{R}_+}$).

When $X = \mathbb{R}^d$ with $d \geq 1$, examples of compact stopping sets include, in infinite volume (see [4] and [5] for other examples)

- the minimal closed ball centered in the origin and containing exactly $n \geq 1$ points;
- the Poisson–Voronoi flower, which is the union of balls centered at the vertices of the Voronoi polygon containing the point 0 and exactly two other process points;
- the closed complement of the convex hull of a Poisson point process inside a convex subset of $\mathbb{R}^d$.

The latter example is a stopping set because the addition of a point within the convex hull will not modify its shape, in other words whether a compact K contains $A(\omega)$ is equivalent to whether K can contain all edges of the convex hull, and this can be decided based on the sole knowledge of the positions of configuration points contained in K.

A stopping set $A(\omega)$ is said to be *nonincreasing* if

$$A(\omega \cup \{x\}) \subset A(\omega), \qquad \omega \in \Omega^X,\ x \in X,$$

which implies that, in particular,

$$D_x \mathbf{1}_A(y) \leq 0, \qquad x, y \in X.$$

A stopping set $A(\omega)$ is said to be *stable* if

$$x \in A(\omega) \quad \Longrightarrow \quad x \in A(\omega \cup \{x\}), \qquad \omega \in \Omega^X,\ x \in X, \tag{1}$$

i.e. $D_x \mathbf{1}_A(x) \geq 0$ for all $x \in X$. In particular, for $A(\omega)$ a stable and nonincreasing stopping set, we have

$$D_x \mathbf{1}_A(x) = 0, \qquad x \in X.$$

The above monotonicity and stability conditions are not restrictive in practice because they are satisfied by common examples of stopping sets.

- The closed complement $A(\omega)$ of the convex hull of a Poisson point process inside a convex subset of $\mathbb{R}^d$ is a stable and nonincreasing stopping set. The stability follows from the fact that the addition of a point $x \in A(\omega)$ to ω creates a new vertex in the convex hull of $\omega \cup \{x\}$. On the other hand, $A(\omega)$ is nonincreasing because the addition of any configuration point can only make the convex hull larger.
- The Poisson–Voronoi flower is also a stable stopping set, which is nonincreasing because each disk is defined by three points while only one of them is displaced by the addition of a new configuration point and the modified disk can only have a smaller radius.

The minimal closed ball centered in the origin and containing exactly $n \geq 1$ points is also a stable and nonincreasing stopping set. The stability property depends on the openness or closedness of the stopping set. For example, the closed complement of the convex hull is stable, while the open complement is not stable according to (1).

The following lemma will be needed for the proof of Proposition 2 below.

Lemma 1. *Let $A(\omega)$ be a nonincreasing stopping set. Then for any $\mathcal{F}_A$-measurable random variable $F(\omega)$, we have*

$$D_x F(\omega) = 0, \qquad x \in A^c(\omega),\ \omega \in \Omega^X.$$

Proof. Consider $B \in \mathcal{F}$ such that

$$B \cap \{\omega\colon A(\omega) \subset K\} \in \mathcal{F}_K;$$

hence,

$$D_x(\mathbf{1}_B(\omega)\mathbf{1}_{\{A(\omega)\subset K\}}) = 0, \qquad x \in K^c$$

for all $K \in \mathcal{K}(X)$ and $\omega \in \Omega^X$. Now let $\omega \in \Omega^X$ and $x \in A^c(\omega)$. There exists $K \in \mathcal{K}(X)$ such that

$$x \in K^c \subset A^c(\omega)$$

and, in particular, $A(\omega \cup \{x\}) \subset A(\omega) \subset K$ since $A(\omega)$ is nonincreasing; hence,

$$\begin{aligned} D_x\mathbf{1}_B(\omega) &= \mathbf{1}_B(\omega \cup \{x\}) - \mathbf{1}_B(\omega) \\ &= \mathbf{1}_B(\omega \cup \{x\})\mathbf{1}_{\{A(\omega\cup\{x\})\subset K\}} - \mathbf{1}_B(\omega)\mathbf{1}_{\{A(\omega)\subset K\}} \\ &= D_x\mathbf{1}_{\{B\cap\{A\subset K\}\}}(\omega) \\ &= 0, \end{aligned}$$

and we extend the statement from $B \in \mathcal{F}_A$ to any $\mathcal{F}_A$-measurable $F(\omega)$ by a monotone class argument.

In particular, from Lemma 1, we have

$$D_x\mathbf{1}_{A(\omega)}(y) = 0, \qquad y \in X,\ x \in A^c(\omega),\ \omega \in \Omega^X, \tag{2}$$

by taking $F = \mathbf{1}_{A(\omega)}(y) \in \mathcal{F}_A$ for $y \in X$.

4. Girsanov identities

Proposition 1 is a Girsanov identity for random, nonadapted shifts of a Poisson point process which extends [10, Proposition 2.1] by including a density F. Recall that the adapted Girsanov identity for a Poisson point process on $X = \mathbb{R}_+$ can be stated as

$$\mathbb{E}\bigg[F \exp\bigg(-\int_0^T u_t\sigma(\mathrm{d}t)\bigg) \prod_{t\in\omega\cap[0,T]} (1+u_t)\bigg] = \mathbb{E}[F],$$

provided that

$$\mathbb{E}\bigg[F \exp\bigg(\int_0^T u_t\sigma(\mathrm{d}t)\bigg) \prod_{t\in\omega\cap[0,T]} (1+u_t)\bigg] < \infty$$

for $(u_t)_{t\in\mathbb{R}_+}$ an adapted process such that $u_t > -1$, $t \in \mathbb{R}_+$, and F an independent nonnegative random variable which is measurable with respect to sigma-algebra generated by the future increments $(N_t - N_s)_{T\leq s\leq t}$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ after time T.

Proposition 1. *Consider $\phi\colon \Omega^X \times X \to \mathbb{R}_+$ a nonnegative process and $F(\omega)$ a nonnegative random variable such that*

$$\begin{aligned} &D_{\Theta_0} F(\omega) D_{\Theta_1} \phi(\omega, x_1) \cdots D_{\Theta_k} \phi(\omega, x_k) = 0, \\ &\sigma^{\otimes k}(dx_1, \ldots, dx_k) \quad \textit{almost everywhere (a.e.)} \end{aligned} \tag{3}$$

for all $\omega \in \Omega^X$, $k \geq 1$, and all families $\Theta_1, \ldots, \Theta_k$ of (possibly empty) subsets of $\{x_1, \ldots, x_k\}$ with union $\Theta_0 \cup \Theta_1 \cup \cdots \cup \Theta_k = \{x_1, \ldots, x_k\}$. Then under the condition

$$\mathbb{E}\bigg[F(\omega) \exp\bigg(\int_X \phi(\omega, x)\sigma(\mathrm{d}x)\bigg) \prod_{x\in\omega} (1 + \phi(\omega, x))\bigg] < \infty,$$

we have the Girsanov identity

$$\mathbb{E}[F(\omega)] = \mathbb{E}\bigg[F(\omega) \exp\bigg(-\int_X \phi(\omega, x)\sigma(\mathrm{d}x)\bigg) \prod_{x\in\omega} (1 + \phi(\omega, x))\bigg].$$

The proof of Proposition 1 is given in Appendix A.

In Proposition 2 below we show that (3) is satisfied by the indicator functions of stopping sets. Given $A(\omega)$ a stopping set we define the stopped sigma-algebra

$$\mathcal{F}_A = \sigma(B \in \mathcal{F} : B \cap \{\omega\colon A(\omega) \subset K\} \in \mathcal{F}_K,\ K \in \mathcal{K}(X));$$

see [15, Definition 1].

Next, we show that the indicator function of a stable and nonincreasing stopping set $A(\omega)$ satisfies (3); see also [3, Proposition 3.3] for a particular situation.

Proposition 2. *For any stable and nonincreasing stopping set $A(\omega)$, (3) is satisfied by $\phi(\omega, x) := \mathbf{1}_{A^c(\omega)}(x)$ and any $\mathcal{F}_A$-measurable random variable $F(\omega)$.*

Proof. Let $x_1, \ldots, x_k \in X$. We consider the following cases.

Case 1: $\{x_1, \ldots, x_k\} \cap A^c(\omega) \neq \varnothing$. First, if there exists $i \in \{1, \ldots, k\}$ such that $x_i \in A^c(\omega)$, then $x_i \in A^c(\omega \cup \eta)$ for any $\eta \subset \{x_1, \ldots, x_k\}$ because $A(\omega)$ is nonincreasing, and this shows that $D_{x_i} \mathbf{1}_{A(\omega\cup\eta)}(x) = 0$ for all $x \in X$ by (2); hence, $D_\Theta \mathbf{1}_{A(\omega)}(x) = D_\Theta \mathbf{1}_{A^c(\omega)}(x) = 0$ whenever $\{x_i\} \subset \Theta \subset \{x_1, \ldots, x_k\}$. It follows that

$$D_{\Theta_1} \mathbf{1}_{A^c(\omega)}(x_1) \cdots D_{\Theta_k} \mathbf{1}_{A^c(\omega)}(x_k) = 0$$

provided that $\Theta_1 \cup \cdots \cup \Theta_k \neq \varnothing$. If $\Theta_1 \cup \cdots \cup \Theta_k = \varnothing$ then $\Theta_0 = \{x_1, \ldots, x_k\}$ and we can assume again that $\{x_1, \ldots, x_k\} \subset A^c(\omega)$, since otherwise we would have

$$\begin{aligned} D_{\Theta_0} F(\omega) D_{\Theta_1} \mathbf{1}_{A^c(\omega)}(x_1) \cdots D_{\Theta_k} \mathbf{1}_{A^c(\omega)}(x_k) &= (D_{\Theta_0} F(\omega)) \mathbf{1}_{A^c(\omega)}(x_1) \cdots \mathbf{1}_{A^c(\omega)}(x_k) \\ &= 0. \end{aligned}$$

Under the condition $\{x_1, \ldots, x_k\} \subset A^c(\omega)$, we have $D_{x_i} F(\omega \cup \eta) = 0$ for all $i = 1, \ldots, k$ by Lemma 1 since $A^c(\omega\cup\eta) \supset A^c(\omega)$ for any $\eta \subset \{x_1, \ldots, x_k\}$, and this shows that $D_{\Theta_0} F(\omega) = 0$ due to the relation

$$D_\Theta F(\omega) = D_{\Theta\setminus\{x_i\}} D_{x_i} F(\omega) = \sum_{\eta\subset\Theta\setminus\{x_i\}} (-1)^{|\Theta|-|\eta|} D_{x_i} F(\omega \cup \eta) = 0,$$

where the above summation is taken over all (possibly empty) subsets η of $\Theta \setminus \{x_i\}$.

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Case 2: $\{x_1,\dots,x_k\}\cap A^c(\omega)=\varnothing$. Next, if $\{x_1,\dots,x_k\}\subset A(\omega)$ then it follows from Lemma 2 below that there exists $x_e\in\{x_1,\dots,x_k\}$ such that $x_e\in A(\omega\cup\{x_1,\dots,x_k\})$. Hence, since $A(\omega)$ is nonincreasing, we have

$$\mathbf{1}_{A(\omega\cup\eta)}(x_e)=\mathbf{1}_{A(\omega)}(x_e)=1\quad\text{for all }\eta\subset\{x_1,\dots,x_k\},$$

and $D_\Theta\mathbf{1}_{A(\omega)}(x_e)=0$ for all nonempty $\Theta\subset\{x_1,\dots,x_k\}$, by the relation

$$\begin{aligned}D_\Theta\mathbf{1}_{A(\omega)}(x_e)&=\sum_{\eta\subset\Theta}(-1)^{|\Theta|+1-|\eta|}\mathbf{1}_{A(\omega\cup\eta)}(x_e)\\&=\mathbf{1}_{A(\omega)}(x_e)\sum_{\eta\subset\Theta}(-1)^{|\Theta|+1-|\eta|}\\&=\mathbf{1}_{A(\omega)}(x_e)(1-1)^{|\Theta|+1}\\&=0,\end{aligned}$$

where the summation above is taken over all (possibly empty) subsets η of Θ. As a consequence, a factor in the product

$$D_{\Theta_1}\mathbf{1}_{A^c(\omega)}(x_1)\cdots D_{\Theta_k}\mathbf{1}_{A^c(\omega)}(x_k)=\prod_{l=1}^k D_{\Theta_l}\mathbf{1}_{A^c(\omega)}(x_l)$$

has to vanish when $\Theta_1\cup\cdots\cup\Theta_k\neq\varnothing$. In the $\Theta_1\cup\cdots\cup\Theta_k=\varnothing$ case, we can show as in case 1 above that $D_{\Theta_0}F(\omega)=0$, which concludes the proof.

The next lemma has been used in the proof of Proposition 2.

Lemma 2. *Let $A(\omega)$ be a stable and nonincreasing stopping set. For any $\omega\in\Omega^X$ and $x_1,\dots,x_k\in A(\omega)$, there exists $i\in\{1,\dots,k\}$ such that $x_i\in A(\omega\cup\{x_1,\dots,x_k\})$.*

Proof. Assume that $\{x_1,\dots,x_k\}\subset A^c(\omega\cup\{x_1,\dots,x_k\})$. We will show that

$$A^c(\omega\cup\{x_1,\dots,x_k\})=A^c\Big(\omega\cup\bigcup_{i=j}^k\{x_i\}\Big),\tag{4}$$

by induction on $j=1,\dots,k+1$ with the convention $\cup_{i=k+1}^k\{x_i\}=\varnothing$. This leads to $A^c(\omega\cup\{x_1,\dots,x_k\})=A^c(\omega)$ for $j=k+1$, and to $x_j\in A^c(\omega)$, $j=1,\dots,k$, which is a contradiction since we assumed that $\{x_1,\dots,x_k\}\subset A(\omega)$.

Relation (4) clearly holds for $j=1$ and we suppose that it holds for some $j\in\{1,\dots,k\}$. By assumption, we have $x_j\in A^c(\omega\cup\{x_1,\dots,x_k\})$, which implies that

$$x_j\in A^c(\omega\cup\{x_1,\dots,x_k\})=A^c\Big(\omega\cup\bigcup_{i=j}^k\{x_i\}\Big);$$

hence, $x_j\in A^c(\omega\cup\cup_{i=j+1}^k\{x_i\})$ by the stability condition (1). Consequently, by (2) or Lemma 1, we have

$$A^c(\omega\cup\{x_{j+1},\dots,x_k\})=A^c\Big(\omega\cup\bigcup_{i=j}^k\{x_i\}\Big)$$

since $A(\omega)$ is a stable and nonincreasing stopping set.

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5. Laplace transforms of stopping sets

In the following consequence of Propositions 1 and 2 we start by recovering the conditional moment generating function of $\sum_{x\in\omega} g(x)\mathbf{1}_{A^c(\omega)}(x)$, given $\mathcal{F}_A$ for g in the space $\mathcal{C}_c(X)$ of continuous functions with compact support in X. Given that this moment generating function characterizes the point process distribution, we recover the intuitive fact that given $\mathcal{F}_A$, the restriction of ω to $A^c(\omega)$ is a Poisson point process with intensity $\mathbf{1}_{A^c(\omega)}(x)\sigma(\mathrm{d}x)$ when $A(\omega)$ is a stopping set.

Proposition 3. *For any stable and nonincreasing stopping set $A(\omega)$, we have*

$$\mathbb{E}\bigg[\exp\bigg(\sum_{x\in\omega\cap A^c(\omega)} g(x)\bigg)\,\bigg|\,\mathcal{F}_A\bigg] = \mathbb{E}\bigg[\exp\bigg(\int_{A^c(\omega)} (\mathrm{e}^{g(x)}-1)\sigma(\mathrm{d}x)\bigg)\,\bigg|\,\mathcal{F}_A\bigg]$$

for all nonnegative $g \in \mathcal{C}_c(X)$.

Proof. Taking $f(x) = \mathrm{e}^{g(x)} - 1$, $x \in X$, by Propositions 1 and 2, we have

$$\mathbb{E}\bigg[F(\omega)\exp\bigg(-\int_{A_c(\omega)} f(x)\sigma(\mathrm{d}x)\bigg)\prod_{x\in\omega\cap A^c(\omega)}(1+f(x))\bigg] = \mathbb{E}[F(\omega)] \tag{5}$$

for any $\mathcal{F}_A$-measurable bounded random variable $F(\omega)$. Next, letting

$$F = G\exp\bigg(\int_{A^c(\omega)} f(x)\sigma(\mathrm{d}x)\bigg) = G\exp\bigg(\int_X f(x)\sigma(\mathrm{d}x) - \int_{A(\omega)} f(x)\sigma(\mathrm{d}x)\bigg),$$

where G is a $\mathcal{F}_A$-measurable bounded random variable, we obtain

$$\mathbb{E}\bigg[G\prod_{x\in\omega\cap A^c(\omega)}(1+f(x))\bigg] = \mathbb{E}\bigg[G\exp\bigg(-\int_{A^c(\omega)} f(x)\sigma(\mathrm{d}x)\bigg)\bigg].$$

As a consequence of Propositions 1 and 2 with $\phi := z\mathbf{1}_{A^c(\omega)}$, $z > 0$, we also obtain the following corollary.

Corollary 1. *Consider $A(\omega)$ a stable and nonincreasing stopping set and $F(\omega)$ a nonnegative $\mathcal{F}_A$-measurable random variable with*

$$\mathbb{E}[F(\omega)\mathrm{e}^{z\sigma(A^c)}(1+z)^{\omega(A^c)}] < \infty \quad \textit{for some } z > 0.$$

We have the Girsanov identity

$$\mathbb{E}[F(\omega)] = \mathbb{E}[F(\omega)\mathrm{e}^{-z\sigma(A^c)}(1+z)^{\omega(A^c)}], \qquad z > 0. \tag{6}$$

The gamma Laplace transform $\mathbb{E}[\mathrm{e}^{-zT_n}] = (1+z)^{-n}$ of the nth Poisson jump time T_n can be recovered as a straightforward application of Girsanov identities to the stopping set $A = [0, T_n]$ with respect to the standard Poisson process filtration.

The following consequence of (6) on the conditional Laplace transform is consistent with the gamma-type results of [9, Theorem 2] and [15, Theorem 2].

Corollary 2. *Let $A(\omega)$ be a stable and nonincreasing stopping set. We have the conditional Laplace transform*

$$\mathbb{E}[\mathrm{e}^{-z\sigma(A)} \mid \omega(A) = n] = \bigg(\frac{1}{(1+z)^n}\bigg)\bigg(\frac{\mathbb{P}_z(\{\omega(A)=n\})}{\mathbb{P}(\{\omega(A)=n\})}\bigg), \qquad z > 0,\ n \in \mathbb{N}, \tag{7}$$

where $\mathbb{P}_z$ denotes the Poisson point process distribution with intensity $z\sigma(\mathrm{d}x)$.

Proof. Taking $F = \mathbf{1}_{\{\omega(A)=n\}} \mathrm{e}^{-z\sigma(A)} \in \mathcal{F}_A$, by (6), we obtain

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-z\sigma(A)} \mathbf{1}_{\{\omega(A)=n\}}] &= \mathbb{E}[\mathrm{e}^{-z(\sigma(X)-\sigma(A^c))} \mathbf{1}_{\{\omega(A)=n\}}] \\
&= \mathrm{e}^{-z\sigma(X)} \mathbb{E}[(1+z)^{\omega(A^c)} \mathbf{1}_{\{\omega(A)=n\}}] \\
&= \frac{1}{(1+z)^n} \mathrm{e}^{-z\sigma(X)} \mathbb{E}[(1+z)^{\omega(X)} \mathbf{1}_{\{\omega(A)=n\}}] \\
&= \frac{1}{(1+z)^n} \mathbb{P}_z(\{\omega(A)=n\}).
\end{aligned}$$

When $\mathbb{P}_z(\{\omega(A)=n\})$ does not depend on $z > 0$ as assumed in [15], from Corollary 2 we recover the gamma Laplace transform

$$\mathbb{E}[\mathrm{e}^{-z\sigma(A)} \mid \omega(A)=n] = \frac{1}{(1+z)^n}, \qquad z > 0,$$

conditionally to the number $n \geq 0$ of points in A.

Stopping sets in finite volume. In the $\sigma(X) < \infty$ case, taking $F = \mathrm{e}^{-z\sigma(A)}$ in Corollary 1 we can derive the Laplace transform

$$\mathbb{E}[\mathrm{e}^{-z\sigma(A)}] = \mathrm{e}^{-z\sigma(X)} \mathbb{E}[(1+z)^{\omega(A^c)}], \qquad z > 0$$

of the random variable $\sigma(A(\omega))$, where $A(\omega)$ is a stable and nonincreasing stopping set. More generally, from (6), we have

$$\mathbb{E}[f(\sigma(A^c))\mathbf{1}_{\{\omega(A)=n\}}] = \mathbb{E}[f(\sigma(A^c))\mathrm{e}^{-z\sigma(A^c)} \mathbf{1}_{\{\omega(A)=n\}} (1+z)^{\omega(A^c)}] \quad \text{for all } z > 0. \tag{8}$$

In the next proposition we provide a more explicit form for (8) by denoting $\sigma_n^c(x_1, \ldots, x_n)$ the volume content of the complement $A^c(\omega)$ in X when $A \cap \omega = \{x_1, \ldots, x_n\}$ has $n \in \mathbb{N}$ points.

Proposition 4. *Assume that $\sigma(X) < \infty$ and consider a stable and nonincreasing stopping set $A(\omega)$. We have*

$$\mathbb{E}[f(\sigma(A^c(\omega)))\mathbf{1}_{\{\omega(A)=n\}}] = \frac{\mathrm{e}^{-\sigma(X)}}{n!} \int_{X^n} \mathrm{e}^{\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n)) \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n), \tag{9}$$

$n \geq 1$ for f bounded and measurable on $\mathbb{R}$, where

$$\mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) := \mathbf{1}_{\{A(\{x_1,\ldots,x_n\}) \supset \{x_1,\ldots,x_n\}\}} \sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_n), \qquad n \geq 1.$$

Proof. By (8), we have, conditioning on the number k of points in $A^c(\omega)$,

$$\begin{aligned}
&\mathbb{E}[f(\sigma(A^c))\mathrm{e}^{-z\sigma(A^c)} \mathbf{1}_{\{\omega(A)=n\}} (1+z)^{\omega(A^c)}] \\
&\quad = \frac{\mathrm{e}^{-\sigma(X)}}{n!} \sum_{k=0}^{\infty} \frac{(1+z)^k}{k!} \\
&\qquad \times \int_{X^n} (\sigma_n^c(x_1, \ldots, x_n))^k \mathrm{e}^{-z\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n)) \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)
\end{aligned}$$

$$= \frac{\mathrm{e}^{-\sigma(X)}}{n!} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{1}{(k-l)!} \frac{z^l}{l!}$$
$$\times \int_{X^n} (\sigma_n^c(x_1, \ldots, x_n))^k \mathrm{e}^{-z\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n)) \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$$
$$= \frac{\mathrm{e}^{-\sigma(X)}}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=0}^{\infty} \frac{z^l}{l!}$$
$$\times \int_{X^n} (\sigma_n^c(x_1, \ldots, x_n))^{m+l} \mathrm{e}^{-z\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n)) \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$$
$$= \frac{\mathrm{e}^{-\sigma(X)}}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{X^n} (\sigma_n^c(x_1, \ldots, x_n))^m f(\sigma_n^c(x_1, \ldots, x_n)) \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n),$$

which yields (9) by (8).

By (9), we have

$$\mathbb{E}[\mathrm{e}^{-z\sigma(A)} \mathbf{1}_{\{\omega(A)=n\}}] = \frac{\mathrm{e}^{-(1+z)\sigma(X)}}{n!} \int_{X^n} \mathrm{e}^{(1+z)\sigma_n^c(x_1,\ldots,x_n)} \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$$
$$= \frac{1}{n!} \int_0^{\sigma(X)} \mathrm{e}^{-(1+z)x} \nu_n(\mathrm{d}x), \tag{10}$$

where $\nu_n(\mathrm{d}x)$ is the image measure on $[0, \sigma(X)]$ of $\mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$ by

$$(x_1, \ldots, x_n) \mapsto \sigma(X) - \sigma_n^c(x_1, \ldots, x_n)$$

with $\nu_1(\mathrm{d}x) = \nu_2(\mathrm{d}x) = \delta_{\sigma(X)}(\mathrm{d}x)$. Hence,

$$\mathbb{P}(\{\omega(A) = n\}) = \frac{\mathrm{e}^{-\sigma(X)}}{n!} \int_{X^n} \mathrm{e}^{\sigma_n^c(x_1,\ldots,x_n)} \mu_n(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) \tag{11}$$
$$= \frac{1}{n!} \int_0^{\sigma(X)} \mathrm{e}^{-x} \nu_n(\mathrm{d}x) \tag{12}$$

and the probability distribution of the random variable $\sigma(A^c)$ on $\{\omega(A) = n\}$ is given by

$$\frac{1}{n!} \mathrm{e}^{-x} \nu_n(\mathrm{d}x), \qquad n \geq 1.$$

Consequently, we have the following corollary of Proposition 4 which, in comparison with Corollary 3, provides an expression for the ratio $\mathbb{P}_z(\{\omega(A) = n\})/\mathbb{P}(\{\omega(A) = n\})$, $z > 0$.

Corollary 3. *Assume that $\sigma(X) < \infty$. For any stable and nonincreasing stopping set $A(\omega)$, we have*

$$\mathbb{E}[\mathrm{e}^{-z\sigma(A)} \mid \omega(A) = n] = \frac{\int_0^{\sigma(X)} \mathrm{e}^{-(1+z)x} \nu_n(\mathrm{d}x)}{\int_0^{\sigma(X)} \mathrm{e}^{-x} \nu_n(\mathrm{d}x)}, \qquad z \in \mathbb{R}_+,\ n \in \mathbb{N}. \tag{13}$$

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Proof. By (7), (10), and (12), we have

$$\begin{aligned}\mathbb{E}[e^{-z\sigma(A)} \mid \omega(A) = n] &= \frac{1}{(1+z)^n} \frac{\mathbb{P}_z(\{\omega(A) = n\})}{\mathbb{P}(\{\omega(A) = n\})} \\ &= \frac{\mathbb{E}[e^{-z\sigma(A)} \mathbf{1}_{\{\omega(A)=n\}}]}{\mathbb{P}(\{\omega(A) = n\})} \\ &= \frac{\int_0^{\sigma(X)} e^{-(1+z)x} \nu_n(dx)}{\int_0^{\sigma(X)} e^{-x} \nu_n(dx)}.\end{aligned}$$

The above analysis also yields the Laplace transform of $\sigma(A)$.

Corollary 4. *Assume that $\sigma(X) < \infty$. For any stable and nonincreasing stopping set $A(\omega)$, we have the Laplace transform*

$$\begin{aligned}&\mathbb{E}[e^{-z\sigma(A)}] \\ &\quad = e^{-\sigma(X)} + e^{-(1+z)\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} e^{(1+z)\sigma_n^c(x_1,\dots,x_n)} \mu_n(dx_1, \dots, dx_n), \qquad z > 0.\end{aligned} \tag{14}$$

Proof. By Proposition 4, we have

$$\begin{aligned}&\mathbb{E}[f(\sigma(A^c))] \\ &\quad = \mathbb{E}[f(\sigma(A^c)) e^{-z\sigma(A^c)} (1+z)^{\omega(A^c)}] \\ &\quad = e^{-\sigma(X)} f(0) + e^{-\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} e^{\sigma_n^c(x_1,\dots,x_n)} f(\sigma_n^c(x_1, \dots, x_n)) \mu_n(dx_1, \dots, dx_n).\end{aligned}$$

Next, we consider some examples of applications for Corollaries 3 and 4.

Convex hulls of Poisson point processes. The closed complement $A(\omega)$ of the (open) convex hull $A^c(\omega)$ of a Poisson point process in a convex domain X of finite volume in $\mathbb{R}^d$ is a stable and nonincreasing stopping set; see Section 3 and Figure 1 for an illustration.

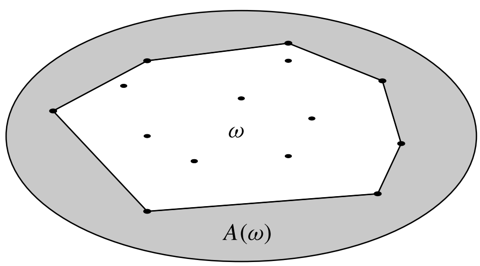

FIGURE 1: Convex hull of a Poisson point process.

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When $n = 1, 2$, we clearly have $\mathbb{P}(\{\omega(A) = n\}) = \mathbb{P}(\{\omega(X) = n\}), \sigma_1^c(x_1) = 0$, and $\sigma_2^c(x_1, x_2) = 0$. When $n = 3$, $\sigma_3^c(x_1, x_2, x_3)$ is given by, when $X \subset \mathbb{R}^2$ by Heron's formula,

$$\begin{aligned}
&\sigma_3(x_1, x_2, x_3) \\
&\quad = \tfrac{1}{4}\sqrt{(|x_1 - x_2| + |x_1 - x_3| + |x_2 - x_3|)(-|x_1 - x_2| + |x_1 - x_3| + |x_2 - x_3|)} \\
&\qquad \times \sqrt{(|x_1 - x_2| - |x_1 - x_3| + |x_2 - x_3|)(|x_1 - x_2| + |x_1 - x_3| - |x_2 - x_3|)},
\end{aligned}$$

$x_1, x_2, x_3 \in X$, which can also be used to compute $\sigma_n^c(x_1, \dots, x_n)$ for any convex polytope by triangulation. In higher dimensions, Heron's formula can be replaced with simplex volumes that can be computed with the help of the Cayley–Menger determinants, and the Laplace transform of $\sigma(A)$ can be computed from (14). By Corollaries 3 and 4 and the expression of $\sigma_n^c(x_1, \dots, x_n)$, we can compute the conditional and unconditional Laplace transforms $\mathbb{E}[\mathrm{e}^{-z\sigma(A)} \mid \omega(A) = n]$ and $\mathbb{E}[\mathrm{e}^{-z\sigma(A)}]$, $z > 0, n \in \mathbb{N}$.

Last Poisson jump time. When $X = [0, T]$, the last Poisson jump time T_{N_T} before time T, with $T_{N_T} = 0$ if $N_T = 0$, is not a stopping time for the forward filtration of the Poisson process; however, T_{N_T} can be seen as the first jump time of the time reversed Poisson process $(N_{T-t})_{t\in[0,T]}$ and the process

$$u(t, \omega) = z\mathbf{1}_{[T_{N_T}, T]}(t) = z(1 - \mathbf{1}_{[0, T_{N_T})}(t)), \qquad t \in \mathbb{R}_+,$$

is backward predictable on $[0, T]$. In this case, by (5) with $A = [T_{N_T}, T]$ and $\sigma(\mathrm{d}t) = \mathrm{d}t$, $F = f(T_{N_T})$ and $\phi(t) = \mathbf{1}_{[0,T]}(t)$, we simply have $\mu_1(\mathrm{d}t_1) = \mathrm{d}t_1$ and $\mu_n(\mathrm{d}t_1, \dots, \mathrm{d}t_n) = 0$, $n \geq 2$; hence, by (14), we have

$$\begin{aligned}
&\mathbb{E}[f(\sigma(A^c))\mathrm{e}^{-z\sigma(A^c)}(1+z)^{\omega(A^c)}] \\
&\quad = f(0)\mathbb{P}(\{N_T = 0\}) + \mathbb{E}[f(T_{N_T})\mathrm{e}^{-zT_{N_T}}(1+z)^{N_T - 1}\mathbf{1}_{\{N_T \geq 1\}}] \\
&\quad = f(0)\mathrm{e}^{-\sigma(T)} + \mathrm{e}^{-T}\int_X \mathrm{e}^{\sigma_1^c(t)} f(\sigma_1^c(t))\mu_1(\mathrm{d}t) \\
&\quad = \mathrm{e}^{-T} f(0) + \int_0^T \mathrm{e}^{-(T-t)} f(t)\,\mathrm{d}t.
\end{aligned}$$

Hence, taking $f(x) = \mathrm{e}^{xz}\mathbf{1}_{(0,\infty)}(x)$, from (13), we recover the Laplace transform

$$\mathbb{E}[\mathrm{e}^{-z(T - T_{N_T})} \mid N_T \geq 1] = \frac{1}{1+z}\left(\frac{\mathrm{e}^T - \mathrm{e}^{-zT}}{\mathrm{e}^T - 1}\right), \qquad z > -1,$$

of the truncated exponential distribution on $[0, T]$.

Annuli in finite volume. In the case where X is a ball centered at 0 in $\mathbb{R}^d$ we can consider the stable and nonincreasing stopping set $A(\omega) = B_m(\omega) \cap X$, where $B_m(\omega)$ is the smallest closed ball centered at the origin and containing $m \geq 1$ process points in ω. Here, we have

$$\sigma_n^c(x_1, \dots, x_n) = \mathbf{1}_{\{n \geq m\}}(\sigma(X) - v_d(\max(|x_1|, \dots, |x_n|))), \qquad n \in \mathbb{N},$$

where $v_d(r)$ is the volume of the d-dimensional ball with radius r. In the $d = 1$ case, $X = [0, T]$ and $A = [0, T_m \wedge T]$, we have

$$\sigma_n^c(x_1, \dots, x_n) = \mathbf{1}_{\{n \geq m\}}(T - \max(x_1, \dots, x_n)), \qquad n \in \mathbb{N}.$$

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For $n = 1, \ldots, m$, we have $\mathbb{P}(\{\omega(A) = n\}) = \mathbb{P}(N_T = n)$ and $\sigma(X) = T$, while for $n \geq m+1$ we have $\mathbb{P}(\{\omega(A) = n\}) = \mathbb{P}(T_n \leq T)$. By Proposition 9,

$$\begin{aligned}
&\mathbb{E}[f(\sigma(A))\mathrm{e}^{-z\sigma(A^c)}(1+z)^{\omega(A^c)}\mathbf{1}_{\{\omega(A)=n\}}] \\
&\quad = \frac{1}{n!}\int_{X^n} \mathrm{e}^{\sigma(X)-\sigma_n^c(x_1,\ldots,x_n)} f(\sigma(X) - \sigma_n^c(x_1, \ldots, x_n))\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&\quad = \frac{1}{n!}\int_0^T \cdots \int_0^T \mathrm{e}^{-\max(t_1,\ldots,t_n)} f(\max(t_1, \ldots, t_n))\,\mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&\quad = \frac{1}{(n-1)!}\int_0^T \mathrm{e}^{-t} f(t) t^{n-1}\,\mathrm{d}t
\end{aligned}$$

and (13) becomes the Laplace transform

$$\mathbb{E}[\mathrm{e}^{-zT_m} \mid T_m < T] = \frac{1}{(1+z)^m}\frac{\mathbb{P}_z(N_T \geq m)}{\mathbb{P}(N_T \geq m)} = \frac{1 - \mathrm{e}^{-(1+z)T}\sum_{k=0}^{m-1}((1+z)T)^k/k!}{(1+z)^m(1 - \mathrm{e}^{-T}\sum_{k=0}^{m-1} T^k/k!)}$$

of the truncated gamma distribution on $[0, T]$.

Appendix A. Anticipating Girsanov identities

In this appendix we provide the proof of Proposition 1, which extends the argument of [10, Proposition 2.1] to take into account a density $F(\omega)$. We will use the multiple Poisson stochastic integral

$$I_n(f_n)(\omega) := \int_{\Delta_n} f_n(x_1, \ldots, x_n)(\omega(\mathrm{d}x_1) - \sigma(\mathrm{d}x_1)) \cdots (\omega(\mathrm{d}x_n) - \sigma(\mathrm{d}x_n)),$$

where f_n is a symmetric function of n variables in the space $L^2_\sigma(X^n)$ of functions on X^n which are square-integrable with respect to $\sigma^{\otimes n}$, and

$$\Delta_n = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for all } i \neq j\}.$$

The proof of Proposition 1 will be based on the following Lemmas 3 and 4. In the sequel we let $u^{\otimes n}(x_1, \ldots, x_n) := u(x_1) \cdots u(x_n)$, $x_1, \ldots, x_n \in X$ for $u \in L^2_\sigma(X)$.

Lemma 3. *Let $A(\omega)$ be a random set and $F(\omega)$ a bounded random variable. We have*

$$\mathbb{E}[F(\omega)I_n(\mathbf{1}_{A(\omega)}^{\otimes n})] = \mathbb{E}\bigg[\int_{X^n} D_{x_1} \cdots D_{x_n}\bigg(F(\omega)\prod_{p=1}^n \mathbf{1}_{A(\omega)}(x_p)\bigg)\sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_n)\bigg].$$

Proof. For all (possibly random) disjoint subsets $A_1(\omega), \ldots, A_n(\omega)$ of X with finite measure, denoting by $\mathbf{1}_{A_1^{k_1}(\omega)} \circ \cdots \circ \mathbf{1}_{A_n^{k_n}(\omega)}$ the symmetrization in $k_1 + \cdots + k_n$ variables of the function $\mathbf{1}_{A_1^{k_1}(\omega)} \otimes \cdots \otimes \mathbf{1}_{A_n^{k_n}(\omega)}$, we have

$$I_N(\mathbf{1}_{A_1^{k_1}(\omega)} \circ \cdots \circ \mathbf{1}_{A_n^{k_n}(\omega)}) = \prod_{i=1}^n C_{k_i}(\omega(A_i), \sigma(A_i))$$

between the multiple Poisson integrals and the Charlier polynomial defined as

$$C_n(x, \lambda) = \sum_{k=0}^n x^k \sum_{l=0}^n \binom{n}{l}(-\lambda)^{n-l}s(l, k), \qquad x, \lambda \in \mathbb{R}.$$

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See [13, Section 4.3.3], where $s(k,l)=(1/l!)\sum_{i=0}^{l}(-1)^i\binom{l}{i}(l-i)^k$ is the Stirling number of the first kind (see [1, p. 824]), i.e. $(-1)^{k-l}s(k,l)$ is the number of permutations of k elements which contain exactly l permutation cycles, $n\in\mathbb{N}$. By the moment identity

$$\mathbb{E}[F(\omega)(\omega(A))^k]=\mathbb{E}\bigg[\int_{X^j}\varepsilon^+_{\mathfrak{s}_j}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_j))\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_j)\bigg];$$

see [11, Proposition 3.1] or [12, Theorem 1], where $\varepsilon^+_{\mathfrak{s}_k}$ is the addition operator defined on any random variable $F\colon\Omega^X\longrightarrow\mathbb{R}$ by

$$\varepsilon^+_{\mathfrak{s}_k}F(\omega)=F(\omega\cup\{s_1,\ldots,s_k\}),\qquad \omega\in\Omega^X,\ s_1,\ldots,s_k\in X,$$

and using the Stirling inversion formula

$$\sum_{k=l}^{n}S(n,k)s(k,l)=\sum_{k=0}^{n}S(n,k)s(k,l)=\mathbf{1}_{\{n=l\}},\qquad n,l\geq 0;$$

see, for example, [1, p. 825], we have

$$\begin{aligned}
&\mathbb{E}[F(\omega)I_n(\mathbf{1}^{\otimes n}_{A(\omega)})]\\
&\quad=\mathbb{E}[F(\omega)C_n(\omega(A),\sigma(A))]\\
&\quad=\sum_{l=0}^{n}\sum_{k=0}^{n}\mathbb{E}\bigg[F(\omega)\Big(\omega(A)\Big)^k\binom{n}{l}(-\sigma(A))^{n-l}s(l,k)\bigg]\\
&\quad=\sum_{l=0}^{n}\binom{n}{l}s(l,k)\sum_{k=0}^{n}\\
&\qquad\times\sum_{j=0}^{k}S(k,j)\mathbb{E}\bigg[\int_{X^j}\varepsilon^+_{\mathfrak{s}_j}(F(\omega)(-\sigma(A))^{n-l}\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_j))\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_j)\bigg]\\
&\quad=\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}\sum_{j=0}^{n}\sum_{k=0}^{n}s(l,k)S(k,j)\\
&\qquad\times\mathbb{E}\bigg[\int_{X^{n-l+j}}\varepsilon^+_{\mathfrak{s}_j}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_{n-l+j}))\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_{n-l+j})\bigg]\\
&\quad=\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}\mathbb{E}\bigg[\int_{X^n}\varepsilon^+_{\mathfrak{s}_l}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_n))\sigma(dx_1)\cdots\sigma(\mathrm{d}x_n)\bigg]\\
&\quad=\mathbb{E}\bigg[\int_{X^n}D_{x_1}\cdots D_{x_n}\bigg(F(\omega)\prod_{p=1}^{n}\mathbf{1}_{A(\omega)}(x_p)\bigg)\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_n)\bigg].
\end{aligned}$$

The next lemma is needed in the proof of Proposition 1.

Lemma 4. *Assume that $\phi\colon\Omega^X\times X\to\mathbb{R}_+$ is a nonnegative process and $F(\omega)$ is a nonnegative random variable satisfying (3). Then for all bounded nonnegative random processes $\phi\colon X\times\Omega^X\to\mathbb{R}_+$ with compact support, we have*

$$\mathbb{E}[F(\omega)I_n(\phi^{\otimes n})]=0.$$

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Proof. (i) We start with a random set $A(\omega)$ and a nonnegative random variable $F(\omega)$ that satisfy the condition

$$D_{\Theta_0} F(\omega) D_{\Theta_1} \mathbf{1}_{A(\omega)}(x_1) \cdots D_{\Theta_k} \mathbf{1}_{A(\omega)}(x_k) = 0, \qquad \sigma^{\otimes k}(\mathrm{d}x_1, \ldots, \mathrm{d}x_k) \quad \text{a.e., } \omega \in \Omega^X,$$

for all $k \geq 1$, whenever $\Theta_0 \cup \Theta_1 \cup \cdots \cup \Theta_k = \{x_1, \ldots, x_k\}$, $\omega \in \Omega^X$. By Lemma 3, we have

$$\mathbb{E}[F(\omega) I_n(\mathbf{1}_{A(\omega)}^{\otimes n})] = \mathbb{E}\bigg[\int_{X^n} D_{s_1} \cdots D_{s_n}\bigg(F(\omega) \prod_{p=1}^{n} \mathbf{1}_{A(\omega)}(s_p)\bigg)\sigma(\mathrm{d}s_1) \cdots \sigma(\mathrm{d}s_n)\bigg]. \tag{15}$$

Next, we have

$$\begin{aligned} & D_{x_1} \cdots D_{x_k}(F(\omega)\mathbf{1}_{A(\omega)}(x_1) \cdots \mathbf{1}_{A(\omega)}(x_k)) \\ &\quad = \sum_{\Theta_1 \cup \cdots \cup \Theta_k = \{1, \ldots, k\}} D_{\Theta_1}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)) D_{\Theta_2}\mathbf{1}_{A(\omega)}(x_2) \cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k), \end{aligned}$$

where the above sum runs over all (possibly empty) subsets $\Theta_1, \ldots, \Theta_k$ of $\{1, \ldots, k\}$ such that $\Theta_1 \cup \cdots \cup \Theta_k = \{1, \ldots, k\}$. In addition,

$$D_{\Theta_1}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)) = \sum_{\eta_1 \cup \eta_2 = \Theta_1} D_{\eta_1} F(\omega) D_{\eta_2} \mathbf{1}_{A(\omega)}(x_k),$$

where the sum runs over all sets η_1, η_2 such that $\eta_1 \cup \eta_2 = \Theta_1$; hence,

$$\begin{aligned} & D_{\Theta_1}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)) \cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k) \\ &\quad = \sum_{\eta_1 \cup \eta_2 = \Theta_1} D_{\eta_1} F(\omega) D_{\eta_2}\mathbf{1}_{A(\omega)}(x_1) D_{\Theta_2}\mathbf{1}_{A(\omega)}(x_2) \cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k) \\ &\quad = 0, \qquad \sigma^{\otimes k}(\mathrm{d}x_1, \ldots, \mathrm{d}x_k)\text{-a.e.,} \end{aligned}$$

$\omega \in \Omega^X$, for all $k \geq 1$, which yields

$$\mathbb{E}[F(\omega) I_n(\mathbf{1}_{A(\omega)}^{\otimes n})] = 0 \tag{16}$$

by (15).

(ii) Assuming without loss of generality that ϕ takes values in $[0, 1]$ we consider the step process approximation

$$\begin{aligned} 0 \leq \phi_m(\omega, t) &:= \sum_{k=0}^{2^m - 1} \frac{k}{2^m} \mathbf{1}_{\{k/2^m \leq \phi(\omega,t) < (k+1)/2^m\}} \\ &= \sum_{k=1}^{2^m} \frac{k}{2^m} \mathbf{1}_{B_k(\omega)}(t) \leq \phi(\omega, t), \qquad t \in X,\ m \geq 1, \end{aligned}$$

where $B_k(\omega) = \{t \colon k/2^m \leq \phi(\omega, t) < (k+1)/2^m\}$, $k = 0, 1, \ldots, 2^m - 1$. By the polarization identity

$$h_1 \circ \cdots \circ h_n = \frac{1}{n!} \sum_{k=1}^{k=n} (-1)^{n-k} \sum_{l_1 < \cdots < l_k} (h_{l_1} + \cdots + h_{l_k})^{\circ n},$$

we can extend (16) to $\phi_m^{\otimes n}$ as $\mathbb{E}[F(\omega) I_n(\phi_m^{\otimes n}(\omega, \cdot))] = 0$ for all $m \geq 1$, and the extension to the general case follows by dominated convergence as m goes to ∞.

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Proof of Proposition 1. As in the proof of [10, Proposition 2.1] we apply Lemma 4 to a step function approximation $\phi_m(\omega, t)$ of $\phi(\omega, t)$ and, by Fubini's theorem, deduce that

$$\begin{aligned}\mathbb{E}\bigg[F(\omega)\mathrm{e}^{-\int_K \phi_m(\omega,x)\sigma(\mathrm{d}x)} \prod_{x\in K\cap\omega}(1+\phi_m(\omega,x))\bigg] &= \mathbb{E}[F(\omega)] + \mathbb{E}\bigg[F(\omega)\sum_{n=1}^{\infty}\frac{1}{n!}I_n(\mathbf{1}_{K^n}(\cdot)\phi_m^{\otimes n}(\omega,\cdot))\bigg] \\ &= \mathbb{E}[F(\omega)] + \sum_{n=1}^{\infty}\frac{1}{n!}\mathbb{E}[F(\omega)I_n(\mathbf{1}_{K^n}(\cdot)\phi_m^{\otimes n}(\omega,\cdot))] \\ &= \mathbb{E}[F(\omega)]\end{aligned}$$

and we complete the proof by the same steps as in [10]

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