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A MEAN ERGODIC THEOREM FOR MULTIPARAMETER SUPERADDITIVE PROCESSES ON BANACH LATTICES

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Introduction. Let E be a Banach Lattice. We will consider E to be weakly sequentially complete and to have a weak unit u. Thus we may represent E as a lattice of real valued functions defined on a measure space $(\mathcal{X}, \mathcal{F}, \mu)$. There is a set $R \subset \mathcal{X}$ such that R supports a maximal invariant function Φ for a postive contraction T on E [**5**]. Let $N = \mathcal{X} - R$ be the complement of R. Akcoglu and Sucheston showed that $\mathcal{X}_N(\frac{1}{n}\sum_{i=0}^{n-1} T^i f) \wedge g \longrightarrow 0$ for any $f, g \in E_+$, where E_+ is the positive cone of E. If in addition a monotone condition (UMB) is satisfied, then the same authors showed [**4**] that $\mathcal{X}_R(\frac{1}{n}\sum_{i=0}^{n-1} T^i f)$ converges in norm. A sequence $\{f_n\}_{n\geqq 0} \in E$ is called *superadditive* with respect to a positive contraction T if for $n, k \geqq 0, f_{n+k} \geqq T^k f_n + f_k$. A *moderately* superadditive sequence is one such that $\lim\inf_n \|\frac{1}{n}\sum_{i=0}^{n-1}(f_{i+1} - Tf_i)\| < \infty$. If $\{f_n\}$ is moderately superadditive we have also $\mathcal{X}_N(\frac{1}{n}f_n) \wedge g \longrightarrow 0$ for all $g \in E_+$, and $\mathcal{X}_R(\frac{1}{n}f_n)$ converging in norm. Millet and Sucheston [**13**] had expanded the theory to general multiparameter cases. For k arbitrarily many positive commuting contractions $T_1, T_2, \ldots, T_k$ on E, there is also a set $R \subset \mathcal{X}$ such that it supports a maximal invariant (under the T_i's) function Φ. If $N = \mathcal{X} - R$, then we have $\mathcal{X}_N(\frac{1}{n_1 n_2 \cdots n_k}\sum_{i_1=0}^{n_1-1}\cdots\sum_{i_k=0}^{n_k-1} T_1^{i_1}\cdots T_k^{i_k} f) \wedge g \longrightarrow 0$, for all $f, g \in E_+$. For the superadditive case, only L_1 results are known. Using a Markovian semi-group of operators, Akcoglu and Sucheston [**3**] showed that a bounded superadditive process converges in norm on the support of an invariant function; the convergence is stochastically to zero on the complement of the support.

This paper will show the mean convergence theorems for multiparameter superadditive processes.

In order to simplify the equations and notations involved all the theorems and proofs will be stated in a two-parameter setting. The extension to general multiparameter case is mostly obvious. It should be noted that the definition we use for superadditive process here is due to Krengel and Derriennic [**9**].

1. Preliminaries

Definition 1.1. Let E be a Banach Space. Assume that E satisfy the following.

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(a) There is a partial ordering '$\leqq$' defined on E.
(b) For every pair of $x, y \in E, x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$ both exist in E. We will define: $x^+ = x \vee 0, x^- = -x \vee 0$ and $|x| = x^+ + x^-$.
(c) For any pair of $x, y \in E$ such that $|x| \leqq |y|$ we have $\|x\| \leqq \|y\|$. The partial ordering '$\leqq$' is said to be *norm compatible.*

Such a E is called a Banach Lattice. We denote its positive cone by E_+ and its conjugate space by E^*.

Example. All L_p spaces are Banach Lattices. Simply consider the usual ordering of functions, the usual definition for sup and inf of functions and the L_p norms. Basic properties of Banach Lattices can be found in [**12**] and [**14**]. We will only consider Banach Lattice that has a weak unit and is weakly sequentially complete, that is they satisfy:

(A) There is an element $u \in E_+$ such that if $f \in E_+$ and if $u \wedge f = 0$, then $f = 0$. Such a u is called a weak unit.
(B) Every norm bounded increasing sequence in E has a strong limit. This implies order continuity, so every order interval $[f, g] = \{h : f \leqq h \leqq g\}$ is weakly compact ([**12**] p. 28).

If we consider $L_p[0,1]$ say, then u can be any function in L_p having support of measure 1. Note that if u is a weak unit, then $\lim_{k\to\infty} f \wedge ku = f$. For any Banach Lattice satisfying (A) and (B), we may apply the following representation theorem from [**12**] p. 25.

THEOREM 1.2. *Let E be an order continuous (condition (B)) Banach lattice which has weak unit (condition (A)). Then there exists a probability space $(\mathcal{X}, \mathcal{F}, \mu)$, an (in general not closed) ideal $\tilde{X}$ (an ideal $\tilde{X}$ is a linear subspace for which $x \in \tilde{X}$ whenever $|x| \leqq |\tilde{x}|$ for some $\tilde{x} \in \tilde{X}$) of $L_1(\mathcal{X}, \mathcal{F}, \mu)$ and a lattice norm $\|\cdot\|_{\tilde{X}}$ on $\tilde{X}$ so that*

(a) E is order isometric to $(\tilde{X}, \|\cdot\|_{\tilde{X}})$.
(b) $\tilde{X}$ is dense in $L_1(\mathcal{X}, \mathcal{F}, \mu)$ and $L_\infty(\mathcal{X}, \mathcal{F}, \mu)$ is dense in $\tilde{X}$.
(c) $\|f\|_1 \leqq \|f\|_{\tilde{X}} \leqq 2\|f\|_\infty$, whenever $f \in L_\infty(\mathcal{X}, \mathcal{F}, \mu)$.
(d) The dual of the isometry given in (a) maps E^* onto the Banach Lattice $\tilde{X}^*$ of all μ measurable functions g for which

$$\|g\|_{\tilde{X}^*} = \sup\left\{\int_X fgd\mu \ \|f\|_{\tilde{X}} \leqq 1\right\} < \infty.$$

The value taken by the functional corresponding to g at $f \in \tilde{X}$ is $\int_X fgd\mu$.

This says that we may assume our Banach lattice E to be a lattice of (equivalence class of) real valued measurable functions on a σ finite measure space

$(\mathcal{X}, \mathcal{F}, \mu)$. Henceforth elements of E will be denoted f, g and h, etc, to signify the fact that E is a function space. We will consider two-parameter cases, which means that we will consider two operators T and S on E. An operator $T : E \longrightarrow E$ is a contraction if $\|T\| \leqq 1$, and is positive if $T : E_+ \longrightarrow E_+$. In this paper T and S will always denote two positive commuting contractions on E. For a sequence $\{f_{(\bar{n})}\} \in E, (\bar{n})$ is actually a double subscript (n_1, n_2) with n_i a non-negative integer for $i = 1, 2$. We write $\frac{1}{(\bar{n})} f_{(\bar{n})}$ to denote $\frac{1}{n_1 n_2} f_{(n_1, n_2)}$. $\lim_{\bar{n} \to \infty} f_{(\bar{n})}$ means the limit of the sequence $\{f_{(\bar{n})}\}$ as each of the indices n_1 and n_2 tends to infinity independently of each other. We will use '$\longrightarrow$' for strong convergence, '$\xrightarrow{\omega}$'for weak convergence and '$\uparrow$', '$\downarrow$' for monotone convergence. We now define an additive sequence.

Definition 1.3. A sequence $\{f_n\} \in E_+$ is called *additive* with respect to an operator T if there exist some $f \in E_+$ such that $f_n = \sum_{i=0}^{n-1} T^i f$. For two-parameter case, $f_{(\bar{n})}$ is additive with respect to two commuting operators T and S if $f_{(\bar{n})} = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} T^i S^j f$ for some $f \in E_+$. In their paper [**2**] Akcoglu and Sucheston introduced the notion of *truncated limit.*

Definition 1.4. Let $\{f_n\}$ be a sequence in E. A function Φ is called a truncated limit of $\{f_n\}$ if for each positive integer k we have $\lim_{n \to \infty} f_n \wedge ku = \Phi_k$ exist and $\lim_{k \to \infty} \Phi_k \uparrow \Phi$. We then write $TLf_n = \Phi$. For *weak truncated limit* we only require $f_n \wedge ku \xrightarrow{\omega} \Phi_k$. We write $WTLf_n = \Phi$. A sequence $\{f_n\}$ is called TL null if $TL\,|f_n| = 0$. For general multiparameter case we replace the single index n by $(n_1, n_2, \ldots, n_k)$ in the definition.

Definition 1.5. A non-negative sequence $\{f_n\}$ is said to converge *stochastically to zero* if for $g \in E_+$, we have $f_n \wedge g \longrightarrow 0$ or another way of saying this is that $TLf_n = 0$. Properties and theorems concerning TL limit can be found in [**5**] and [**6**]. We state the three following lemmas without proof.

LEMMA 1.6 ((1.2) of [**13**]). *Let U be a strictly positive element in E^* and let $\{f_n\}$ be a sequence in E_+ such that* $\lim_{n \to \infty} U(f_n) = 0$. *Then*

(a) $TLf_n = 0$ and
(b) The strong limit of f_n as $n \longrightarrow \infty$ is 0 if $\sup_n f_n \in E$.

LEMMA 1.7 ((1.9) of [**5**]). *Let E satisfy (A) and (B). Let $f_n, g_n \in E_+$, $WTLf_n = \Phi$, $WTLg_n = \Gamma$.*

(a) If $WTL(f_n + g_n) = \Psi$ exist then $\Psi = \Phi + \Gamma$.
(b) If $T : E \longrightarrow E$ is a positive linear operator and $Tf_n = g_n$ then $T\Phi \leqq \Gamma$.

LEMMA 1.8 ((1.8) of [**5**]). *If $\{f_n\} \geqq 0$ is a sequence of functions in a Banach Lattice E satisfying (A) and (B) and* $\sup \|f_n\| = M < \infty$, *then there is a subsequence $\{f_{n_i}\}$ such that $WTLf_{n_i} = \Phi$ exists. If $\{f_n\}$ is not a TL null sequence,* *then this subsequence can be chosen so that* $\Phi \neq 0$.

2. Two-parameter results. Let T and S be two positive commuting contractions on E. We will investigate the convergence of : $\{\frac{1}{(\bar{n})} f_{(\bar{n})}\}$; $f_{(\bar{n})}$ an additive sequence. In the one parameter case (one operator T), this is just the Cesáro averages $A_n(T)f = \frac{1}{n} \sum_{i=0}^{n-1} T^i f$ for the operator T and the function f. We write $A_{(\bar{n})}f$ for $\frac{1}{(\bar{n})} f_{(\bar{n})}$ to be consistent with the one parameter notation. In [**13**] Millet and Sucheston showed that if there exist non TL null additive sequence then subinvariant functions can be found. To obtain an invariant function for the general multiparameter case we need to impose a monotone condition (C) [**5**] on the lattic:

(C) For every $\Phi \in E_+$ and for every number $\alpha > 0$, there is a number $B = B(\Phi, \alpha) > 0$ such that if $g \in E_+, \|g\| \leqq 1, 0 \leqq f \leqq \Phi$ and if $\|f\| \geqq \alpha$, then $\|f + g\| \geqq \|g\| + B$.

With this Millet and Sucheston proved the following theorem.

THEOREM 2.1 (2.5 of [**13**]). *Let E satisfy (A), (B) and (C). Let T and S be two positive commuting contractions on E. Then there is a function Φ with maximal support such that Φ is invariant under T and S. Moreover let R be the support of Φ and N be the complement of R. Then $X_N A_{(\bar{n})} f$ converges stochastically to zero for $f \in E_+$.*

This gives us the existence of an invariant function on a (C) lattice. The existence of invariant function is important because many useful results can be deduced from it. First of all we have that for a function f having support on that of an invariant function, $A_{(\bar{n})}(T, S)f$ converges in norm.

LEMMA 2.2 (2.1 of [**13**]). *Let Φ be a T, S invariant function in E_+. T and S are two positive commuting contractions on E. Let R be the support of Φ. Let f be a function in E_+ such that its support is included in R. Then $A_n(T)f \longrightarrow A_\infty(T)f, A_n(S)f \longrightarrow A_\infty(S)f$ and $A_{(\bar{n})}(T, S)f \longrightarrow A_\infty(T)A_\infty(S)f$.*

Another condition that we may impose on E is called (C_1). *[**5**]*

(C_1) *If $f, g \in E_+$ and $f \neq 0$, then* $\|f + g\| > \|g\|$.

This is readily seen to be a weaker condition than (C). *If there is an invariant function, however,* (C_1) *is sufficient to conclude the following lemma.*

LEMMA 2.3. *Assume that the Banach lattice E satisfies the conditions (A), (B) and (C_1). Given $\Phi \in E_+$ with $T\Phi = \Phi$, and $S\Phi = \Phi$, and a number $\alpha > 0$. Then there is a number $\sigma = \sigma(\Phi, \alpha) > 0$ such that if $0 \leqq f \leqq \Phi$ and $\|f\| \geqq \alpha$, then $\lim_{(\bar{n})\to\infty} \|A_{(\bar{n})}f\| \geqq \sigma$.*

Proof. The proof of this resembles that of lemma (2.3) of [**5**]. For $g \in E$, if $\lim_{(\bar{n})\to\infty} A_{(\bar{n})} g$ exists we will denote it be $\bar{g}$. If the lemma is false then there is an invariant function $\Phi \in E_+$, an $\alpha > 0$, and a sequence $\{f_n\}$ in E_+ such that for each $n, f_n \leqq \Phi$, $\|f_n\| > \alpha, \bar{f_n}$ exists and $\lim_{n\to\infty} \|\overline{f_n}\| = 0$. Passing to a subsequence, we may assume that $\|\overline{f_n}\| \leqq \epsilon_n$, with $\epsilon_n \longrightarrow 0$ and $\sum_n \epsilon_n < \infty$. Let $g_n = \bigvee_{k=n}^{\infty} f_k$ and $g = \lim \downarrow g_n$. Then as each of the $\|f_n\| > \alpha$; $\|g_n\| > \alpha$, and $\|g\| > \alpha$ as well. For $\epsilon > 0$ it is possible to find a m such that $g_n = \bigvee_{k=n}^{m} f_k + h_m$ with $\|h_m\| \leqq \epsilon$. Let $g'_{(n,m)} = \bigvee_{k=n}^{m} f_k$. Then $g_n - g'_{(n,m)} = h_m$. By definition, $g_n, g'_{(n,m)}$ and h_m all have support on that of Φ so we may apply lemma (2.2), then taking the norm to obtain

$$\|\overline{g_n} - \overline{g'_{(n,m)}}\| \leqq \|\overline{h_m}\| < \epsilon.$$

Note also $\|\overline{g_n}\| \downarrow \|\bar{g}\|$. Now $g'_{(n,m)} = \bigvee_{k=n}^{m} f_k \leqq \sum_{k=n}^{m} f_k$. So if we take the average and then the limit as $(\bar{r}) \longrightarrow \infty$, we have $\lim_{(\bar{r})\to\infty} A_{(\bar{r})} g'_{(n,m)} \leqq \lim_{(\bar{r})\to\infty} \sum_{k=n}^{m} A_{(\bar{r})} f_k$, or just simply $\overline{g'_{(n,m)}} \leqq \sum_{k=n}^{m} \overline{f_k}$. Since ϵ is arbitrary, we then have

$$\|\overline{g_n}\| \leqq \sum_{k=n}^{m} \|\overline{f_k}\| + \epsilon \leqq \sum_{k=n}^{\infty} \|\overline{f_k}\| \leqq \sum_{k=n}^{\infty} \epsilon_k < \infty,$$

which is a decreasing sequence, so $\|\bar{g}\| = 0$ as well. Consider

$$\|\Phi\| = \|A_{(\bar{n})}(\Phi + g - g)\| \leqq \|A_{(\bar{n})}(g)\| + \|A_{(\bar{n})}(\Phi - g)\|.$$

The first term tends to zero as $(\bar{n}) \longrightarrow \infty$ as $\|\bar{g}\| = 0$; the last term is less than $\|\Phi - g\|$ as T, S are contractions. We are left with $\|\Phi\| \leqq \|\Phi - g\|$. As g is non-negative, (C_1) forces g to be zero. This contradicts the fact that $\|g\| \geqq \alpha$, so $\lim_{(\bar{n})\to\infty} \|A_{(\bar{n})} f\| \geqq \sigma > 0$. ∎

We will now introduce yet a stronger monotone condition (UMB) (first introduced by Birkoff in [**8**]).

(UMB) For every number $\alpha > 0$, there is a number $B = B(\alpha) > 0$ such that $\|f + g\| \geqq \|g\| + B$ whenever $f, g \in E_+$, $\|g\| \leqq 1$ and $\|f\| \geqq \alpha$. For convenience we also have that $B(0) = 0$.

The (UMB) condition is stated in many different forms. In an Orlicz Space it is equivalent to the Δ_2 condition (see [**7**]). One very convenient form which will be used is the following:

(2.4) If $0 \leqq \Psi \leqq \Psi$ and $\|\Phi\| \leqq M, \|\Psi\| \geqq \alpha$ then

$$\|\Phi - \Psi\| \leqq \|\Phi\| - MB(\alpha/M).$$

Clearly the (UMB) condition implies the condition (C), which in turn implies the condition (C_1). Condition (B) is also a consequence of (UMB). However, for the sake of explicitness, these conditions will still be mentioned separately. In [**13**] Millet and Sucheston proved the following theorem for a (UMB) lattice.

THEOREM 2.5. *Let E satisfy (A), (B) and (UMB). Let Φ be an invariant function under two positive commuting contractions T and S. Let R be the support of Φ. Then for any function $f \in E_+, X_R A_{(\bar{n})} f$ converges strongly to an invariant function.*

3. Superadditive results. We adopted the definition of Krengel and Derriennic [**9**] for a superadditive process. Let '$\mathcal{C}' = \{(a_1, a_2) = \bar{a}, a_i$ non-negative integer$\}$. '$\mathcal{I}' = \{[\bar{a}, \bar{b}), \bar{a}$ and $\bar{b} \in \mathcal{C}, a_i \leqq b_i, i = 1, 2\}$. $[\bar{a}, \bar{b}) = \{\bar{c} \mid \bar{c} \in \mathcal{C}, a_i \leqq c_i < b_i, i = 1, 2\}$. For $\bar{u} = (u_1, u_2) \in \mathcal{C}$, we write $\bar{T}^{\bar{u}}$ to denote $T^{u_1} S^{u_2}$. Since T and S commute, $\mathcal{T} = \{\bar{T}^{\bar{u}}$ where $\bar{u} \in \mathcal{C}\}$ is actually a semi-group of positive bounded linear operators on E; ie, we have $\bar{T}^{\bar{u}} o \bar{T}^{\bar{v}} = \bar{T}^{\bar{u}+\bar{v}}$ where $\bar{u}, \bar{v} \in \mathcal{C}$. A set function $F : I \in \mathcal{I} \longrightarrow F_I \in E$ is called a superadditive process (with respect to T and S) if the following two conditions are satisfied.

(3.1) $\bar{T}^{\bar{u}} F_I = F_{I+\bar{u}}$ whenever $I \in \mathcal{I}$ and $\bar{u} \in \mathcal{C}$. That is, let $I = [\bar{a}, \bar{b})$, then $I + \bar{u} = [(a_1 + u_1, a_2 + u_2), (b_1 + u_1, b_2 + u_2))$.

(3.2) If I_1, I_2 are disjoint sets in $\mathcal{I}$ and if $I_1 \cup I_2$ is also in $\mathcal{I}$, then $F_{I_1 \cup I_2} \geqq F_{I_1} + F_{I_2}$.

To simplify notations we write $F_{(\bar{n})}$ for $F_{[\bar{0}, \bar{n})}$. We will consider non-negative superadditive processes. Applying (3.1) in (3.2) we then obtain the following useful form of (3.2):

$$F_{(\bar{m}+\bar{n})} \geqq F_{(\bar{m})} + \bar{T}^{\bar{m}} F_{(\bar{n})}.$$

for all $(\bar{n})$ and $(\bar{m})$. Let $\{F_{(\bar{n})}\}$ be a non-negative superadditive process in E. We write $\frac{1}{(\bar{n})} F_{(\bar{n})}$ to mean $\frac{1}{n_1 n_2} F_{(\bar{n})}$ for non-negative integers $n_1 n_2$. Then the boundedness of $\frac{1}{(\bar{n})} F_{(\bar{n})}$ for all $(\bar{n})$ by an invariant function Φ implies the convergence of the sequence $\{\frac{1}{(\bar{n})} F_{(\bar{n})}\}$. First we need a generalization of equation (3.2) so that we can deal with more than two rectangles at a time.

LEMMA 3.3. *Let E be a Banach Lattice. Let $\{F_{(\bar{n})}\}$ be a superadditive process on E_+ with respect to two positive commuting operators T and S on E. Let $(\bar{n})$ and $(\bar{m})$ be given such that there exist k_1, k_2 and that $m_1 = n_1 \cdot k_1, m_2 = n_2 \cdot k_2$. Then*

$$F_{[\bar{0}, \bar{m})} \geqq \sum_{i=0}^{k_1 - 1} \sum_{j=0}^{k_2 - 1} T^{in_1} S^{jn_2} F_{[\bar{0}, \bar{n})}.$$

Proof. It is easy to see that

$$\bigcup_{i=0}^{k_1-1}\bigcup_{j=0}^{k_2-1}[(in_1,\ jn_2),\ ((i+1)n_1,\ (j+1)n_2)) = [\bar{0},\ (k_1n_1,\ k_2n_2))$$
$$= [\bar{0},\ \bar{m}).$$

We have cut up the rectangle $[\bar{0},\bar{m})$ into k_1k_2 smaller rectangles, it is then possible to apply equation (3.2) to two of the rectangles at a time and obtain $F_{[\bar{0},\bar{m})} \geqq \sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-1} F_{[(in_1,jn_2),((i+1)n_1,(j+1)n_2))}$. However by condition (3.1) we know that for each (i,j), $T^{in_1}S^{jn_2}F_{[\bar{0},\bar{n})} = F_{[(in_1,jn_2),((i+1)n_1,(j+1)n_2))}$, hence the lemma is proved. ∎

LEMMA 3.4. *Let E be a Banach lattice satisfying (A), (B) and (C). Let F be a superadditive process on E_+ with respect to two positive commuting contractions T and S on E. Let Φ be an invariant function such that $\frac{1}{(\bar{n})}F_{(\bar{n})} \leqq \Phi$ for all $(\bar{n})$. We define $g_{(\bar{n})} = \frac{1}{n_1n_2}F_{(\bar{n})}$ and $\lim_{(\bar{n})\to\infty}A_{(\bar{n})}g_{(\bar{k})} = \overline{g_{(\bar{k})}}$ if it exists. Then $\lim_{(\bar{n})\to\infty}\|\overline{g_{(\bar{k})}} - \overline{g_{(\bar{k})}} \wedge g_{(\bar{n})}\| = 0$ for any fixed $(\bar{k})$.*

Proof. With (C) we have the existence of invariant functions. Let Φ be a T,S invariant function. By lemma (2.2) $\overline{g_{(\bar{k})}}$ exists for all $(\bar{k})$. For a fixed $(\bar{k})$, consider $\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} T^iS^jF_{(\bar{k})}$ with $(\bar{n}) > (\bar{k})$. If $(\bar{n})$ is sufficiently large, we may write

$$n_1 = m_1k_1 + r_1 \quad 0 \leqq r_1 < k_1; \qquad n_2 = m_2k_2 + r_2 \quad 0 \leqq r_2 < k_2.$$

To estimate $\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} T^iS^jF_{(\bar{k})}$, we rewrite:

$$\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} T^iS^jF_{(\bar{k})} \leqq \sum_{u=0}^{k_1-1}\sum_{v=0}^{k_2-1}\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} T^{ik_1+u}S^{jk_2+v}F_{(\bar{k})}.$$

By lemma (3.3), we have for $(0,0) \leqq (u,v) < (k_1,k_2)$, $\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} T^{ik_1+u}S^{jk_2+v}F_{(\bar{k})} \leqq F_{(\bar{n}+3\bar{k})}$. Since there are $k_1 \cdot k_2$ of these inequalities in the above equation, it can be rewritten as:

$$\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} T^iS^jF_{(\bar{k})} \leqq (k_1\cdot k_2)F_{(\bar{n}+3\bar{k})}, \quad \text{or}$$

$$A_{(\bar{n})}g_{(\bar{k})} \leqq \frac{1}{(\bar{n})}F_{(\bar{n}+3\bar{k})} = \frac{(n_1+3k_1)(n_2+3k_2)}{n_1n_2}g_{(\bar{n}+3\bar{k})}.$$

If $(\bar{n})$ is sufficiently larger than $(\bar{k})$, one can replace $(\bar{n}-3\bar{k})$ for $(\bar{n})$ to get

$$A_{(\bar{n}-3\bar{k})}g_{(\bar{k})} \leqq \frac{(n_1\cdot n_2)}{(n_1-3k_1)\cdot(n_2-3k_2)}g_{(\bar{n})}$$
$$\leqq g_{(\bar{n})} + \frac{3n_1k_2+3n_2k_1-9k_1k_2}{(n_1-3k_1)(n_2-3k_2)}\Phi$$

as $g_{(\bar{n})} \leqq \Phi$ for all $(\bar{n})$. By splitting up the sum appropriately we have

$$\begin{aligned} A_{(\bar{n})}g_{(\bar{k})} &= \frac{(n_1-3k_1)(n_2-3k_2)}{n_1n_2} A_{(\bar{n}-3\bar{k})}g_{(\bar{k})} \\ &\quad + \frac{1}{n_1n_2}\sum_{i=0}^{n_1-3k_1-1}\sum_{j=n_2-3k_2}^{n_2-1} T^iS^jg_{(\bar{k})} \\ &\quad + \frac{1}{n_1n_2}\sum_{i=n_1-3k_1}^{n_1-1}\sum_{j=0}^{n_2-3k_2-1} T^iS^jg_{(\bar{k})} \\ &\quad + \frac{1}{n_1n_2}\sum_{i=n_1-3k_1}^{n_1-1}\sum_{j=n_2-3k_2}^{n_2-1} T^iS^jg_{(\bar{k})} \\ &\leqq A_{(\bar{n}-3\bar{k})}g_{(\bar{k})} + \frac{1}{n_1n_2}[3k_2n_1+3k_1n_2+27k_1k_2]\Phi \\ &\leqq g_{(\bar{n})} + \frac{1}{(n_1-3k_1)(n_2-3k_2)}[6k_2n_1+6k_1n_2+18k_1k_2]\Phi. \end{aligned}$$

Now if $a, b, c \geqq 0$ and $a \leqq b + c$, then $a - (a \wedge b) \leqq c$. Hence $A_{(\bar{n})}g_{(\bar{k})} - A_{(\bar{n})}g_{(\bar{k})} \wedge g_{(\bar{n})} \leqq \frac{6n_1k_2+6k_2n_1+18k_1k_2}{(n_1-3k_1)(n_2 3k_2)}\Phi$. Now as Φ is fixed, letting $(\bar{n}) \longrightarrow \infty$ we get $\lim_{(\bar{n})\longrightarrow\infty}\|A_{(\bar{n})}g_{(\bar{k})} - A_{(\bar{n})}g_{(\bar{k})} \wedge g_{(\bar{n})}\| = 0$. Therefore if $\epsilon > 0$ is given, it is possible to write

$$\begin{aligned} \|\overline{g_{(\bar{k})}} - \overline{g_{(\bar{k})}} \wedge g_{(\bar{n})}\| &\leqq \|g_{(\bar{k})}^{-} - A_{(\bar{n})}g_{(\bar{k})}\| + \|A_{(\bar{n})}g_{(\bar{k})} - A_{(\bar{n})}g_k \Lambda g_{(\bar{n})}\| \\ &\quad + \|\overline{g_{(\bar{k})}} \wedge g_{(\bar{n})} - A_{(\bar{n})}g_{(\bar{k})} \wedge g_{(\bar{n})}\| \\ &< \epsilon. \end{aligned}$$

■

LEMMA 3.5. *Let E be a Banach lattice satisfying the conditions (A), (B) and (C). Let $\{F_{(\bar{n})}\}$ be a superadditive process on E_+ with respect to two positive commuting contractions T and S. Let Φ be a maximal invariant function under T and S. If $\frac{1}{n_1n_2}F_{(\bar{n})} \leqq \Phi$ for all $(\bar{n})$, and we define $g_{(\bar{n})}, \overline{g_{(\bar{k})}}$ as in lemma (3.4), then* $\liminf_{(\bar{k})\longrightarrow\infty}\limsup_{(\bar{n})\longrightarrow\infty}\|g_{(\bar{n})} - g_{(\bar{n})} \wedge \overline{g_{(\bar{k})}}\| = 0$.

Proof. Let $\alpha(\bar{k}) = \limsup_{(\bar{n})\longrightarrow\infty}\|g_{(\bar{n})} - g_{(\bar{n})} \wedge \overline{g_{(\bar{k})}}\|$. Assume that $\liminf_{(\bar{k})\longrightarrow\infty}\alpha(\bar{k}) > \alpha > 0$. Then there exists a $(\bar{k}_0)$ such that if $(\bar{k}) \geqq (\bar{k}_0)$ then $\alpha(\bar{k}) > \alpha > 0$. Let $(\bar{n}_1) = (\bar{k}_0)$. For $\epsilon > 0$ we have by lemma (3.4) that there exists $(\bar{N}_1)$ such that for $(\bar{n}) \geqq (\bar{N}_1)$, $\|\overline{g_{(\bar{n}_1)}} - g_{(\bar{n})}\Lambda\overline{g_{(\bar{n}_1)}}\| < \epsilon$; since $\alpha(\bar{n}_1) > \alpha$ one can actually pick a large enough $(\bar{n})$, calling it $(\bar{n}_2)$ such that we have

$$\|g_{(\bar{n}_2)} - g_{(\bar{n}_2)} \wedge \overline{g_{(\bar{n}_1)}}\| > \alpha \quad \text{and} \quad \|\overline{g_{(\bar{n}_1)}} - g_{(\bar{n}_2)} \wedge \overline{g_{(\bar{n}_1)}}\| < \epsilon$$

for any $\epsilon > 0$. To simplify notation we define for each index j,

$$Q_j = \overline{g_{(n_j-1)}} - g_{(\bar{n}_j)} \wedge \overline{g_{(n_j-1)}}, \quad P_j = g_{(\bar{n}_j)} - g_{(\bar{n}_j)} \wedge \overline{g_{(n_j-1)}}.$$

Choose a sequence of $\{\epsilon(i) > 0\}$ such that $\sum_i \epsilon(i) < \infty$. Repeating the process above we obtain a sequence of indices $\{(\bar{n}_1), (\bar{n}_2), \ldots, \}$ and two sequences of P_i, Q_i such that

$$\|Q_i\| < \epsilon(i) \quad \|P_i\| > \alpha \quad g_{(\bar{n}_i)} = \overline{g_{(\bar{n}_{i-1})}} + P_i - Q_i$$

for each i. By construction P_i and Q_i are both less than or equal to Φ. By lemma (2.2) $\bar{P}_i = \lim_{(\bar{n})\to\infty} A_{(\bar{n})}P_i, \bar{Q}_i = \lim_{(\bar{n})\to\infty} A_{(\bar{n})}Q_i$ both exist. By lemma (2.3) $\|\bar{P}_i\| > \sigma(\Phi, \alpha) > 0$, and it is easy to see that $\|\bar{Q}_i\| \leqq \epsilon(i)$. So by taking the average we have $\overline{g_{(\bar{n}_i)}} = \overline{g_{(\bar{n}_{i-1})}} + \overline{P_i} - \overline{Q_i}$ with $\|\overline{P_i}\| > \sigma, \|\overline{Q_i}\| < \epsilon(i)$. So

$$\|\overline{g_{(\bar{n}_i)}} - \overline{g_{(n_{i-1})}}\| = \|\overline{P_i} - \overline{Q_i}\| > \sigma - \epsilon(i), \tag{3.5.1}$$

which does not converge to zero. Now $\overline{g_{(\bar{n}_i)}} + \bar{Q}_i \geqq \overline{g_{(n_i-1)}}$; as $\overline{P_i} \geqq 0$. So

$$\overline{g_{(\bar{n}_1)}} \leqq \overline{g_{(\bar{n}_2)}} + \overline{Q_2} \leqq \overline{g_{(\bar{n}_3)}} + \overline{Q_2} + \overline{Q_3} \leqq \cdots$$

The sequence $\{\overline{g_{(\bar{n}_i)}} + \sum_{j=2}^{i} \overline{Q_j}\}$ is hence an increasing sequence in i. Now $\overline{g_{(\bar{n}_i)}} + \sum_{j=2}^{i} \overline{Q_j}$ is a norm bounded increasing sequence in i. By (B), this sequence converges strongly. The sequence $\{\sum_{j=2}^{i} \bar{Q}_j\}$ is also a norm bounded increasing sequence and converges as well. The sequence $\{\overline{g_{(n_i)}}\}$ hence also converges strongly, which contradicts (3.5.1). ∎

Combining lemma (3.4) and (3.5) we obtain that for any positive superadditive process that is bounded by an invariant function, the average of the process will converge in norm.

THEOREM 3.6. *Let E be a Banach lattice satisfying (A), (B) and (C). Let F be a superadditive process on E_+ with respect to two positive commuting contractions T and S on E. If there exists a T, S invariant function Φ such that $\frac{1}{n_1 \cdot n_2} F_{(\bar{n})} \leqq \Phi$ for all $\bar{n}$ where $\bar{n} = (n_1, n_2)$ then $\frac{1}{n_1 \cdot n_2} F_{(\bar{n})}$ converges strongly.*

Proof. Let $\epsilon > 0$ be given. Let $g_{(\bar{n})}$ be as defined in (3.4). Using lemma (3.5) and the definition of $\alpha(\bar{k})$ we can find $(\bar{k}_0)$ such that $\alpha(\bar{k}_0) < \epsilon$. For this $(\bar{k}_0)$ find $(\bar{n}_0)$ such that for $(\bar{n}) > (\bar{n}_0)$, $\|\overline{g_{(\bar{k}_0)}} - (g_{(\bar{n})} \wedge \overline{g_{(\bar{k}_0)}})\| < \epsilon$ by Lemma (3.4). By the definition of $\alpha(k_0)$ it is possible to find $(\bar{n}_1)$ such that for $(\bar{n}) > (\bar{n}_1)$, $\|g_{(\bar{n})} - (g_{(\bar{n})} \wedge \overline{g_{(\bar{k}_0)}}\| < \epsilon$. Let $(\bar{n}_2) = \max((\bar{n}_0), (\bar{n}_1))$. Then for $(\bar{n}), (\bar{m}) > (\bar{n}_2)$,

$$\begin{aligned} \|g_{(\bar{n})} - g_{(\bar{m})}\| &< \|g_{(\bar{n})} - \overline{g_{(\bar{k}_0)}} \wedge g_{(\bar{n})}\| + \|\overline{g_{(\bar{k}_0)}} \wedge g_{(\bar{n})} - \overline{g_{(\bar{k}_0)}}\| \\ &\quad + \|\overline{g_{(\bar{k}_0)}} - g_{(\bar{m})} \wedge \overline{g_{(\bar{k}_0)}}\| + \|\overline{g_{(\bar{k}_0)}} \wedge g_{(\bar{m})} - g_{(\bar{m})}\| \\ &< 4\epsilon. \end{aligned}$$

By (3.5), ϵ can be arbitrarily small by picking $(\bar{k}_0)$ sufficiently large. Hence $g_{(\bar{n})}$ converges. ∎

COROLLARY 3.7. *Let E, T, S and Φ be as defined in lemma (3.6). Let $\{F_{(\bar{n})}\}$ be an arbitrary non-negative superadditive process on E, then $\frac{1}{(\bar{n})} F_{(\bar{n})} \Lambda \Phi$ converges strongly.*

Proof. Define $F'_{(\bar{n})} = F_{(\bar{n})} \wedge (n_1 n_2 \Phi)$. Then $1/(n_1 n_2) F'_{(\bar{n})} \leqq \Phi$. $\{F'_{(\bar{n})}\}$ can be easily shown to be another superadditive process. Now just apply (3.6) to $F'_{(\bar{n})}$. ∎

Definition 3.8. Let $\{F_{(\bar{n})}\}$ be a non-negative superadditive process on a Banach lattice E with respect to two positive commuting contractions T and S on E. Let

$$\Phi_{(n_1,n_2)} \text{ or } \Phi_{(\bar{n})} = \frac{1}{n_1 n_2} \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} F_{(i+1,j+1)} - TSF_{(i,j)}.$$

$\{F_{(\bar{n})}\}$ is said to be *moderately superadditive* if $M = \lim\inf_{(\bar{n})\to\infty} \|\Phi_{\bar{n}}\|$ is finite. In the multiparameter case we would define for $(\bar{n}) = (n_1, n_2 \ldots n_m)$,

$$\Phi_{(n_1,n_2,\ldots n_m)} = \frac{1}{n_1 n_2 \cdots n_m} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \cdots \sum_{i_m=0}^{n_m-1} F_{(i_1+1,i_2+1,\ldots i_m+1)} - T_1 T_2 \cdots T_m F_{(i_1,i_2,\ldots i_m)},$$

and M is defined in the obvious way corresponding to the two-parameter model. For $\bar{u} \in C$ we write $\Phi_{(\bar{n})+(\bar{u})}$ to mean $T^{u_1} S^{u_2} \Phi_{(n_1,n_2)}$ and $\Phi_{(\bar{n}+\bar{u})}$ as $\Phi_{(n_1+u_1,n_2+u_2)}$.

LEMMA 3.9. *If a, b and c are three non-negative elements in a Banach lattice E then the following inequalities hold:*

(3.9.1) $a + b - (a+b) \wedge 2c \leqq (a - a \wedge c) + (b - b \wedge c)$,

(3.9.2) $a - a \wedge c \leqq a + b - (a+b) \wedge c$.

The proof of these are obvious and will be omitted here.

LEMMA 3.10. *Suppose $\{\alpha_n\}$ is a sequence in E_+ such that there exists another sequence $\{\gamma_n\}$ in E_+ with $\alpha_n \leqq \gamma_n$ for all n. If the weak limit of $\{\alpha_n\}$ exists and is equal to α, then $\|\alpha\| \leqq \lim\inf_{n\to\infty} \|\gamma_n\|$.*

Proof. Let $\mathcal{L} = \{a\alpha,\ a$ is a complex number$\}$. Define a functional f on $\mathcal{L}$ by $f(a\alpha) \equiv a\|\alpha\|$. Then $\|f\| = 1$, and $f(\alpha) = \|\alpha\|$. By the Hahn-Banach Theorem there is another functional F defined on the entire Banach Lattice E such that $\|F\| = 1$ and that $F(\alpha) = \|\alpha\|$. As $\alpha_n \xrightarrow{\omega} \alpha$, we have

$$F(\alpha_n) \longrightarrow F(\alpha) \quad \text{and} \quad \|F(\alpha_n)\| \longrightarrow \|\alpha\|.$$

But we also have $|F(\alpha_n)| \leqq \|F\|\|\alpha_n\| \leqq \|\alpha_n\| \leqq \|\gamma_n\|$ for each n. Hence

$$\|\alpha\| = \liminf_n \|F(\alpha_n)\| \leqq \liminf_n \|\gamma_n\|. \qquad \blacksquare$$

LEMMA 3.11. *Given two non-negative sequences $\{\alpha(\bar{n})\}$ and $\{\gamma(\bar{n})\}$ that converge weakly to α and γ respectively, then*

$$(\alpha(\bar{n}) - \gamma(\bar{n}))^{\pm} \xrightarrow{\omega} (\alpha - \gamma)^{\pm}.$$

Proof. The proof of this can be found in p. 51 of [**14**].

LEMMA 3.12. *Let E be a Banach lattice satisfying the conditions (A), (B) and (UMB). Let $\{\psi_{(\bar{n})}\}$ be a sequence in E_+ such that there exists a number K with $\|\psi_{(\bar{n})}\| \leqq K$ for each $(\bar{n})$. Let T and S be two positive commuting contractions on E. Let $(\bar{u}) = (u_1, u_2)$ be fixed, and assume*

(3.12.1) $WTL\psi_{(\bar{n})} = \Psi$ *and* $WTL\psi_{(\bar{n})+(\bar{u})} = \Psi_{(\bar{u})}$

both exist. Now let v be the weak unit and define

(3.12.2) $\sigma = \lim_{j\longrightarrow\infty} (\limsup_{(\bar{n})\longrightarrow\infty} \|\psi_{(\bar{n})} - (\psi_{(\bar{n})} \wedge jv)\|)$,

(3.12.3) $\sigma_{(\bar{u})} = \lim_{j\longrightarrow\infty} (\limsup_{(\bar{n})\longrightarrow\infty} \|\psi_{(\bar{n})+(\bar{u})} - (\psi_{(\bar{n})+(\bar{u})} \wedge jv)\|)$.

Then we have $\sigma \geqq \sigma_{(\bar{u})}$, and if $\|\Psi_{(\bar{u})} - T^{u_1}S^{u_2}\Psi\| > \alpha > 0$, then

(3.12.4) $\sigma - \sigma_{(\bar{u})} \geqq KB(\alpha/K)$,

where $B(\alpha/K)$ is the factor derived from the (UMB) condition.

Proof. We shall use the symbol $\psi_{(\bar{n})}$ and $\psi_{(\bar{n})+(\bar{u})}$ and these are defined in a similar way as that of definition (3.8). Let k be a fixed integer, arbitrarily chosen. Let

$$f_{(n_1,n_2)} = \psi_{(n_1,n_2)} \wedge kv \quad \text{and} \quad g_{(n_1,n_2)} = \psi_{(n_1,n_2)} - f_{(n_1,n_2)}.$$

We substitute $T^{u_1}S^{u_2}f_{(\bar{n})}, T^{u_1}S^{u_2}g_{(\bar{n})}$ and jv for a, b and c into lemma (3.9) to get

(3.12.5) $$\begin{aligned} T^{u_1}S^{u_2}g_{(\bar{n})} - (T^{u_1}S^{u_2}g_{(\bar{n})} \wedge 2jv) &\leqq \psi_{(\bar{n})+(\bar{u})} - (\psi_{(\bar{n})+(\bar{u})} \wedge 2jv) \\ &\leqq [T^{u_1}S^{u_2}f_{(\bar{n})} - T^{u_1}S^{u_2}f_{(\bar{n})} \wedge jv] \\ &\quad + [T^{u_1}S^{u_2}g_{(\bar{n})} - T^{u_1}S^{u_2}g_{(\bar{n})} \wedge jv]. \end{aligned}$$

By definition we may find for sufficiently large j, a function h_j such that $k\bar{T}^{\bar{u}}v \leqq jv + h_j$, with $\|h_j\| \longrightarrow 0$. Hence we can write:

$$\lim_{j\to\infty} (\limsup_{(\bar{n})\to\infty} \|T^{u_1}S^{u_2}f_{(\bar{n})} - (T^{u_1}S^{u_2}f_{(\bar{n})} \wedge jv)\|) = 0.$$

Using this and (3.12.5), it is seen that by taking limits, we in fact have

$$\text{(3.12.6)}\quad \sigma_{(\bar{u})} = \lim_{j\to\infty} (\limsup_{(\bar{n})\to\infty} \|\bar{T}^{\bar{u}}g_{(\bar{n})} - (\bar{T}^{\bar{u}}g_{(\bar{n})} \wedge jv)\|).$$

Assume that $\|\Psi_{(\bar{n})} - \bar{T}^{\bar{u}}\Psi\| > \alpha > 0$. We will proceed to show that if k is chosen large enough, there exists $(\bar{n}_0)$, j_0 such that $\|\bar{T}^{\bar{u}}g_{(\bar{n})} \wedge jv\| > \alpha$ for all $(\bar{n}) \geqq (\bar{n}_0)$, and $j \geqq j_0$. Given $\epsilon > 0$, there exists k_0 such that if $k \geqq k_0$,

$$w \lim_{(\bar{n})\to\infty} f_{(\bar{n})} = w \lim_{(\bar{n})\to\infty} (\psi_{(\bar{n})} \wedge kv) = \Psi_k \text{ exists satisfying } \|\Psi - \Psi_k\| < \epsilon.$$

Let $\Psi_{(\bar{u}),j} = w \lim_{(\bar{n})\to\infty}(\bar{T}^{\bar{u}}\psi_{(\bar{n})} \wedge jv)$, then $\lim_{j\to\infty} \Psi_{(\bar{u}),j} = \Psi_{(\bar{u})}$. We also have $w \lim_{(\bar{n})\to\infty}[\bar{T}^{\bar{u}}\psi_{(\bar{n})} \wedge jv - \bar{T}^{\bar{u}}f_{(\bar{n})}] = \Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k$. It is easy to see that $(\bar{T}^{\bar{u}}\psi_{(\bar{n})} \wedge jv - \bar{T}^{\bar{u}}f_{(\bar{n})})^+ \leqq \bar{T}^{\bar{u}}g_{(\bar{n})} \wedge jv$. So by lemma (3.10) and (3.11) we obtain

$$\text{(3.12.7)}\quad \|(\Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k)^+\| \leqq \liminf_{(\bar{n})\to\infty} \|\bar{T}^{\bar{u}}g_{(\bar{n})} \wedge jv\|.$$

By the definition of $\Psi_{(\bar{u})}$, there exists a j_1 such that for $j > j_1$, $\|\Psi_{(\bar{u})} - \Psi_{(\bar{u}),j}\| < \epsilon$. So

$$\begin{aligned}\|\Psi_{(\bar{u})} - \bar{T}^{\bar{u}}\Psi\| &\leqq \|\Psi_{(\bar{u})} - \Psi_{(\bar{u}),j}\| + \|\Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k\| + \|\bar{T}^{\bar{u}}\Psi_k - \bar{T}^{\bar{u}}\Psi\| \\ &\leqq \|\Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k\| + 2\epsilon\end{aligned}$$

for any $j \geqq j_1$ and $k \geqq k_0$. Rewritting, we get:

$$\text{(3.12.8)}\quad \|\Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k\| > \|\Psi_{(\bar{u})} - \bar{T}^{\bar{u}}\Psi\| - 2\epsilon.$$

The next step is to show that there exists a j_2 such that $\|(\Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k)^-\| < \epsilon$ for $j \geqq j_2$. As v is a weak unit, we can find j sufficiently large and a $h_{(\bar{n}),j}$ such that $k\bar{T}^{\bar{u}}v \leqq jv + h_{(\bar{n}),j}$. Hence once k is chosen, we may write $\bar{T}^{\bar{u}}f_{(\bar{n})} = h_{(\bar{n}),j} + \bar{T}^{\bar{u}}f_{(\bar{n})} \wedge jv$ with $\|h_{(\bar{n}),j}\| < \epsilon$. We claim that $h_{(\bar{n}),j} \geqq [(\bar{T}^{\bar{u}}\psi_{(\bar{n})} \wedge jv) - \bar{T}^{\bar{u}}f_{(\bar{n})}]^-$. This can be checked easily. Again applying lemma (3.10) and (3.11) we have

$$\begin{aligned}w \lim_{(\bar{n})\to\infty} \|(\bar{T}^{\bar{u}}\Psi_{(\bar{n})} \wedge jv - \bar{T}^{\bar{u}}f_{(\bar{n})})^-\| &= \|(\Psi_{(\bar{u}),j} - \bar{T}^{\bar{u}}\Psi_k)^-\| \\ &\leqq \liminf_{(\bar{n})\to\infty} \|h_{(\bar{n}),j}\| < \epsilon\end{aligned}$$

if $j \geqq j_2$. Now let $j_0 = \max(j_1, j_2)$. Combining this with equations (3.12.7) and (3.12.8) we obtain

$$\text{(3.12.9)} \quad \liminf_{(\bar{n})\to\infty} \|\bar{T}^{\bar{u}} g_{(\bar{n})} \Lambda j v\| > \|\Psi_{(\bar{u})} - \bar{T}^{\bar{u}}\Psi\| - 3\epsilon$$

for $j > j_0$ and $k > k_0$. First we will show (3.12.4), and here we assumed $\|\Psi_{(\bar{u})} - \bar{T}^{\bar{u}}\Psi\| > \alpha > 0$. By equation (3.12.9) we have, for k sufficiently large, $(\bar{n}_0)$ and j_0 such that for $(\bar{n}) \geqq (\bar{n}_0)$ and $j \geqq j_0$, $\|\bar{T}^{\bar{u}} g_{(\bar{n})} \wedge jv\| > \alpha$ as $\epsilon > 0$ is arbitrary. (Really we have $\|\bar{T}^{\bar{u}} g_{(\bar{n})} \wedge jv\| > \alpha - \epsilon = \alpha' > 0$. ϵ can be made arbitrarily small by picking k_0 sufficiently large.) Let

$$\Psi = \bar{T}^{\bar{u}} g_{(\bar{n})} \wedge jv \qquad \|\Psi\| > \alpha \text{ for } j, (\bar{n}) \geqq j_0, (\bar{n}_0),$$

$$\Phi = \bar{T}^{\bar{u}} g_{(\bar{n})} \qquad \|\Phi\| \leqq \|g_{(\bar{n})}\| \leqq \|\psi_{(\bar{n})}\| \leqq K.$$

By equation (2.4), $\|\bar{T}^{\bar{u}} g_{(\bar{n})} - (\bar{T}^{\bar{u}} g_{(\bar{n})} \wedge jv)\| \leqq \|\bar{T}^{\bar{u}} g_{(\bar{n})}\| - KB(\frac{\alpha}{K})$, for $j, (\bar{n}) \geqq j_0, (\bar{n}_0)$. By the definition of σ, for $\epsilon > 0$, one can find $k_1, (\bar{n}_1)$ such that for $k, (\bar{n}) \geqq k_1, (\bar{n}_1)$,

$$\|\bar{T}^{\bar{u}} g_{(\bar{n})}\| \leqq \|g_{(\bar{n})}\| \equiv \|\psi_{(\bar{n})} - (\psi_{(\bar{n})} \wedge kv)\| < \sigma + \epsilon.$$

So for $k \geqq k_1$, $\limsup_{(\bar{n})\to\infty} \|\bar{T}^{\bar{u}} g_{(\bar{n})}\| \leqq \sigma + \epsilon$. Taking $\lim_{j\to\infty} \limsup_{(\bar{n})\to\infty}$ of this, and using (3.12.6) we get $\sigma_{(\bar{u})} \leqq \sigma + \epsilon - KB(\frac{\alpha}{K})$. As ϵ is arbitrary, we have $\sigma - \sigma_{(\bar{u})} \geqq KB(\frac{\alpha}{K})$. (Again we really have $\sigma_{(\bar{u})} \leqq \sigma + \epsilon - KB(\frac{\alpha'}{K})$. However as k becomes much larger than k_0 and k_1 we will have $\epsilon \to 0$ and $\alpha' \to \alpha$. The important point is that there is a positive difference between the two.) To complete the proof note that by definition $\|\bar{T}^{\bar{u}} g_{(\bar{n})} - \bar{T}^{\bar{u}} g_{(\bar{n})} \wedge jv\| \leqq \|\bar{T}^{\bar{u}} g_{(\bar{n})}\| < \sigma + \epsilon$; if we have $k \geqq k_1$ in defining $g_{(\bar{n})}$. Taking limits as j and $(\bar{n})$ go to infinity, with ϵ arbitrary, we get $\sigma_{(\bar{u})} \leqq \sigma$ as desired. ∎

LEMMA 3.13. *Let E be a Banach lattice satisfying the conditions (A), (B) and (UMB). Let T and S be two positive commuting contractions on E. Let $\{\psi_{(\bar{n})}\}$ be a sequence in E_+ such that $\|\psi_{(\bar{n})}\| \leqq K$ for all $(\bar{n})$ and some finite number K. Assume that $\lambda_{(i,j)} = WTL(T^i S^j \psi_{(\bar{n})})$ exists for each pair $(i, j) \geqq (0, 0)$. Let*

$$\epsilon_{(n_1,n_2)} = \sup_{(k_1,k_2)\geqq(0,0)} \|\lambda_{(n_1+k_1,n_2+k_2)} - T^{k_1} S^{k_2} \lambda_{(n_1,n_2)}\|$$

for each pair $(n_1, n_2) \geqq (0, 0)$. Then $\lim_{(\bar{n})\to\infty} \epsilon_{(\bar{n})} = 0$.

Proof. If $\{\epsilon_{(\bar{n})}\}$ does not converge to zero, then there exists a number $\alpha > 0$ such that for infinitely many $(\bar{n})$, the corresponding $\epsilon_{(\bar{n})} > \alpha$. In any case, let $\{(\bar{n}_1), (\bar{n}_2), \ldots\}$ be a enumerated set of such $(\bar{n})$'s. We can pick and a set of

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$\{(\bar{k}_i)\}$ such that $0 \leqq (\bar{n}_i) < (\bar{n}_i + \bar{k}_i) \leqq (\overline{n_{i+1}})$ and $\|\lambda_{(\bar{n}_i+\bar{k}_i)} - \bar{T}^{\bar{k}_i}\lambda_{(\bar{n}_i)}\| > \alpha$ for each i. Now define

$$\sigma_{(i,j)} = \lim_{l\to\infty} (\limsup_{(\bar{n})\to\infty} \|T^i S^j \psi_{(\bar{n})} - (T^i S^j \psi_{(\bar{n})} \wedge lv)\|)$$

By the results of lemma (3.12), we see that $\sigma_{(i_1,i_2)} \geqq \sigma_{(i_2,j_2)} \geqq 0$ if $i_1 \leqq i_2$, $j_1 \leqq j_2$. So the sequence $\{\sigma(i,j)\}$ is non-increasing. Set $\Psi = \lambda_{(\bar{n}_i)}$ and $\Psi_{(\bar{k}_i)} = \lambda_{(\bar{n}_i+\bar{k}_i)}$. Then we have for each pair of $(\bar{n}_i, \bar{k}_i)$, $\|\Psi_{(\bar{k}_i)} - \bar{T}^{\bar{k}_i}\Psi\| > \alpha$. By (3.12.4) we have $\sigma_{(\bar{n}_i)} - \sigma_{(\bar{n}_i+\bar{k}_i)} \geqq KB(\alpha/K) > 0$. This means that $\sigma(i,j)$ will eventually be negative as $i, j \to \infty$, this is a contradiction as by definition $\sigma_{(\bar{n})}$ cannot be negative. ∎

In the one parameter case, Akcoglu and Sucheston in [**4**] introduced a sequence of asymptotic dominants. Basically it is a sequence of functions in E such that their averages almost dominates the superadditive process in question. We define for the multiparameter system a corresponding sequence of asymptotic dominants.

Definition 3.14. Let $\{F_{(\bar{n})}\}$ be a non-negative superadditive process on a Banach lattice E. A sequence $\lambda_{(\bar{k})}$ in E will be called a sequence of **asymptotic dominants** for $\{F_{(\bar{n})}\}$ if there are functions $\Psi_{(\bar{n})}^{(\bar{k})}$ such that:

(3.14.1) $\dfrac{1}{n_1 \cdot n_2} F_{(\bar{n})} \leqq A_{(\bar{n})}\lambda_{(\bar{k})} + \Psi_{(\bar{n})}^{(\bar{k})}$ and

(3.14.2) $\lim_{(\bar{k})\to\infty} (\limsup_{(\bar{n})\to\infty} \|\Psi_{(\bar{n})}^{(\bar{k})}\|) = 0.$

The extension to multi-parameter case is again obvious. We simply replace both of the doubly indexed $(\bar{n}), (\bar{k})$ by any m-dimensional $(\bar{n})$ and $(\bar{k})$ in equations (3.14.1) and (3.14.2).

Remarks 3.15. We will proceed to show that if $F_{(\bar{n})}$ is moderately superadditive, then it has a sequence of asymptotic dominants. Moreover, the sequence $\lambda_{(k_i)}$ will have support on that of the invariant function Φ. By lemma (2.2) then the sequence $A_{(\bar{n})}\lambda_{(k_i)}$ will converge. By picking $(\bar{k}_i)$ large we will have $\frac{1}{n_1 n_2} F_{(\bar{n})}$ dominated (up to ϵ in norm) by a convergence sequence. Its own convergence then follows.

LEMMA 3.16. *Let E be a Banach Lattice satisfying conditions (A), (B) and (UMB). Let $\{F_{(\bar{n})}\}$ be a positive superadditive process in E with respect to two positive commuting contractions T and S on E. Let $\Phi_{(\bar{m})} = \frac{1}{m_1 \cdot m_2} \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} (F_{(i+1,j+1)} - TSF_{(i,j)})$. Then $m_1 m_2 \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} T^i S^j \Phi_{(m_1,m_2)} \geqq [(m_1 - n_1 + 1) \cdot (m_2 - n_2 + 1)] F_{(\bar{n})}$ for $(m_1, m_2) \geqq (n_1, n_2)$.*

Proof. First of all observe the following. Let $\{a_n\}$ and $\{b_n\}$ ($n \in I$, a set of indices) be two sequences in E such that $a_n \geqq b_n$ for all n. Suppose we would like to show that $\sum_{n\in I}(a_n - b_n) \geqq f$. Then

(a) as each term is non-negative in the summation, it is sufficient to show that the sum over some of the indices will majorize f. That is, if we can find a subset $I_0 \in I$ such that $\sum_{n\in I_0}(a_n - b_n) \geqq f$ we are done. The reader should not be alarmed if many non-negative terms are discarded in the proof.

(b) Suppose $a_0, a_1, \ldots, a_m \in \{a_n\}$ is identical to $b_k, b_{k+1}, \ldots, b_{k+m} \in \{b_n\}$. Then consider

$$\sum_n (a_n - b_n) = \sum_{\substack{n=0,\ldots,m\\ n=k,\ldots,k+m}} (a_n - b_n) + \sum_{\substack{n\neq 0,\ldots,m\\ n\neq k,\ldots,k+m}} (a_n - b_n).$$

By (a) we will show that the first term is sufficient to give us the desired inequality.

(c) From the definition of superadditivity we see that $T^{i_0}S^{j_0}F_{(u_0,v_0)} \geqq T^{i_1}S^{j_1}F_{(u_1,v_1)}$ if we have $u_0 \geqq u_1, v_0 \geqq v_1, u_0 + i_0 \geqq u_1 + i_1$ and lastly $v_0 + j_0 \geqq v_1 + j_1$.

In our proof, we will simplify the notation so that $a_{i,j}^{u,v} \equiv T^iS^jF_{(u,v)}$. This is translated to be $a_{i_0,j_0}^{u_0,v_0} \geqq a_{i_1,j_1}^{u_1,v_1}$ if u_i, v_i etc satisfy the said conditions. We will now look at which terms cancel out each other in the summation

$$m_1m_2 \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} T^iS^j\Phi_{(m_1,m_2)} = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}\sum_{u=0}^{m_1-1}\sum_{v=0}^{m_2-1} T^iS^jF_{(u+1,v+1)} - T^{i+1}S^{j+1}F_{(u,v)}.$$

We see that $\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\sum_{u=0}^{m_1-2}\sum_{v=0}^{m_2-2} a_{i,j}^{u+1,v+1}$ and $\sum_{i=0}^{n_1-2}\sum_{j=0}^{n_2-2}\sum_{u=1}^{m_1-1}\sum_{v=1}^{m_2-1} a_{i+1,j+1}^{u,v}$ cancel each other. Therefore using (b) we will consider only the following:

$$\sum_{i=0}^{n_1-2}\sum_{j=0}^{n_2-2}\sum_{u=1}^{m_1-1}\sum_{v=1}^{m_2-1} a_{i,j}^{u+1,v+1} - \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\sum_{u=0}^{m_1-2}\sum_{v=0}^{m_2-2} a_{i+1,j+1}^{u,v}.$$

We would like to write $[{}_{k_1}^{k_2}][{}_{r_1}^{r_2}]$ to mean $\sum_{u=k_1}^{k_2}\sum_{v=r_1}^{r_2}$. We will further partition the sum into many blocks according to the indices i, j. Consider the case when $i = 0$ and $j = 0$. We are looking at:

$$(3.16.1)\quad \begin{bmatrix} m_1-2 \\ 0 \end{bmatrix}\begin{bmatrix} m_2-2 \\ 0 \end{bmatrix}(a_{0,0}^{u+2,v+2}-a_{2,2}^{u,v})$$
$$= \left(\begin{bmatrix} n_1-3 \\ 0 \end{bmatrix}\begin{bmatrix} n_2-3 \\ 0 \end{bmatrix}+\begin{bmatrix} n_1-3 \\ 0 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}\right)(a_{0,0}^{u+2,v+2}-a_{2,2}^{u,v})$$
$$+\left(\begin{bmatrix} m_1-2 \\ n_1-2 \end{bmatrix}\begin{bmatrix} n_2-3 \\ 0 \end{bmatrix}+\begin{bmatrix} m_1-2 \\ n_1-2 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}\right)(a_{0,0}^{u+2,v+2}-a_{2,2}^{u,v}).$$

The terms $a_{0,0}^{u+2,v+2}-a_{2,2}^{u,v}$ are all non-negative. We will drop all of them except the lower right block of (3.16.1), namely

$$(3.16.2)\quad \begin{bmatrix} m_1-2 \\ n_1-2 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}(a_{0,0}^{u+2,v+2}-a_{2,2}^{u,v}).$$

We will leave this momentarily and move onto $i=1$ and $j=0$. This block will be cut up in a slightly different way.

$$(3.16.3)\quad \begin{bmatrix} m_1-2 \\ 0 \end{bmatrix}\begin{bmatrix} m_2-2 \\ 0 \end{bmatrix}(a_{1,0}^{u+2,v+2}-a_{3,2}^{u,v})$$
$$= \left(\begin{bmatrix} m_1-2 \\ 0 \end{bmatrix}\begin{bmatrix} n_2-3 \\ 0 \end{bmatrix}+\begin{bmatrix} n_1-4 \\ 0 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}\right)(a_{1,0}^{u+2,v+2}-a_{3,2}^{u,v})$$
$$+\left(\begin{bmatrix} m_1-2 \\ m_1-2 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}+\begin{bmatrix} m_1-3 \\ n_1-3 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}\right)(a_{1,0}^{u+2,v+2}-a_{3,2}^{u,v}).$$

Consider again only the lower right hand block of (3.16.3). We will combine the positive part of this with the negative part of (3.16.2) getting $[{m_1-3 \atop n_1-3}][{m_2-2 \atop n_2-2}]a_{1,0}^{u+2,v+2}-[{m_1-2 \atop n_1-2}][{m_2-2 \atop n_2-2}]a_{2,2}^{u,v}$ which can be rewritten as

$$\begin{bmatrix} m_1-3 \\ n_1-3 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}(a_{1,0}^{u+2,v+2}-a_{2,2}^{u+1,v}).$$

We will drop these and all the other three summations in (3.16.3) as they are easily seen to be non-negative. What we are left with is the positive term of (3.16.2) and the negative term of the lower right block of (3.16.3). That is the following:

$$(3.16.4)\quad \begin{bmatrix} m_1-2 \\ n_1-2 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}a_{0,0}^{u+2,v+2}-\begin{bmatrix} m_1-3 \\ n_1-3 \end{bmatrix}\begin{bmatrix} m_2-2 \\ n_2-2 \end{bmatrix}a_{3,2}^{u,v}.$$

From here we may try to move say from $(i,j) = (1,0)$ to $(i,j) = (1,1)$. We can go through a similar procedure in cutting up this block appropriately. Then combining part of it and (3.16.4) we will obtain the following:

$$\begin{bmatrix} m_1 - 2 \\ n_1 - 2 \end{bmatrix}\begin{bmatrix} m_2 - 2 \\ n_2 - 2 \end{bmatrix} a_{0,0}^{u+2,v+2} - \begin{bmatrix} m_1 - 3 \\ n_1 - 3 \end{bmatrix}\begin{bmatrix} m_2 - 3 \\ n_2 - 3 \end{bmatrix} a_{3,3}^{u,v}$$

where all other non-negative terms are again discarded. By a simple induction it is possible to move from $(i,j) = (0,0)$ to $(i,j) = (n_1 - 2, n_2 - 2)$. It does not matter which path we take. We will eventually arrive at $(i,j) = (n_1 - 2, n_2 - 2)$ and be left with the following terms:

$$\begin{bmatrix} m_1 - 2 \\ n_1 - 2 \end{bmatrix}\begin{bmatrix} m_2 - 2 \\ n_2 - 2 \end{bmatrix} a_{0,0}^{u+2,v+2} - \begin{bmatrix} m_1 - n_1 \\ n_1 - n_1 \end{bmatrix}\begin{bmatrix} m_2 - n_2 \\ n_2 - n_2 \end{bmatrix} a_{n_1,n_2}^{u,v}.$$

We may now write it out in the origin notation and collect terms by making the appropriate changes to get:

$$\text{(3.16.5)} \quad \sum_{u=0}^{m_1-n_1} \sum_{v=0}^{m_2-n_2} (F_{(u+n_1,v+n_2)} - T^{n_1} S^{n_2} F_{(u,v)}).$$

By superadditivity, each of the difference terms in (3.16.5) is positive and is greater than or equal to $F_{(\bar{n})}$. And there are a total of $(m_1 - n_1 + 1)(m_2 - n_2 + 1)$ terms. Hence if we only consider the terms in (3.16.5) and drop off all the others, we would have the inequality we desire. ∎

THEOREM 3.17. *Let E be a Banach lattice satisfying the conditions (A), (B) and (UMB). Let $\{F_{(\bar{n})}\}$ be a superadditive process in E_+ with respect to two positive commuting contractions T and S on E. Let $\{F_{(\bar{n})}\}$ be moderately superadditive. Then $F_{(\bar{n})}$ has a sequence of asymptotic dominants.*

Proof. Since M is finite, it is possible to find a sequence $\{(\bar{m}_k)\}$ such that

$$\bar{M} = \sup_{(\bar{m}_k)} \|\Phi_{(\bar{m}_k)}\| < \infty$$

where $(\bar{m}_k) \leqq (\overline{m_{k+1}})$. We may assume that $\lambda(i,j) = WTL_{(\bar{m}_k)\to\infty}(T^i S^j \Phi_{(\bar{m}_k)})$ exists for each $(i,j), i,j = 0,1,2,\ldots$ (To get this use lemma (1.8); and if necessary, by going to a subsequence of $\Phi_{(\bar{m}_k)}$.) We will also assume $\Phi_{(\bar{m}_k)}$ is not TL null, so by (1.8) $\lambda(i,j) \neq 0$ for each (i,j). We claim that $\lambda(i,j)$ is a sequence of asymptotic dominants for $\{F_{(\bar{n})}\}$. To prove this, consider any fixed $(\bar{n})$ and a k such that $(\bar{m}_k) \geqq (\bar{n})$. By lemma (3.16) we get for $(\bar{m}_k) \geqq (\bar{n})$,

$$m_{k_1} \cdot m_{k_2} \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} T^i S^j \Phi_{(\bar{m}_k)} \geqq [(m_{k_1} - n_1 + 1)(m_{k_2} - n_2 + 1)] F_{(\bar{n})}$$

This can be rewritten to get:

$$\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} T^i S^j \Phi_{(\bar{m}_k)} \geqq \frac{[(m_{k_1}-n_1+1)(m_{k_2}-n_2+1)]}{m_{k_1}m_{k_2}} F_{(\bar{n})}.$$

Taking the limit as $(\bar{m}_k) \longrightarrow \infty$ we obtain: $\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} \lambda(i,j) \geqq F_{(\bar{n})}$. By lemma (1.7) we know that there exist non-negative functions $P_{(i,j)}^{(k_1,k_2)}$ for all (k_1,k_2) such that

(3.17.1) $\lambda(i+k_1, j+k_2) - T^{k_1}S^{k_2}\lambda(i,j) = P_{(i,j)}^{(k_1,k_2)}$.

Lemma (3.13) concludes that for all $\epsilon > 0$, there exists (i_0, j_0) such that $\|P_{(i,j)}^{(k_1,k_2)}\| < \epsilon$ for all $(i,j) \geqq (i_0,j_0), (\bar{k}) \geqq 1$. Let $(\bar{n}) \geqq (i,j) \geqq (i_0,j_0)$. We have

$$\begin{aligned}\frac{1}{n_1n_2}F_{(\bar{n})} &\leqq \frac{1}{n_1n_2}\sum_{u=0}^{n_1-1}\sum_{v=0}^{n_2-1}\lambda(u,v)\\ &= \frac{1}{n_1n_2}\left(\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\lambda(u,v)+\sum_{u=0}^{i-1}\sum_{v=j}^{n_2-1}\lambda(u,v)\right)\\ &\quad+\frac{1}{n_1n_2}\left(\sum_{u=1}^{n_1-1}\sum_{v=0}^{j-1}\lambda(u,v)+\sum_{u=i}^{n_1-1}\sum_{v=j}^{n_2-1}\lambda(u,v)\right).\end{aligned}$$

Changing variables for the last term and adding extra terms, the equation becomes:

$$\begin{aligned}\frac{1}{n_1n_2}F_{(\bar{n})} \leqq \frac{1}{n_1n_2}&\left[\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\lambda(u,v)+\sum_{u=0}^{i-1}\sum_{v=j}^{n_2-1}\lambda(u,v)\right.\\ &\left.+\sum_{u=i}^{n_1-1}\sum_{v=0}^{j-1}\lambda(u,v)\right]+\frac{1}{n_1n_2}\sum_{u=0}^{n_1-1}\sum_{v=0}^{n_2-1}\lambda(u+i,\ v+j).\end{aligned}$$

By equation (3.17.1), $\lambda(u+i, v+j) = T^uS^v\lambda(i,j) + P_{(i,j)}^{(u,v)}$. So we may rewrite the last equation as

$$\frac{1}{n_1n_2}F_{(\bar{n})} \leqq \frac{1}{n_1n_2}\left[\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\lambda(u,v)+\sum_{u=0}^{i-1}\sum_{v=j}^{n_2-1}\lambda(u,v)\right.$$
$$\left.+\sum_{u=i}^{n_1-1}\sum_{v=0}^{j-1}\lambda(u,v)\right]+A_{(\bar{n})}\lambda(i,j)+\frac{1}{n_1n_2}\sum_{u=0}^{n_1-1}\sum_{v=0}^{n_2-1}P_{(i,j)}^{(u,v)}$$
$$= A_{(\bar{n})}\lambda(i,j)+\Psi_{(\bar{n})}^{(\bar{i})}.$$

where

$$\Psi_{(\bar{n})}^{(\bar{i})} = \frac{1}{n_1n_2}\left[\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\lambda(u,v)+\sum_{u=0}^{i-1}\sum_{v=j}^{n_2-1}\lambda(u,v)\right.$$
$$\left.+\sum_{u=i}^{n_1-1}\sum_{v=0}^{j-1}\lambda(u,v)\right]+\frac{1}{n_1n_2}\sum_{u=0}^{n_1-1}\sum_{v=0}^{n_2-1}P_{(i,j)}^{(u,v)}.$$

It remains to show that $\lim_{(\bar{i})\to\infty}(\limsup_{(\bar{n})\to\infty}\|\Psi_{(\bar{n})}^{(\bar{i})}\|)=0$. Consider each of the first three terms of the last equation, as $\|\lambda(i,j)\|<\bar{M}<\infty$ for each pair of (i,j) each of those terms tends to zero as $(\bar{n})\to\infty$. (Note that in the summation of each of the terms, at most one but not both of the indices n_1, n_2 appears.) As we had set $(i,j)>(i_0,j_0)$, each of the term in $\frac{1}{n_1n_2}\sum_{u=0}^{n_1-1}\sum_{v=0}^{n_2-1}P_{(i,j)}^{(u,v)}$ has norm less than or equal to ϵ. So the sum is less than or equal to ϵ as there are a total of $n_1\cdot n_2$ terms. Hence, for any $\epsilon>0$, it is possible to pick (i_0,j_0) such that $\limsup_{(\bar{n})\to\infty}\|\Psi_{(\bar{n})}^{(\bar{i})}\|<\epsilon$ for $(i,j)\geqq(i_0,j_0)$. As ϵ is arbitrary, $\lim_{(i,j)\to\infty}(\limsup_{(\bar{n})\to\infty}\|\Psi_{(\bar{n})}^{(i,j)}\|)=0$. So $\lambda(i,j)$ constructed this way is a sequence of asymptotic dominants for $F_{(\bar{n})}$. ∎

4. Multiparameter mean ergodic theorem.

THEOREM 4.1. *Let E be a Banach Lattice satisfying the conditions (A), (B) and (UMB). Let $\{F_{(\bar{n})}\}$ be a moderately superadditive process on E_+ with respect to two positive commuting contractions T and S. Then let Δ be a maximal invariant function and $N=\{x\mid\Delta(x)=0\}$ and then $(X_N\frac{1}{n_1n_2}F_{(\bar{n})})\wedge g=0$ strongly for each $g\in E^+$.*

Proof. Given any $\epsilon>0$, we use theorem (3.17) to obtain a $\lambda\in E_+$ and $\Psi_{(\bar{n})}\in E_+$ such that $\frac{1}{n_1n_2}F_{(\bar{n})}\leqq A_{(\bar{n})}\lambda+\Psi_{(\bar{n})}$ and such that $\limsup_{(\bar{n})\to\infty}\|\Psi_{(\bar{n})}\|<\epsilon$. So

$$\left(X_N\frac{1}{n_1n_2}F_{(\bar{n})}\right)\wedge g \leqq X_N(A_{(\bar{n})}\lambda+\Psi_{(\bar{n})})\wedge g$$
$$\leqq X_NA_{(\bar{n})}\lambda\wedge g+\Psi_{(\bar{n})}.$$

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By theorem (2.1), $X_N A_{(\bar{n})}\lambda \wedge g \longrightarrow 0$, so as $\limsup_{(\bar{n})\to\infty} \|\Psi_{(\bar{n})}\| < \epsilon$, then for all $\epsilon > 0$ we have $\lim_{(\bar{n})\to\infty} \|(X_N \frac{1}{n_1 n_2} F_{(\bar{n})}) \wedge g\| < \epsilon$. Hence $(X_N \frac{1}{n_1 n_2} F_{(\bar{n})}) \wedge g \longrightarrow 0$ strongly for all $g \in E^+$. ∎

THEOREM 4.2. *Let E be a Banach lattice satisfying the conditions (A), (B) and (UMB). Let $\{F_{(\bar{n})}\}$ be a moderately superadditive in E_+ process with respect to two positive commuting contractions T and S. Let Φ be an invariant function under T and S. Let P be the projection onto the support $S(\Phi)$ of Φ, then $P(\frac{1}{n_1 \cdot n_2})F_{(\bar{n})}$ converges strongly.*

Proof. Once again we have

$$\frac{1}{n_1 \cdot n_2} F_{(\bar{n})} \leqq A_{(\bar{n})}\lambda + \Psi_{(\bar{n})}$$

with $\Psi_{(\bar{n})} \in E_+$ for all $(\bar{n})$, and $\limsup_{(\bar{n})\to\infty} \|\Psi_{(\bar{n})}\| < \epsilon$. Hence

$$P \frac{1}{n_1 n_2} F_{(\bar{n})} \leqq PA_{(\bar{n})}\lambda + P\Psi_{(\bar{n})} \leqq PA_{(\bar{n})}\lambda + \Psi_{(\bar{n})}.$$

By theorem (2.5), $PA_{(\bar{n})}\lambda$ in fact will converge to an invariant function ξ. By a simple calculation $PA_{(\bar{n})}\lambda \wedge \xi \longrightarrow \xi$ as well. We can also prove that

$$P \frac{1}{n_1 n_2} F_{(\bar{n})} - P \frac{1}{n_1 n_2} F_{(\bar{n})} \Lambda \xi \leqq PA_{(\bar{n})}\lambda - PA_{(\bar{n})}\lambda \wedge \xi + \Psi_{(\bar{n})}.$$

We then write

$$\begin{aligned}
\limsup_{(\bar{n})\to\infty} \|P \frac{1}{n_1 n_2} F_{(\bar{n})} - P \frac{1}{n_1 n_2} F_{(\bar{n})} \wedge \xi\| \\
&\leqq \limsup_{(\bar{n})\to\infty} \|PA_{(\bar{n})}\lambda - PA_{(\bar{n})}\lambda \wedge \xi + \Psi_{(\bar{n})}\| \\
&\leqq \limsup_{(\bar{n})\to\infty} \|PA_{(\bar{n})}\lambda - \xi\| \\
&\quad + \limsup_{(\bar{n})\to\infty} \|PA_{(\bar{n})}\lambda \wedge \xi - \xi\| \\
&\quad + \limsup_{(\bar{n})\to\infty} \|\Psi_{(\bar{n})}\| \\
&\leqq \limsup_{(\bar{n})\to\infty} \|\Psi_{(\bar{n})}\| < \epsilon.
\end{aligned}$$

As $\epsilon > 0$ is arbitrarily chosen,

$$\limsup_{(\bar{n})\to\infty} \left\| P \frac{1}{n_1 n_2} F_{(\bar{n})} - P \frac{1}{n_1 n_2} F_{(\bar{n})} \wedge \xi \right\| = 0.$$

Now ξ is an invariant function under the operators T and S. $\{F_{(\bar{n})}\}$ is a superadditive sequence and so is the sequence $\{PF_{(\bar{n})}\}$. So by corollary (3.7) we have

$P \frac{1}{n_1 n_2} F_{(\bar{n})} \wedge \xi$ converging strongly. Hence $P \frac{1}{n_1 n_2} F_{(\bar{n})}$ converges strongly as well.

We will now state the general multiparameter superadditive theorem:

THEOREM 4.3. *Let E be a Banach lattice satisfying the conditions (A) (B) and (UMB), Let $\{F_{(\bar{n})}\}$ be a superadditive process in E_+ with respect to k positive commuting contractions $T_1, T_2, \dots T_k$. Let R be the support of a maximal invariant function Φ in E_+ under $T_1, T_2, \dots T_k$. N is the complement of R. Let the following condition be satisfied:*

$$\liminf_{(\bar{n})} \left\| \frac{1}{n_1 n_2 \cdots n_k} \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_k=0}^{n_k-1} (F_{(i_1+1,\dots i_k+1)} - T_1 \cdots T_k F_{(i_1,\dots i_k)}) \right\| < \infty.$$

Then

$$X_R \frac{1}{n_1 n_2 \cdots n_k} F_{(\bar{n})}$$

converges strongly, and for all $g \in E_+$,

$$\left(X_N \frac{1}{n_1 n_2 \cdots n_k} F_{(\bar{n})} \right) \wedge g \longrightarrow 0$$

In the one parameter case Akcoglu and Sucheston obtained further results (theorem 4.6 of [**4**]) if one additional condition is satisfied — namely that the convergence of the Cesáro averages for all $f \in E_+$.

LEMMA 4.4. *Let E be a Banach lattice satisfying the conditions (A), (B) and (UMB). Let $\{F_{(\bar{n})}\}$ be a moderately superadditive process in E_+ with respect to two positive commuting operators T and S. Let T and S be chosen so that $A_{(\bar{n})}(T,S)f$ converges in norm for every $f \in E_+$. Then $\frac{1}{n_1 n_2} F_{(\bar{n})}$ converges in norm to an invariant function.*

Proof. Let Φ be the maximal invariant function with support R and N be the complement of R. By theorem (4.2) we have $X_R \frac{1}{n_1 n_2} F_{(\bar{n})}$ converging strongly. It is sufficient to show that $X_N \frac{1}{n_1 n_2} F_{(\bar{n})}$ also converges (to zero in norm). Once again we obtain a sequence of asymptotic dominants for $\{F_{(\bar{n})}\}$ such that

$$(4.4.1) \quad \frac{1}{n_1 n_2} F_{(\bar{n})} \leqq A_{(\bar{n})}\lambda + \Psi_{(\bar{n})}$$

with $\lim\sup_{(\bar{n})\to\infty} \|\Psi_{(\bar{n})}\| < \epsilon$. Now $A_{(\bar{n})}\lambda$ converges as λ is in E_+. So $X_N A_{(\bar{n})}\lambda$ converges strongly as well. From theorem (2.1) we get that $TLX_N A_{(\bar{n})}\lambda = 0$. If all the limits exist, then

$$\liminf_{(\bar{n})\to\infty} g_{(\bar{n})} \leqq TLg_{(\bar{n})} \leqq \limsup_{(\bar{n})\to\infty} g_{(\bar{n})}$$

So if a sequence has a limit, it is in fact the TL limit. So $X_N A_{(\bar{n})}\lambda$ converges to zero. Going back to equation (4.4.1), we first multiply it with X_N and then take $\limsup_{(\bar{n})\to\infty}$ of the whole equation. Note that $\limsup_{(\bar{n})\to\infty}\|X_N A_{(\bar{n})}\lambda\| = 0$, and $\limsup_{(\bar{n})\to\infty}\|\Psi_{(\bar{n})}\| < \epsilon$. So $X_N \frac{1}{n_1 n_2} F_{(\bar{n})}$ converges strongly to zero as well, as ϵ is arbitrary. Rewriting equation (4.4.1) we obtain

$$X_R \frac{1}{n_1 n_2} F_{(\bar{n})} \leqq X_R A_{(\bar{n})}\lambda + X_R \Psi_{(\bar{n})} \leqq X_R A_{(\bar{n})}\lambda + \Psi_{(\bar{n})}$$

with $R = X - N$. Now let $\lim_{(\bar{n})\to\infty} X_R \frac{1}{n_1 n_2} F_{(\bar{n})} = \Psi$. As $\epsilon > 0$ is arbitrary,

$$\begin{aligned}\lim_{(\bar{n})\to\infty} X_R \frac{1}{n_1 n_2} F_{(\bar{n})} &= \limsup_{(\bar{n})\to\infty} X_R \frac{1}{n_1 n_2} F_{(\bar{n})} \\ &= X_R \bar{\lambda} + h\end{aligned}$$

where $\bar{\lambda} = \lim_{(\bar{n})\to\infty} A_{(\bar{n})}\lambda$, and $\|h\| \leqq \epsilon$. Since $X_N \frac{1}{n_1 n_2} F_{(\bar{n})} \longrightarrow 0$ we have

$$\text{(4.4.2)} \quad \lim_{(\bar{n})\to\infty} \frac{1}{n_1 n_2} F_{(\bar{n})} = \lim_{(\bar{n})\to\infty} X_R \frac{1}{n_1 n_2} F_{(\bar{n})} \leqq X_R \bar{\lambda} + h \leqq \bar{\lambda} + h.$$

Since $\epsilon > 0$ is arbitrary, $\frac{1}{(\bar{n})} F_{(\bar{n})} \leqq \bar{\lambda}$ and by theorem (3.6) we know that $\frac{1}{(\bar{n})} F_{(\bar{n})}$ converges in norm. Though not obvious, it is not difficult to show that the limit of $\frac{1}{(\bar{n})} F_{(\bar{n})}$ is invariant as well. ∎

It can be shown and is known that for a reflexive Banach Lattice, $A_{(\bar{n})}f$ converges for all $f \in E_+$ if T and S are contractions. Hence for a reflexive Banach Lattice we have the following theorem in the multiparameter form.

THEOREM 4.5. *Let E be a reflexive Banach lattice satisfying the conditions (A), (B) and (UMB). Let $\{F_{(\bar{n})}\}$ be a superadditive process in E_+ with respect to k positive commuting operators $T_1, T_2, \ldots, T_k$. Let*

$$\liminf_{(\bar{n})\to\infty} \left\| \frac{1}{n_1 \cdots n_k} \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_k=0}^{n_k-1} \left(F_{(i_1+1,\ldots i_k+1)} - T_1 \cdots T_k F_{(i_1,\ldots i_k)} \right) \right\| = M < \infty.$$

Then $\frac{1}{(\bar{n})} F_{(\bar{n})}$ converges in norm to an invariant function.

The material of this paper is condensed from the author's doctoral thesis at the University of Toronto, 1988.

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