Canad. Math. Bull. Vol. **15** (1), 1972

MATHEMATICAL NOTES

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SOME REMARKS ON WEIGHTS OF PERMUTATIONS

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This note concerns the smallest number $w(\sigma)$ of transpositions of which a permutation σ may be written as a product. The discussion is mainly concerned with the value of this number on the product of two permutations.

1. If σ is a finite permutation, by a *minimal expression* for σ, we mean the expression of σ as the product of a minimal number of transpositions. $w(\sigma)$, the *weight* of σ, refers to the number of transpositions involved in a minimal expression. For example, if s is a cycle of length h, it is easy to see that $w(s)=h-1$. We observe that $w(\sigma^{-1})=w(\sigma)$ and that $w(\tau^{-1}\sigma\tau)=w(\sigma)$.

We describe $\sigma=s_1\ldots s_n$ as a *disjoint presentation* if $s_1,\ldots,s_n$ are disjoint cycles. For disjoint presentations $\sigma=s_1\ldots s_n$, $\tau=s'_1\ldots s'_m$, we introduce letters $A_1,\ldots,A_n$; $B_1,\ldots,B_m$ corresponding to $s_1,\ldots,s_n$; $s'_1,\ldots,s'_m$ and we construct the matching for which A_i, B_j are joined by a segment labelled k when s_i, s'_j have the symbol k in common. We note that, in general, two letters may be joined by more than one segment. When no two letters are joined by more than one segment, we say that σ, τ are *simply matched*. To avoid confusion, the word *cycle* is reserved for permutation terminology, whereas a cycle in the connectivity sense will be termed a *closed chain*.

A sequence $(\sigma_1,\ldots,\sigma_h)$ of permutations is called *concurrent* if $w(\sigma_1\ldots\sigma_h) =w(\sigma_1)+\cdots+w(\sigma_h)$; $\sigma_1\ldots\sigma_h$ is called the *product* of the sequence. For example, a sequence of disjoint cycles may be readily seen to be concurrent. It is not difficult to see that, by the procedure of commuting disjoint permutations, any minimal expression assumes the form of a juxtaposition of minimal expressions for the disjoint cycles of a disjoint presentation. As another example, useful in the computations below, the sequence of three cycles

$$\text{(A)}\qquad \{(a_1a_2\ldots a_i),\quad (a_{i+1}a_{i+2}\ldots a_n),\quad (a_ia_n)\}$$

is concurrent with product $(a_1a_2\ldots a_n)$. Clearly, if a sequence is concurrent, so also is any subsequence formed from a consecutive subset.

As a further example, if two cycles s_1, s_2 have a single symbol in common, (s_1, s_2) is concurrent. An easy deduction from this shows that if the matching between permutations σ, τ contains no closed chains, then (σ, τ) is concurrent. We assert that the converse is also true (Assertion 1).

2. We say that ρ is a *middle term* for σ, τ if there are permutations σ_1, τ_1 where (σ_1, ρ), (ρ^{-1}, τ_1) are concurrent with products σ, τ. By treating the case where ρ is a transposition, we may easily see, via (A), that simply matched permutations possess no nontrivial middle term. (A simple example of simply matched non-concurrent permutations is given by $\sigma=(12)(34)$, $\tau=(13)(24)$.)

Conversely, if σ, τ, with disjoint presentations $s_1 \ldots s_n$, $s'_1 \ldots s'_m$, are not simply matched, choose a pair of symbols i, j both occurring in two of the cycles in question. On applying (A) and its dual, we find concurrent pairs $(\sigma', (ij))$, $((ij), \tau')$ with products σ, τ. By repeating this procedure on σ', τ' and so on, we find concurrent pairs (σ_1, ρ), (ρ^{-1}, τ_1) with products σ, τ and with σ_1, τ_1 simply matched.

For permutations σ, τ, we have, in general, $w(\sigma\tau)=w(\sigma)+w(\tau)-2k$, say; we describe $2k$ as the *depletion* of the pair. We wish to explain how the depletion $2k$ occurs. A very simple case, for example, is offered when (σ_1, ρ), (ρ^{-1}, τ_1) are concurrent with products σ, τ and where (σ_1, τ_1) is concurrent. It is clear from the above remarks that this does not explain all cases of depletion although it could be used to reduce the explanation to that of simply matched permutations. Another case, which we shall refer to as the *basic case*, occurs when

$$\sigma = (a_1a_2)(a_3a_4)\ldots(a_{2n-1}a_{2n}) = H, \text{ say,}$$

$$\tau = (a_3a_2)(a_5a_4)\ldots(a_1a_{2n}) = K, \text{ say,}$$

where the symbols $a_1, \ldots, a_{2n}$ are distinct. The depletion is, of course, 2.

We say that $(\sigma_1, \ldots, \sigma_h)$ is *simply depleted* if $w(\sigma_1, \ldots, \sigma_p, \sigma_{p+1})=w(\sigma_1 \ldots \sigma_p)+w(\sigma_{p+1})-2$ for each p with $1 \leq p < h$. The third example of depletion of σ, τ occurs when there is a concurrent sequence $(\rho_1, \ldots, \rho_k, \tau_1)$ with product τ such that $(s, \rho_1^{-1}, \ldots, \rho_k^{-1})$ is simply depleted and (σ, τ_1) is concurrent. The depletion is $2k$ and we assert, conversely, that all cases of depletion $2k$ occur in this form for a suitable choice of $\rho_1, \ldots, \rho_k$ (Assertion 2). We further assert that every simple depletion occurs in the following special way: for every pair σ, τ with depletion 2, there are permutations σ_1, τ, ϕ, ψ where (σ_1, ϕ) and (ψ, τ_1) are concurrent with products σ, τ, respectively, and where ϕ, ψ constitute a basic case (Assertion 3).

3. Let $\sigma=s_1\ldots s_r$, $\tau=s'_1\ldots s'_m$ be disjoint presentations. Choose a closed chain of the matching defined by σ and τ that is minimal in the sense that no proper subset of the vertices carried by the closed chain is carried by a closed chain. Write this closed chain as $S=[a_1a_2\ldots a_{2n}]$, where each of (a_1, a_2), $(a_3, a_4), \ldots, (a_{2n-1}, a_{2n})$ is a pair of symbols occurring in a cycle s_i and each of (a_2, a_3), $(a_4, a_5), \ldots, (a_{2n}, a_1)$ is a pair of symbols occurring in a cycle s'_j. Minimality ensures that each of the cycles for σ or τ, whose associated letter A_i or B_j occurs in the closed chain S contains exactly one pair (a_i, a_{i+1}) (i.e. mod $2n$). For $i=1, 2, \ldots, n$, denote by c_i the cycle for

σ containing $(a_{2i-1,2i})$ and by c_i' the cycle for τ containing (a_{2i}, a_{2i+1}). One sees that minimality also ensures that any symbol in common between c_i, c_j' occurs among $a_1, \ldots, a_{2n}$. Indeed, if b were a common symbol and $i \leq j$ then S could be replaced by the smaller $[a_{2i-1}a_{2i} \ldots a_{2j}b]$; the case $j \geq i$ is dealt with similarly. It then follows, by an easy computation, that $c_1 \ldots c_n c_1' \ldots c_n'$ is a product of two disjoint cycles together involving all N symbols, say, occurring in $c_1, \ldots, c_n$ and all M symbols occurring in $c_1', \ldots, c_n'$. Hence the weight of the product is $N+M-2n-2$ and we conclude that

$$w(c_1 \ldots c_n, c_1' \ldots c_n') = w(c_1) + \cdots + w(c_n) + w(c_1') + \cdots + w(c_n') - 2.$$

Assertion 1 follows immediately.

With H, K as in §2, application of (A) shows that there are permutations σ_1', τ_1 such that (σ_1', H), (K, τ_1) are concurrent with products σ, τ. Assertion 3 then follows. Since $HK = (a_1a_3 \ldots a_{2n-1})^{-1}(a_2a_4 \ldots a_{2n})$, the converse of Assertion 1 shows that (σ_1', HK) is concurrent with product σ_1, say, and hence that (σ, K) is simply depleted. We repeat the construction with σ, τ replaced by σ_1, τ_1 and so on and Assertion 2 follows.

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https://doi.org/10.4153/CMB-1972-022-x Published online by Cambridge University Press