Canad. Math. Bull. Vol. **20** (4), 1977

SOME GENERALIZATIONS OF AN IDENTITY OF SUBHANKULOV

BY

D. SURYANARAYANA* AND DAVID T. WALKER

ABSTRACT. In 1957, M. A. Subhankulov established the following identity

$$\sum_{r=1}^{\infty} \frac{\mu(r_1r_2)r_2^2}{J_2(r)} \sum_{d \mid (n,r)} d\mu\left(\frac{r}{d}\right) = \mu^2(n)\frac{\pi^2}{6},$$

where $r = r_1r_2^2$, $(r_1, r_2) = 1$; μ is the Möbius function and J_2 is the Jordan totient function of order 2. Since the Ramanujan trigonometrical sum $C(n, r) = \sum_{d \mid (n,r)} d\mu(r/d)$, we rewrite the above identity using $C(n, r)$.

In this paper, we give a generalization of Ramanujan's sum, which generalizes some of the earlier generalizations mainly due to E. Cohen, and prove a theorem from which we deduce some generalizations of the above identity.

§1. Introduction. In 1957, M. A. Subhankulov [10] established the following curious identity (in a slightly different form and notation):

(1) $$\sum_{r=1}^{\infty} \frac{\mu(r_1r_2)r_2^2}{J_2(r)} \sum_{d \mid (n,r)} d\mu\left(\frac{r}{d}\right) = \mu^2(n)\frac{\pi^2}{6},$$

where $r = r_1r_2^2$, $(r_1, r_2) = 1$; μ is the well-known Möbius function and J_2 is the Jordan totient function of order 2. The identity (1) may also be found in a subsequent paper of Subankulov written jointly with S. N. Muhatarov (cf. [11], eq. (3)).

Since it is known (cf. [8], Theorem 271) that

(2) $$C(n, r) = \sum_{d \mid (n,r)} d\mu\left(\frac{r}{d}\right)$$

Received by the editors Jan. 4, 1977

AMS (MOS) Subject classifications. Primary 10A20, 10A99.

Key words and phrases: Möbius function, Jordan totient function, generalized Ramanujan's sum, k-free integers, Riemann zeta function.

* On leave from Andhra University, Waltair, India.

and $\zeta(2)=\pi^2/6$, where $\zeta(s)$ is the Riemann Zeta function defined for $s>1$ by $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ and $C(n, r)$ is the Ramanujan trigonometrical sum (cf. [8], §5.6) defined by

$$C(n, r)=\sum_{\substack{x \,(\mathrm{mod}\, r)\\(x, r)=1}} \exp(2\pi ixn/r), \tag{3}$$

we can rewrite the identity (1) as follows:

$$\sum_{r=1}^{\infty} \frac{\mu(r_1r_2)r_2^2}{J_2(r)} C(n, r)=\mu^2(n)\zeta(2). \tag{4}$$

In this paper, we establish some identities as generalizations of the identity (4). For example, we prove that for $k\geq 1$,

$$\sum_{r=1}^{\infty} \frac{\mu(r_1r_2)r_2^{2k}}{J_{2k}(r)} C_k(n, r)=q_{2k}(n)\zeta(2k), \tag{5}$$

where $r=r_1r_2^2$, $(r_1, r_2)=1$; $J_k(r)$ is the Jordan totient function of order k (cf. [7], p. 147; also cf. [2] and [3]) which has the arithmetic evaluation

$$J_k(r)=\sum_{d \mid r} d^k\mu\left(\frac{r}{d}\right)=r^k\prod_{p \mid r}(1-p^{-k}), \tag{6}$$

p a prime, $q_k(r)=1$ or 0 according as $r\in Q_k$ or $r\notin Q_k$, Q_k being the set of all k-free integers (a positive integer r is called k-free, if r is not divisible by p^k for any prime p) and $C_k(n, r)$ is E. Cohen's [1] generalized Ramanujan sum defined by

$$C_k(n, r)=\sum_{\substack{x \,(\mathrm{mod}\, r^k)\\(x, r^k)_k=1}} \exp(2\pi ixn/r^k), \tag{7}$$

the summation being extended over all x modulo r^k, whose greatest common kth power divisor with r^k is 1. E. Cohen (cf. [1], eq. (2.5)) also established the following arithmetic evaluation of $C_k(n, r)$:

$$C_k(n, r)=\sum_{\substack{d^k \mid n\\ d \mid r}} d^k\mu\left(\frac{r}{d}\right). \tag{8}$$

In fact, we first prove a general result, from which we deduce some generalizations of the identity (4) (for example, see Remark 2 and (16) of §3), in the following:

https://doi.org/10.4153/CMB-1977-073-6 Published online by Cambridge University Press

THEOREM. *If α and β are integers each ≥ 2; if k, u, and $n_1, \ldots, n_u$ are integers each ≥ 1; and if $v_p(n)$ is the non-negative integer such that $p^{v_p(n)}$ is the highest power of the prime p that divides n, where $n = (n_1, \ldots, n_u)$, then*

$$(9)\quad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{\alpha(\beta-1)}}{J_\alpha(r)} C_k^{(u)}(n_1, \ldots, n_u, r)$$
$$= \zeta(\alpha) \prod_{\substack{p \\ k \leq v_p(n) < (\beta-1)k}} \left(1 - \frac{1}{p^{\alpha-ku}}\right) \prod_{\substack{p \\ (\beta-1)k \leq v_p(n) < \beta k}} \left(1 - \frac{1}{p^{\alpha-ku}} + \frac{1}{p^{\alpha-(\beta-1)ku}}\right)$$
$$\times \prod_{\substack{p \\ v_p(n) \geq \beta k}} \left(1 - \frac{1}{p^{\alpha-ku}} + \frac{1}{p^{\alpha-(\beta-1)ku}} - \frac{1}{p^{\alpha-\beta ku}}\right),$$

where $r = r_1 r_2^\beta$, $(r_1, r_2) = 1$, and each product is extended over all primes p subject to the restrictions on $v_p(n)$ mentioned under each product.

In the above Theorem $C_k^{(u)}(n_1, \ldots, n_u, r)$ is a generalization of Ramanujan's sum defined as follows:

$$(10)\quad C_k^{(u)}(n_1, \ldots, n_u, r) = \sum_{\substack{(x_i)(\text{mod } r^k) \\ ((x_i), r^k)_k = 1}} \exp(2\pi i(n_1 x_1 + \cdots + n_u x_u)/r^k),$$

where the summation is extended over all x_i modulo r^k, for $i = 1, \ldots, k$, such that the greatest common kth power divisor of $(x_1, \ldots, x_u)$ and r^k is 1. Following the method adopted by M. Sugunamma (cf. [12]), Theorem 1) and E. Cohen (cf. [5], Lemma 2 and cf. [6], p. 30), we get the following arithmetic evaluation:

$$(11)\qquad C_k^{(u)}(n_1, \ldots, n_u, r) = \sum_{\substack{d^k \mid n \\ d \mid r}} d^{ku} \mu\left(\frac{r}{d}\right),$$

where $n = (n_1, \ldots, n_u)$.

REMARK 1. We note here that (10) gives a generalization of some of the known generalizations of Ramanujan's sum. For example, when $n_1 = \cdots = n_u = n$, (10) reduces to $C_k^{(u)}(n, r)$, which is due to M. Sugunamma [12]; when $u = 1$, $n_1 = n$, (10) *reduces to* $C_k(n, r)$ which is due to E. Cohen [1]; when $k = 1$, $n_1 = \cdots = n_u = n$, (10) reduces to $C^{(u)}(n, r)$ which is again due to E. Cohen [4]; and finally when $k = 1$, (10) reduces to $C^{(u)}(n_1, \ldots, n_u, r)$ which is once again due to E. Cohen (cf. [5] and [6]).

§**2. Proof of the Theorem.** Since the general term in the series (9) is a multiplicative function of r and the series is absolutely convergent, it can be expanded into an infinite product of Euler type (cf. [8],Theorem 286). Hence by (6) and (11), we have

$$\sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{\alpha(\beta-1)}}{J_\alpha(r)} C_k^{(u)}(u_1, \ldots, n_u, r)$$

$$= \prod_p \left\{ 1 - \frac{C_k^{(u)}(n_1, \ldots, n_u, p)}{J_\alpha(p)} - \frac{C_k^{(u)}(n_1, \ldots, n_u, p^\beta) p^{\alpha(\beta-1)}}{J_\alpha(p^\beta)} \right\}$$

$$= \prod_{\substack{p \\ v_p(n)<k}} \left(1 + \frac{1}{p^\alpha - 1}\right) \prod_{\substack{p \\ k \le v_p(n) < (\beta-1)k}} \left(1 - \frac{p^{ku}-1}{p^\alpha - 1}\right)$$

$$\times \prod_{\substack{p \\ (\beta-1)k \le v_p(n) < \beta k}} \left(1 - \frac{p^{ku}-1}{p^\alpha - 1} + \frac{p^{(\beta-1)ku}}{p^\alpha - 1}\right) \prod_{\substack{p \\ v_p(n) \ge \beta k}} \left(1 - \frac{p^{ku}-1}{p^\alpha - 1} - \frac{p^{\beta ku} - p^{(\beta-1)ku}}{p^\alpha - 1}\right)$$

$$= \prod_{\substack{p \\ v_p(n)<k}} \left(\frac{1}{1-p^{-\alpha}}\right) \prod_{\substack{p \\ k \le v_p(n) < (\beta-1)k}} \left(\frac{1-p^{-\alpha+ku}}{1-p^{-\alpha}}\right) \prod_{\substack{p \\ (\beta-1)k \le v_p(n) < \beta k}} \left(\frac{1-p^{-\alpha+ku} + p^{-\alpha+(\beta-1)ku}}{1-p^{-\alpha}}\right)$$

$$\times \prod_{\substack{p \\ v_p(n) \ge \beta k}} \left(\frac{1-p^{-\alpha+ku} + p^{-\alpha+(\beta-1)ku} - p^{-\alpha+\beta ku}}{1-p^{-\alpha}}\right)$$

$$= \prod_p \left(\frac{1}{1-p^{-\alpha}}\right) \prod_{\substack{p \\ k \le v_p(n) < (\beta-1)k}} \left(1 - \frac{1}{p^{\alpha-ku}}\right) \prod_{\substack{p \\ (\beta-1)k \le v_p(n) < \beta k}} \left(1 - \frac{1}{p^{\alpha-ku}} + \frac{1}{p^{\alpha-(\beta-1)ku}}\right)$$

$$\times \prod_{\substack{p \\ v_p(n) \ge \beta k}} \left(1 - \frac{1}{p^{\alpha-ku}} + \frac{1}{p^{\alpha-(\beta-1)ku}} - \frac{1}{p^{\alpha-\beta ku}}\right).$$

Now, applying Euler's result that $\zeta(\alpha) = \prod_p (1-p^{-\alpha})$, (cf. [8], Theorem 280), the Theorem follows.

§3. **Some special cases.** Taking $\beta = 2$ in (9), we have the following result: For $\alpha \ge 2$, $k \ge 1$ and $u \ge 1$,

$$\sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{\alpha}}{J_\alpha(r)} C_k^{(u)}(n_1, \ldots, n_u, r) = \zeta(\alpha) \prod_{p^{2k}/n} \left(1 - \frac{1}{p^{\alpha-2ku}}\right), \tag{12}$$

where $r = r_1 r_2^2$, $(r_1, r_2) = 1$ and $n = (n_1, \ldots, n_u)$.

If $n \in Q_{2k}$, then the right side of (12) becomes $\zeta(\alpha)$. On the other hand, if $n \notin Q_{2k}$ and $\alpha = 2ku$, then the right side of (12) becomes zero. Hence from (12), we obtain the following:

$$\sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{\alpha}}{J_\alpha(r)} C_k^{(u)}(n_1, \ldots, n_u, r) = \begin{cases} \zeta(\alpha), & \text{if } n \in Q_{2k} \\ 0, & \text{if } n \notin Q_{2k} \text{ and } \alpha = 2ku. \end{cases} \tag{13}$$

Taking $\alpha = 2ku$ in (13), we have

$$\sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{2ku}}{J_{2ku}(r)} C_k^{(u)}(n_1, \ldots, n_u, r) = q_{2k}(n) \zeta(2ku). \tag{14}$$

https://doi.org/10.4153/CMB-1977-073-6 Published online by Cambridge University Press

As a particular case of (14), taking $n_1 = \cdots = n_u = n$, we have the identity

$$(15)\qquad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{2ku}}{J_{2ku}(r)} C_k^{(u)}(n, r) = q_{2k}(n)\zeta(2ku).$$

REMARK 2. Now, taking $u = 1$ in (15), we have the identity (5). Also, taking $k = 1$ in (15), we have the following identity, which is a generalization of (4), different from (5):

$$(16)\qquad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{2u}}{J_{2u}(r)} C^{(u)}(n, r) = \mu^2(n)\zeta(2u).$$

Taking $\alpha = 2k(u+1)$ in (12), we get the following identity:

$$(17)\qquad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{2k(u+1)}}{J_{2k(u+1)}(r)} C_k^{(u)}(n_1, \ldots, n_u, r) = \zeta(2k(u+1)) \frac{\Phi_{2k}(n)}{n},$$

where $\Phi_k(n)$ is Klee's [9] generalized Euler totient function $\varphi(n)$, which has the arithmetic evaluation

$$(18)\qquad \Phi_k(n) = \sum_{d^k \mid n} \mu(d)(n/d^k) = n \prod_{p^k \mid n} (1 - p^{-k}).$$

As it is clear from (6) and (18) that $\Phi_k(n^k) = J_k(n)$, we obtain, from (17), the following identity:

$$(19)\qquad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{2k(u+1)}}{J_{2k(u+1)}(r)} C_k^{(u)}(n_1^{2k}, \ldots, n_u^{2k}, r) = \zeta(2k(u+1)) \frac{J_{2k}(n)}{n^{2k}}$$

Taking $n_1 = \cdots = n_u = n$ in (19), we have

$$(20)\qquad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^{2k(u+1)}}{J_{2k(u+1)}(r)} C_k^{(u)}(n^{2k}, r) = \zeta(2k(u+1)) \frac{J_{2k}(n)}{n^{2k}}$$

As a particular case of (20), taking $k = 1$ and $u = 1$, we have

$$(21)\qquad \sum_{r=1}^{\infty} \frac{\mu(r_1 r_2) r_2^4}{J_4(r)} C(n^2, r) = \frac{\pi^4}{90} \left(\frac{J_2(n)}{n^2} \right)$$

REMARK 3. We can deduce some results from the Theorem in case $\beta \geq 3$ also. For example, in this case, if $\alpha = ku$ and $k \leq v_p(n) < (\beta - 1)k$ for some prime divisor p of $n = (n_1, \ldots, n_u) > 1$, then the sum of the series in (9) becomes zero.

REFERENCES

1. E. Cohen, *An extension of Ramanujan's sum*, Duke Math. J., **16** (1949), 85–90.
2. E. Cohen, *Some totient functions*, Duke Math. J., **23** (1956), 515–522.
3. E. Cohen, *Generalizations of the Euler φ-function*, Scripta Math., **23** (1957), 157–161.
4. E. Cohen, *Trigonometric sums in elementary number theory*, Amer. Math. Monthly, **66** (1959), 105–117.

5. E. Cohen, *A class of arithmetical functions in several variables with applications to congruences*, Trans. Amer. Math. Soc., **96** (1960), 355–381.

6. E. Cohen, *A trigonometric sum*, Math. Student, **28** (1960), 29–32.

7. L. E. Dickson, *History of the theory of numbers*, Vol. **I**, Chelsea Publishing Company, reprinted, 1952.

8. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Fourth edition, Oxford University Press, 1960.

9. V. L. Klee, *A generalization of Euler's function*, Amer. Math. Monthly, **55** (1948), 358–359.

10. M. A. Subhankulov, *Some asymptotic formulas in additive theory of numbers (Russian)*, Scientific Journal of the Tadzhik University, Vol. **X**, No. **4**, (1957), 15–22.

11. M. A. Subhankulov and S. N. Muhatarov, *Representations of a number as a sum of two square-free numbers* (Russian), Izv. Akad. Nauk UzSSR Ser. Fiz.–Mat. Nauk 1960, No. **4**, 3–10.

12. M. Sugunamma, *Eckford Cohen's generalizations of Ramanujan's trigonometrical sum* C(n, r), Duke Math. J., **27** (1960), 323–330.

Department of Mathematical Sciences
Memphis State University
Memphis, Tennessee 38152