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A theorem on diophantine approximations

Kurt Mahler

Dedicated to Th. Schneider

If S is a set of positive integers which contains $1, 2, \ldots, n-1$, but not n or any multiple of n, where $n \geq 2$, then

$$\sup_{\alpha \in \mathbb{R}} \inf_{s \in S} \|s\alpha\| = 1/n .$$

Here $\mathbb{R}$ is the field of real numbers, and $\|\alpha\|$ denotes the distance of α from the nearest integer.

Let $\mathbb{R}$ be the field of real numbers. For $\alpha \in \mathbb{R}$, denote as usual by $\|\alpha\|$ the distance of α from the nearest integer; thus always

$$0 \leq \|\alpha\| \leq 1/2 .$$

Further let n be any integer not less than 2.

THEOREM. *Let S be a finite or infinite set of positive integers with the following two properties:*

(P_1) *S contains the integers $1, 2, \ldots, n-1$;*

(P_2) *S does not contain any of the integers $n, 2n, 3n, \ldots$.*

Then

$$\sup_{\alpha \in \mathbb{R}} \inf_{s \in S} \|s\alpha\| = 1/n .$$

Proof. Put

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$$f(\alpha|S) = \inf_{s \in S} \|s\alpha\| \ , \quad F(S) = \sup_{\alpha \in R} f(\alpha|S) \ .$$

We have to show that $F(S) = 1/n$.

If S and T are any two sets such that $S \supseteq T$, then evidently

$$f(\alpha|S) \le f(\alpha|T) \quad \text{for every} \quad \alpha \in R \ .$$

Thus, on putting

$$T = \{1, 2, \ldots, n-1\} \ ,$$

certainly $f(\alpha|S) \le f(\alpha|T)$ if S has the property (P_1) as we are assuming. We therefore begin by proving that $f(\alpha|T) \le 1/n$ for all α .

The two linear forms $\alpha x - y$ and x in x and y have the determinant 1 . It follows then from Minkowski's theorem on linear forms that the pair of inequalities

(1) $$|\alpha x - y| \le 1/n \ , \quad |x| < n$$

has a solution in integers x, y not both zero. If $x = 0$, then y does not vanish, and the first inequality (1) gives a contradiction; hence $x \neq 0$. Without loss of generality x is positive, hence by (1) is one of the integers $1, 2, \ldots, n-1$. Further $1/n \le 1/2$ by hypothesis. Hence for $s = x$,

$$\|s\alpha\| = |\alpha x - y| \le 1/n \ ,$$

which implies that $f(\alpha|T) \le 1/n$ for all $\alpha \in R$ and therefore that both

$$F(T) \le 1/n \quad \text{and} \quad F(S) \le 1/n \ .$$

In the other direction, we shall deduce from the assumption (P_2) that $F(S) \ge 1/n$. It suffices to prove that

$$\|s.1/n\| \ge 1/n \quad \text{for all} \quad s \in S \ .$$

This is obvious because s is not a multiple of n and hence the distance of $s.1/n$ from the nearest integer is not 0 , but is an integral multiple of $1/n$.

As an application, denote by T the set of all primes and put $S = T \cup \{1\}$. It is clear that S has both the properties (P_1) and (P_2) with $n = 4$; hence

$$F(S) = 1/4 \ .$$

We assert that also

$$F(T) = \sup_{\alpha\in\mathbb{R}} \inf_{p} \|p\alpha\| = 1/4 , \tag{2}$$

where in the lower bound p runs over all primes.

If this assertion is false, then necessarily $F(T) > F(S) = 1/4$. There is then a number α, say in the interval from 0 to 1, such that

$$\|\alpha\| > 1/4 \quad \text{and} \quad \|p\alpha\| > 1/4 \quad \text{for all primes } p .$$

The first inequality allows us to assume that α lies between $1/4$ and $3/4$, hence by symmetry between $1/4$ and $1/2$. But it is easily verified that

$$\|3\alpha\| \le 1 - 3\alpha \le 1/4 \quad \text{if} \quad 1/4 \le \alpha \le 1/3 ,$$

$$\|3\alpha\| \le 3\alpha - 1 \le 1/4 \quad \text{if} \quad 1/3 \le \alpha \le 2/5 ,$$

$$\|2\alpha\| \le 1 - 2\alpha \le 1/5 \quad \text{if} \quad 2/5 \le \alpha \le 1/2 .$$

Therefore $f(\alpha|T)$ cannot be greater than $1/4$ when α lies between $1/4$ and $3/4$ and so is never greater than $1/4$. Therefore also $F(T) \le 1/4$, and hence $F(T) = 1/4$ because of $F(T) \ge F(S)$.

Note added in proof [26 March 1976]. A study of the proof of the theorem has led me to the following conjecture:

CONJECTURE. *Let m and n be two positive integers such that $2m \le n$. Let S be a finite or infinite set of positive integers with the following two properties:*

(Q_1) *S contains the integers $m, m+1, m+2, \ldots, n-m$;*

(Q_2) *every element of S satisfies the inequality*

$$\|s/n\| \ge m/n .$$

Then

$$\sup_{\alpha\in\mathbb{R}} \inf_{s\in S} \|s\alpha\| = m/n .$$

For $m = 1$ this conjecture is identical with the theorem.

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.