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Special Fibres and Critical Locus for a Pencil of Plane Curve Singularities

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Abstract. We will analyze the relationships between the special fibres of a pencil Λ of plane curve singularities and the Jacobian curve J of Λ (defined by the zero locus of the Jacobian determinant for any fixed basis $\phi, \phi' \in \Lambda$). From the results, we find decompositions of J (and of any special fibre of the pencil) in terms of the minimal resolution of Λ. Using these decompositions and the topological type of any generic pair of curves of Λ, we obtain some topological information about J. More precise decompositions for J can be deduced from the minimal embedded resolution of any pair of fibres (not necessarily generic) or from the minimal embedded resolution of all the special fibres.

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1. Introduction

Let $f, g: \mathbb{C}^2, 0 \to \mathbb{C}, 0$ be two germs of holomorphic functions and let $\Lambda = \{\lambda f + \mu g : \lambda, \mu \in \mathbb{C}\}$ be the pencil defined by f and g. We will assume that the plane curve germs defined by $\{f = 0\}$ and $\{g = 0\}$ do not share any branch. For $w = (w_1 : w_2) \in \mathbb{CP}^1$, let $\phi_w = w_2 f - w_1 g$ be the function of the pencil Λ corresponding to w and let $\Phi_w = \{\phi_w = 0\}$ be the fibre defined by ϕ_w. It is well known that all the fibres Φ_w are equisingular (and in fact have the same resolution) but a finite number, those called *special fibres*. We will denote $\mathrm{Sp}(\Lambda)$ the set of special fibres of Λ. The nonspecial fibres are called *generic*. Notice that, if Φ_w is generic, then its corresponding function ϕ_w is reduced.

In what follows the plane curve germ $\Phi = \{\varphi = 0\}$ defined by the germ of a holomorphic function $\varphi: \mathbb{C}^2, 0 \to \mathbb{C}, 0$ is taken with its reduced structure (except places where otherwise stated). Let $\Psi = \{\psi = 0\}$ be the plane curve germ defined by the holomorphic function $\psi: \mathbb{C}^2, 0 \to \mathbb{C}, 0$ and let $\varphi = \prod_{i=1}^{r} \varphi_i^{r_i}$ (resp. $\psi = \prod_{j=1}^{s} \psi_j^{s_j}$) be the decomposition of φ (resp. of ψ) in irreducible factors in $\mathbb{C}\{x, y\}$. Then $(\Phi, \Psi)_0$

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stands for the intersection multiplicity between Φ and Ψ at the origin. We also use $(\varphi, \psi)_0$ to denote the contact order between the functions φ and ψ at the origin; i.e. $(\varphi, \psi)_0 = \sum_{i,j} r_i s_j(\{\varphi_i = 0\}, \{\psi_j = 0\})_0$. Note that $(\Phi, \Psi)_0 = (\varphi, \psi)_0$ if and only if both φ and ψ are reduced equations for Φ and Ψ. In the same way we define $(\Phi, \psi)_0$ to be $\sum_j s_j(\Phi, \{\psi_j = 0\})_0$.

The *critical locus* (or the *Jacobian curve*) J of the pencil Λ is the reduced plane curve defined by the vanishing of the Jacobian determinant $j(f, g) = f_x g_y - f_y g_x$. If $\phi, \phi' \in \Lambda$, $\phi \neq \phi'$, then we have $j(f, g) = cj(\phi, \phi')$ for some $c \in \mathbb{C}^*$. It means that J depends only on Λ and not on the basis, $\{\phi, \phi' : \phi \neq \phi'\}$, fixed for Λ. Notice that J is the critical locus of the map germ defined by f and g: $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ (in fact for the map germ defined by any pair $\phi, \phi' \in \Lambda$).

Among the relationships between J (or more precisely the branches of J) and the special fibres of Λ, Sp(Λ), we are going to mention two of them. The first one is a direct relationship due to Casas★ (see [Ca], 7.4):

THEOREM A. *A curve Φ_w of Λ is special if and only if*

$$(\phi_w, j(f, g))_0 > \min\{(\phi_{w'}, j(f, g))_0 : w' \in \mathbb{CP}^1\}.$$

Moreover, let γ be a branch of J. Then there exists exactly one $w(\gamma) \in \mathbb{CP}^1$ such that $(\gamma, \phi_{w(\gamma)})_0 > (\gamma, \phi_{w'})_0$ for all $w' \neq w(\gamma)$.

As a consequence $\gamma \mapsto \Phi_{w(\gamma)} = \{\phi_{w(\gamma)} = 0\}$ defines a map from the set of branches of J, $\mathcal{B}(J)$, onto the set of special fibres of Λ:

$$\begin{array}{rcl} \rho \colon \mathcal{B}(J) & \to & Sp(\Lambda) \\ \gamma & \mapsto & \Phi_{w(\gamma)} \end{array}$$

There is another relationship between $\mathcal{B}(J)$ and $Sp(\Lambda)$ which could be deduced from the results of Maugendre in [Ma1] (see also [Ab]). Let $\pi \colon (X, E) \to (\mathbb{C}^2, 0)$ be a modification (i.e. the composition of a finite number of point blowings-up, each one centered at an infinitely near point to the origin 0). For each irreducible component E_α of the exceptional divisor E of π, let $m_f(E_\alpha)$ (resp. $m_g(E_\alpha)$) be the multiplicity of the function $\tilde{f} = f \circ \pi \colon X \to \mathbb{C}$ along E_α (resp. $\tilde{g} = g \circ \pi$). We will denote $q(E_\alpha) = m_f(E_\alpha)/m_g(E_\alpha)$.

Let $\pi' \colon (X', E') \to (\mathbb{C}^2, 0)$ be an embedded resolution of $\{fg = 0\}$; that is π' is a modification such that the support of the divisor of the function $\widetilde{fg} = fg \circ \pi' \colon X \to \mathbb{C}$ is a normal crossing divisor. We will say that an irreducible component, E_α, of E' is a *rupture divisor* if it intersects at least three different components of the total transform of $\{fg = 0\}$; i.e. if $\overline{\pi'^{-1}(\{fg = 0\}) - E_\alpha}$ has at least three connected components.

THEOREM B. *Let $\pi' \colon (X', E') \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\{fg = 0\}$. Then*

★The authors wish to acknowledge E. Casas for a very early version of this result.

$$\left\{\frac{(\gamma, f)_0}{(\gamma, g)_0} : \gamma \in \mathcal{B}(\mathcal{J})\right\} = \{q(E_\alpha) : E_\alpha \text{ a rupture divisor}\}.$$

The above result could be used to separate the branches of J in packages, each one constituted by branches which have same 'Jacobian quotient' $(\gamma, f)_0/(\gamma, g)_0$. As a consequence it permits us, roughly speaking, to obtain some topological information about the branches of J from the topological type of f and g (see [Ma1] and the results at the end of this introduction and Subsection 2.3 for a more precise version of this assertion). Thus, this result can be seen as a generalization of the well-known factorization proved by Merle ([Me]) for the polar curves (i.e. when g is a general linear form).

However, an interesting fact is that in the case where f and g are generic elements of Λ (in this case $\pi'\colon (X', E') \to (\mathbb{C}^2, 0)$ just coincides with the minimal resolution of the pencil Λ) then $q(E_\alpha) = 1$ for every irreducible component E_α of E' and as consequence the above theorem does not give any information. Thus, in order to obtain nontrivial information about $\mathcal{B}(J)$ one needs to use the minimal embedded resolution (and so the topological type) of a pair of special fibres of Λ.

One of the main goals of this paper is to understand the above two relationships and the connection between them. As the results allows a very natural interpretation in terms of exceptional divisors (and its dual graph) we refer generically to [LW] for notations and results about the geometry of pencils. Readers who prefer different languages (such as infinitely near points and Enriques diagrams, for example) can find in [Ca] the main ingredients to establish the adequate translation.

Let $\pi\colon (X, E) \to (\mathbb{C}^2, 0)$ be a modification and let $h = f/g$ be the meromorphic function defined by f and g in a punctured neighbourhood U of the origin of $\mathbb{C}^2$. One can see h as a map $h\colon U \to \mathbb{CP}^1$ defined by $h(z) := (f(z) : g(z))$ and so the fibre Φ_w is the closure of $h^{-1}(w)$ for $w \in \mathbb{CP}^1$. Let $\tilde{h} := h \circ \pi$ be the lifting of h to X. $\tilde{h}$ is a meromorphic function defined in a suitable neighbourhood of E in X but in a finite set of points of the exceptional divisor E. We will say that π is a resolution of Λ if $\tilde{h} : X \to \mathbb{CP}^1$ is a morphism. It is well known that Λ admits a minimal resolution and also that this minimal resolution coincides with the minimal embedded resolution of any pair of generic fibres of the pencil.

We will say that an irreducible component E_α of E is *dicritical* if $\tilde{h}|_{E_\alpha}\colon E_\alpha \to \mathbb{CP}^1$ is defined everywhere and not constant. Note that if E_α is dicritical then $q(E_\alpha) = 1$. However, the condition $q(E_\alpha) = 1$ does not imply that E_α is dicritical. Moreover, the condition $q(E_\alpha) > 1$ (resp. $q(E_\alpha) < 1$) is equivalent to saying that E_α is in the zero divisor (resp. in the pole divisor) of $\tilde{h}$.

Let $\pi\colon (X, E) \longrightarrow (\mathbb{C}^2, 0)$ be the minimal resolution of the pencil Λ and denote $\mathcal{D}$ the dicritical locus; i.e. the union of the dicritical components of E. In this situation we have the following results:

THEOREM 1. *Let Δ be a connected component of $\overline{(E - \mathcal{D})}$. Then $\tilde{h}$ is constant along Δ and if $\tilde{h}|_\Delta \equiv w \in \mathbb{CP}^1$ we have*

(1) *Φ_w is special.*
(2) *There exists a branch φ of Φ_w such that the strict transform of φ intersects Δ.*
(3) *There exists a branch γ of J such that the strict transform of γ intersects Δ.*
(4) $\rho(\gamma) = \Phi_w$.

THEOREM 2. *Let P be a singular point of $\mathcal{D}$ (i.e. the intersection point of two dicritical divisors) and $\tilde{h}(P) = w \in \mathbb{CP}^1$. Then the statements (1) to (4) of the above theorem are also true for w and $\Delta = \{P\}$.*

THEOREM 3. *Let D be a dicritical divisor and let P be a critical point of $\tilde{h}|_D\colon D \to \mathbb{CP}^1$, $w = \tilde{h}(P)$ and we assume that P is a smooth point of E (that means that P is not the intersection point with another divisor). Then we have*

(1) *Φ_w is special.*
(2) *The strict transform of Φ_w is singular or tangent to D at P.*
(3) *There exists a branch γ of J such that the strict transform of γ intersects D at P.*
(4) $\rho(\gamma) = \Phi_w$.

Moreover, if there exists a branch γ of J such that the strict transform of γ intersects D at P, then P is a critical point for $\tilde{h}$.

All the special values $w \in \mathbb{CP}^1$ (i.e. w is such that the corresponding fibre Φ_w is special) are among the ones stated in Theorems 1, 2 and 3 above (see [LW]). On the other hand, according to the last statement in Theorem 3, all the branches of J are in one of the situations described by statement (3) in Theorems 1, 2 and 3. As a consequence J can be decomposed as $J = \cup J_i$ in such a way that J_i consists of the branches which intersect the same connected component Δ of $\overline{(E - \mathcal{D})}$ or the same singular point of $\mathcal{D}$ or the same critical point P in some dicritical divisor D. Moreover, it is clear that the map ρ is constant on the set of branches $\mathcal{B}(J_i)$ (for each J_i).

Let $f = \prod_{i=1}^{p} f_i^{p_i}$ and $g = \prod_{j=1}^{t} g_j^{t_j}$; $f_i \neq g_j$ for any i, j. Let $\pi'\colon (X', E') \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\{fg = 0\}$. We denote by $\mathcal{D}'$ the dicritical locus of E' and by $\widetilde{C(f)}_i$ (resp. $\widetilde{C(g)}_j$) the strict transform of $\{f_i = 0\}$ (resp. $\{g_j = 0\}$) by π'. Let us consider $\Delta^f = \{E_\alpha : q(E_\alpha) > 1\} \cup \{\widetilde{C(f)}_i : p_i > 1\}$ (respectively, $\Delta^g = \{E_\alpha : q(E_\alpha) < 1\} \cup \{\widetilde{C(g)}_i : t_j > 1\}$). Let $\Delta^f = \bigcup_{i=1}^{r_f} \Delta_i^f$ (respectively, $\Delta^g = \bigcup_{i=1}^{r_g} \Delta_i^g$) be the decomposition in connected components of Δ^f (respectively, of Δ^g). In the same way consider $(E')^1 = E' - (\Delta^f \cup \Delta^g \cup \mathcal{D}')$ and let $(E')^1 = \bigcup_{i=1}^{s} \Delta_i$ be the decomposition in connected components of $(E')^1$. It is straightforward to show that if E_α is a divisor not contained in $(E')^1$ and $E_\alpha \cap (E')^1 \neq \emptyset$, then E_α is a dicritical divisor.

Moreover let $P_1, \ldots, P_r$ be the points of $\mathcal{D}'$ which are either singular points of $\mathcal{D}'$ or critical points of $\tilde{h}$. Let us set:

$$CR(\pi') = \{\Delta_1, \ldots, \Delta_3, \Delta_1^f, \ldots, \Delta_{r_f}^f, \Delta_1^g, \ldots, \Delta_{r_g}^g, P_1, \ldots, P_t\}.$$

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The following theorem gives similar information as Theorems 1, 2 and 3 in terms of the minimal embedded resolution of $\{fg = 0\}$. Moreover, we shall use it in order to detail decompositions for the set of branches of J.

THEOREM 4. *Let $\pi'\colon (X', E') \to (\mathbb{C}^2, 0)$ be the minimal resolution of $\{fg = 0\}$ and* $\mathrm{CR}(\pi')$ *defined as above. Then*

(1) *For each $B \in \mathrm{CR}(\pi')$ there exists a unique fibre $\Phi_{w(B)} \in \Lambda$ such that its strict transform by π' intersects B. Moreover, the correspondence $B \mapsto \Phi_{w(B)}$ defines a surjective map v from $\mathrm{CR}(\pi')$ to $Sp(\Lambda)$.*
(2) *For each $B \in \mathrm{CR}(\pi')$ there exists a branch γ of J such that its strict transform by π' intersects B. Moreover, for such a branch γ one has $\rho(\gamma) = \Phi_{w(B)}$.*

As a consequence, the correspondence $\gamma \mapsto u(\gamma)$ which associate with each branch γ of J the (unique) element $u(\gamma)$ of $CR(\pi')$ such that its strict transform intersects $u(\gamma)$, defines a surjective map u from the set of branches $\mathcal{B}(J)$ of J to the set $CR(\pi')$. It is clear that one has $\rho = v \circ u\colon \mathcal{B}(J) \xrightarrow{u} CR(\pi') \xrightarrow{v} Sp(\Lambda)$. Notice that, in general (see the Examples in 2.9), neither u nor v are injective.

The map u gives the following decomposition of the set of branches of the Jacobian curve $\mathcal{B}(J) = \bigcup_{B \in CR(\pi')} u^{-1}(B)$.

In fact, we can give a more precise decomposition of $\mathcal{B}(J)$ using Theorem B above. For each Δ_i^f we consider the subset $\{R_{i,1}^f, \ldots, R_{i,n(f,i)}^f\}$ of rupture divisors of E' belonging to Δ_i^f. For $R_{i,j}^f$ let $RZ_{i,j}^f$ be the maximal connected subset of Δ_i^f so that $R_{i,j}^f \in RZ_{i,j}^f$ and q is constant along $RZ_{i,j}^f$. One knows ([Ma1]) that there exists a branch of J whose strict transform intersects $RZ_{i,j}^f$ for any i, j. One can do the same kind of decomposition for each connected component Δ_j^g of Δ^g. Notice that as q is constant along Δ_i $(1 \leqslant i \leqslant s)$, even if there exists some rupture divisor in it, the above decomposition does not give more information about J. Let us denote $RZ = \{\Delta_1, \ldots, \Delta_s, RZ_{1,1}^f, \ldots, RZ_{r_f,n(f,r_f)}^f RZ_{1,1}^g, \ldots, RZ_{r_g,n(g,r_g)}^g, P_1, \ldots, P_t\}$.

Thus, one can define two surjective maps $u_1 : \mathcal{B}(J) \to RZ$ and $u_2 : RZ \to CR(\pi')$ such that $u = u_1 \circ u_2$. In particular $\mathcal{B}(J)$ can be decomposed as $\mathcal{B}(J) = \bigcup_{Z \in RZ} u_1^{-1}(Z)$. This decomposition, in general, improves the ones in [Ma1] and [Ab]. Notice that the above decomposition could be refined once again by taking similar zones for the minimal embedded resolution of all the special fibres (this gives us the possibility of breaking $u_1^{-1}(Z)$ by means of new rupture zones for the elements $Z \in \{\Delta_1, \ldots, \Delta_s, P_1, \ldots, P_t\}$).

2. Proofs

2.1. THE DUAL GRAPH

Let $f, g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two germs of analytic functions and let $f = \prod_{i=1}^{p} f_i^{p_i}$, $g = \prod_{j=1}^{t} g_j^{t_j}$ be their corresponding factorization in irreducible elements in the ring

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$\mathbb{C}\{x, y\}$. We assume, as at the beginning, that the plane curves $\{f = 0\}$ and $\{g = 0\}$ do not share any branch.

Let $\pi: (X, E) \to (\mathbb{C}^2, 0)$ be an embedded resolution of the reduced curve defined by $\{fg = 0\}$. We can construct the *dual graph*, $G(\pi, fg)$, of π as follows: For each irreducible component of E we put a vertex and two of them are connected by an edge if and only if their associated divisors intersect. Each irreducible component $\tilde{L}$ of the strict transform of $\{fg = 0\}$ is represented by an arrow connected by an edge with the (only) vertex whose corresponding divisor intersects $\tilde{L}$. If $\tilde{L}$ is the strict transform of $\{f_i = 0\}$ (resp. $\{g_j = 0\}$), then we add p_i (resp. t_j) as a weight to the corresponding arrow. For a vertex α we shall denote its corresponding divisor by E_α. Properties defined for divisors (as dicritical, rupture, etc.) will be used as well for their corresponding vertices. For a plane curve germ C and a set of vertices A of $G(\pi, fg)$, the sentence 'C (or $\tilde{C}$) meets A' means that $\tilde{C}$ intersects $\cup_{\alpha \in A} E_\alpha$, being $\tilde{C}$ the strict transform of C by π.

In fact, the dual graph is a tree. It is oriented from the vertex representing the divisor obtained by the blowing-up of 0 in $\mathbb{C}^2$ (we will call '**1**' this vertex) to the arrows. There is a natural partial order on the set of vertices of the dual graph: $\sigma' < \sigma$ if, and only if, in $G(\pi, fg)$, the geodesic from the vertex **1** to the vertex σ passes through the vertex σ'.

Let α be a vertex of $G(\pi, fg)$. We denote $st(\alpha)$ the set constituted by α and the vertices or arrows connected with α by one edge. Note that $\#\mathrm{st}(\alpha) > 3$ if and only if α is a rupture vertex. Vertices with $\#\mathrm{st}(\alpha) = 2$ are called *ends*.

An *arc* in $G(\pi, fg)$ is a completely ordered connected subtree A in such a way that, if $\{\alpha_1 < \cdots < \alpha_r\}$ is the set of vertices of A (we will write $A = \{\alpha_1 < \cdots < \alpha_r\}$ in order to simplify notations) then $\#\mathrm{st}(\alpha_i) = 3$ for $i = 2, \ldots, r-1$.

Remark. The dual graph, with a suitable weight for each vertex (for example one can take the number of blowing-ups done to produce the corresponding divisor or, alternatively, the self-intersection of the divisor), determines the topology of the singularity associated to fg. If, moreover, we distinguish the branches of $\{f = 0\}$ from those of $\{g = 0\}$ (for example by using different colors or different symbols for the corresponding arrows), it determines the topology of the pair (f, g) ([Ma2]).

Remark. If $\pi: (X, E) \to (\mathbb{C}^2, 0)$ is a modification then, in the same way, we can define the dual graph of π, $G(\pi)$, without any reference to strict transforms of curves; i.e. without any arrow. In the same way we use freely for $G(\pi)$ the definitions and notations given in principle for $G(\pi, fg)$.

Let $\pi: (X, E) \to (\mathbb{C}^2, 0)$ be a modification and let $f, g: (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two analytic functions as at the beginning. We define the map $q^f_g: G(\pi) \to \mathbb{Q}$ by $q^f_g(\alpha) = m_f(E_\alpha)/m_g(E_\alpha)$. Notice that, with the notations of the Introduction, $q^f_g(\alpha) = q(E_\alpha)$, but in the sequel we need to use functions 'q^-_-' for different pairs of functions; this justifies the new notations.

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Let α be a vertex of $G(\pi)$ and E_α its corresponding divisor. Let T_α be an irreducible germ of a nonsingular curve which is transversal to the component E_α of E at a nonsingular point P of the total transform $\pi^{-1}(\{fg=0\})$. This means that the point P is not an intersection point of the component E_α with another component of the exceptional divisor or with the strict transform of the curve $\{fg=0\}$. Usually, one says that T_α is a *curvetta* of E_α. Let $\pi(T_\alpha) \subset (\mathbb{C}^2, 0)$ be the projection of the curve germ T_α. Then one has $m_f(E_\alpha) = (\pi(T_\alpha), f)_0$ and $m_g(E_\alpha) = (\pi(T_\alpha), g)_0$. Thus

$$q_g^f(\alpha) = \frac{(\pi(T_\alpha), f)_0}{(\pi(T_\alpha), g)_0}.$$

Remark. Let $\pi\colon (X, E) \to (\mathbb{C}^2, 0)$ be a modification and α a vertex of $G(\pi)$. Then the corresponding divisor E_α is a dicritical one if and only if $q_{\phi'}^{\phi}(\alpha) = 1$ for any pair of elements ϕ, ϕ' of Λ. Note also that the dicritical components are essentially the same for any resolution of Λ, i.e. for any modification π as above, with the condition that $\widetilde{\phi/\phi'}$ is a meromorphic function defined everywhere (this condition is the equivalent of saying that π is a resolution of any pair of generic functions of Λ).

PROPOSITION 1. *Let $\pi\colon (X, E) \longrightarrow (\mathbb{C}^2, 0)$ be a modification, α a vertex of $G(\pi)$ and E_α its corresponding divisor. Assume that the strict transform of $\{fg=0\}$ does not intersect E_α. Then, there exists β in $st(\alpha)$ such that $q_g^f(\beta) > q_g^f(\alpha)$ if and only if there exists β' in $st(\alpha)$ such that $q_g^f(\beta') < q_g^f(\alpha)$.*

Proof. Let $\tilde{C}$ be the strict transform of the curve $C = \{f=0\}$ by π (counted with the corresponding multiplicities). Let $(\tilde{f}) = \tilde{C} + \sum_{a\in G(\pi)} m_f(E_a)E_a$ be the divisor of the lifting $\tilde{f} = f \circ \pi$ of f to the space X of the modification. From the fact $(\tilde{f}) \bullet E_\alpha = 0$ ($X \bullet Y$ stands for the intersection number) and because $\tilde{C} \bullet E_\alpha = 0$ one gets (see [SZ], Proposition 3) $0 = \sum_{\beta\in st(\alpha)} m_f(E_\beta)(E_\beta \bullet E_\alpha)$. Thus, if $N = -(E_\alpha \bullet E_\alpha) > 0$ one has

$$\sum_{\beta\in st(\alpha)-\{\alpha\}} m_f(E_\beta) = N \cdot m_f(E_\alpha). \qquad (*)$$

It is clear that the same equality is true for g instead of f.

Now, let us suppose that $q_g^f(\beta') \geqslant q_g^f(\alpha)$ for each β' in $st(\alpha)$. This is equivalent to

$$m_f(E_{\beta'})m_g(E_\alpha) \geqslant m_f(E_\alpha)m_g(E_{\beta'}).$$

Then we obtain

$$\sum_{\substack{\beta'\in st(\alpha)\\ \beta'\neq\alpha}} m_f(E_{\beta'})m_g(E_\alpha) > \sum_{\substack{\beta'\in st(\alpha)\\ \beta'\neq\alpha}} m_f(E_\alpha)m_g(E_{\beta'}),$$

and using the above formula $(*)$, we reach a contradiction. □

A direct consequence of Proposition 1 is the following:

COROLLARY 1. *Let $A = \{\alpha_1 < \cdots < \alpha_r\}$ be an arc of $G(\pi)$ and assume that the strict transform of $\{fg=0\}$ does not intersect $\bigcup_2^{r-1} E_{\alpha_i}$. Then the map $q_g^f\colon G(\pi) \to \mathbb{Q}$ is*

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strictly monotonous or constant along the arc A. Moreover, if α_r *(or* α_1*) is an end then* q^f_g *is constant along A.*

2.2. PROOF OF THEOREM 1 PARTS (1) AND (2)

Let $\pi\colon (X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of the pencil Λ and let $\mathcal{D}$ be the dicritical locus. Let us denote by Δ a connected component of $(\overline{E - \mathcal{D}})$.

Let $h = f/g$ be the meromorphic function defined in a punctured neighbourhood of the origin in $\mathbb{C}^2$ by f and g and let $\tilde{h} = h \circ \pi$ be the lifting of h to X. As π is a resolution of Λ, $\tilde{h}$ is a meromorphic function defined everywhere in a suitable neighbourhood of E. Moreover, the meromorphic function $\tilde{h}$ is constant along Δ and if $\tilde{h}|_\Delta \equiv w \in \mathbb{CP}^1$, it is known that $\Phi_w = \{\phi_w = 0\}$ is a special fibre of Λ ([LW]).

We consider the functions ϕ_w, $\phi \in \Lambda$, where $\{\phi = 0\}$ is a generic fibre of the pencil and denote $h' = \phi_w/\phi$. As Δ is contained in the zero locus of $\tilde{h}' = h' \circ \pi$ one has that $q^{\phi_w}_\phi(\alpha) > 1$ for any $E_\alpha \subset \Delta$.

Let us suppose that there does not exist any irreducible component ξ of $\Phi_w = \{\phi_w = 0\}$ such that its strict transform by π intersects Δ.

Let $E_\alpha \subset \Delta$ be such that $q^{\phi_w}_\phi(\alpha) \geqslant q^{\phi_w}_\phi(\beta)$ for each $E_\beta \subset \Delta$. Assume that there exists $\alpha' \in \mathrm{st}(\alpha)$ so that $E'_\alpha \not\subset \Delta$; then E'_α must be dicritical and so $q^{\phi_w}_\phi(\alpha') = 1 < q^{\phi_w}_\phi(\alpha)$. Thus, one has a contradiction with Proposition 1. Then, again by Proposition 1, $\bigcup_{\beta \in st(\alpha)} E_\beta \subset \Delta$ and $q^{\phi_w}_\phi(\alpha) = q^{\phi_w}_\phi(\beta)$ for each $\beta \in \mathrm{st}(\alpha)$. As Δ is connected, $q^{\phi_w}_\phi$ is constant in Δ and strictly greater than 1. However, if $E_\beta \subset \Delta$ is such that at least one component β' of $st(\beta)$ satisfies that $E_{\beta'} \not\subset \Delta$ then $E_{\beta'}$ is dicritical and we reach a contradiction with Proposition 1 again.

2.3. STABILITY OF RUPTURE ZONES

Now, in order to prove part 3 of Theorem 1, we need first to prove some technical results.

DEFINITION. Let $\pi\colon (X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of $\{fg = 0\}$. A rupture zone R of $G(\pi, fg)$ (with respect to f and g) is a connected subtree of $G(\pi, fg)$ containing at least one rupture vertex, and such that the map q^f_g is constant on R.

From [Ma2] we have the following result:

THEOREM 5. *Let* $\pi\colon (X, E) \to (\mathbb{C}^2, 0)$ *be the minimal embedded resolution of* $\{fg = 0\}$. *Let* $\{R_1, \dots, R_p\}$ *be the set of rupture zones of* $G(\pi, fg)$. *Then, J can be decomposed as* $J = J_1 \cup \dots \cup J_p$ *in such a way that* $J_i \neq \emptyset$ *for* $i = 1, \dots, p$ *and* γ *is a branch of* J_i *if and only if the strict transform of* γ *by* π *meets* R_i.

In fact, in [Ma2] the above Theorem is given in terms of Waldhausen manifolds. Indeed, it is well known that there exists a bijective correspondence between the rupture vertices of $G(\pi, fg)$ and the Seifert manifolds of the minimal Waldhausen

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decomposition of the complement in S^3_ϵ (it is the sphere of radius ϵ, small enough, centered at the origin of $\mathbb{C}^2$) of the link $K_{fg} = \{fg = 0\} \cap S^3_\epsilon$. Moreover, we have from [Ma1], Theorem 1 and [Ma2], Theorem 1:

THEOREM 6. *The set* $\{q^f_g(\alpha)\}$ *where* α *is a rupture vertex of* $G(\pi, fg)$ *is equal to the set* $\{\mathcal{L}(K_f, v)/\mathcal{L}(K_g, v)\}$, *where* $\mathcal{L}(-,-)$ *denotes the linking number in* S^3_ϵ *and* v *is a leaf of a Seifert manifold of the minimal Waldhausen decomposition of the complement in* S^3_ϵ *of* K_{fg}.

The set $\{\mathcal{L}(K_f, v)/\mathcal{L}(K_g, v)\}$ is called the set of linking quotients. We recall that $v = K_\xi$, where ξ is an irreducible function germ, and that $\mathcal{L}(K_f, v) = (f, \xi)_0$.

As a consequence, a rupture zone R_i, $(1 \leqslant i \leqslant p)$, of $G(\pi, fg)$ corresponds to a finite connected union W_i, $(1 \leqslant i \leqslant p)$, of Seifert fibred manifolds (of the above minimal Waldhausen decomposition) which have the same linking quotient.

In [Ma2] (Proposition 3 and Theorem 7) we prove that for each i, $(1 \leqslant i \leqslant p)$, there exists at least one branch γ of J such that γ intersects W_i. So, in terms of rupture zones it gives Theorem 5.

Remark. Let ϕ, ϕ' be such that $\Phi = \{\phi = 0\}$ and $\Phi' = \{\phi' = 0\}$ are generic fibres of Λ, π: $(X, E) \to (\mathbb{C}^2, 0)$ the minimal resolution of $\{fg = 0\}$ and π': $(X', E') \to (\mathbb{C}^2, 0)$ the one of $\{\phi\phi' fg = 0\}$. The construction of π' from π by a sequence of blowings-up, σ: $(X', E') \to (X, E)$, is well known (see, e.g., [LW], [Ca]): at each step it suffices to blow-up an indetermination point of the lifting of (f/g) in the corresponding space.

More precisely, the lifting $\widetilde{f/g}$ of f/g to X is not defined in a point $P \in X$ if and only if $P = S_1 \cap S_2$ for irreducible components S_1 and S_2 of the total transform of $\{fg = 0\}$ by π with $q^f_g(S_1) > 1$ and $q^f_g(S_2) < 1$ (see [LW]). Note that $q^f_g(S_1) > 1$ (respectively, $q^f_g(S_2) < 1$) is equivalent to saying that $\widetilde{f/g} = 0 = (0 : 1) \in \mathbb{CP}^1$ generically along S_1 (respectively $\widetilde{f/g} = \infty = (1 : 0) \in \mathbb{CP}^1$ generically along S_2). So if one takes different elements ϕ_w, ϕ_v of Λ, the corresponding meromorphic function $\widetilde{\phi_w/\phi_v}$ differs from $\widetilde{f/g}$ up to a projective isomorphism of $\mathbb{CP}^1$ and so takes two different values of $\mathbb{CP}^1$ in S_1 and in S_2. Thus, it could not be defined at $P = S_1 \cap S_2$.

Now to produce π' we start by blowing-up a point $P = S_1 \cap S_2$ as above. In the new divisor, E_α, at most one of the points $E_\alpha \cap S_1$ or $E_\alpha \cap S_2$ is an indetermination point and E_α is a dicritical divisor if and only if the lifting of f/g to the new space is defined at both of them. We repeat the same procedure until (in the corresponding space) the lifting of f/g is a well defined meromorphic function.

For the divisor E_α one has

- α does not belong to any rupture zone with respect to f and g because the new vertices created are not rupture vertices and, as $q^f_g(E_\alpha) \neq q^f_g(S_1)$ and $q^f_g(E_\alpha) \neq q^f_g(S_2)$, α can not be incorporated to any pre-existing rupture zone.
- α does not break any rupture zone of $G(\pi, fg)$ because S_1 and S_2 can not belong to the same rupture zone.

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As a consequence, $G(\pi, fg)$ and $G(\pi', fg)$ have, roughly speaking, the same rupture vertices and the same rupture zones. In a more precise form:

PROPOSITION 2. *The rupture zones of $G(\pi', fg)$ are exactly the strict transforms of those of $G(\pi, fg)$ by $\sigma\colon (X', E') \to (X, E)$. As a consequence statements in Theorem 5 above are valid for $\pi'\colon (X', E') \to (\mathbb{C}^2, 0)$ as well.*

2.4. SEMIGROUP OF VALUES

Let C be the plane branch defined by $\{\varphi = 0\}$ for some irreducible analytic function $\varphi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. The semigroup of values, $S(C)$, of C is the subsemigroup of $\mathbb{N}$ constituted by the intersection multiplicities of C with any plane curve C' provided that C is not a branch of C'. Let $\{\bar{\beta}_0, \ldots, \bar{\beta}_s\}$ be the minimal set of generators of $S(C)$ (i.e the so-called 'maximal contact values' of C) and let $e_i = \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_i)$, $N_i = e_{i-1}/e_i$ for $i = 1, \ldots, s$. It is known that $N_i\bar{\beta}_i < \bar{\beta}_{i+1}$, $i = 0, \ldots, s-1$; $\ell\bar{\beta}_i \notin \langle\bar{\beta}_0, \ldots, \bar{\beta}_{i-1}\rangle$ if $0 < \ell < N_i$ and $N_i\bar{\beta}_i \in \langle\bar{\beta}_0, \ldots, \bar{\beta}_{i-1}\rangle$ for $i = 1, \ldots, s$. Moreover, if $m \in S(C)$ then m can be represented in a unique way as $m = k_0\bar{\beta}_0 + \sum_{j=1}^{s} k_j\bar{\beta}_j$ with $k_0 \geqslant 0$, $0 \leqslant k_j < N_j$ for $j = 1, \ldots, s$. Finally e_i divides m if and only if $m \in \langle\bar{\beta}_0, \ldots, \bar{\beta}_i\rangle = \sum_{j=0}^{i} \mathbb{N}\bar{\beta}_j$ and this condition is also equivalent to saying that, in the unique representation of m, one has $k_j = 0$ for $j = i+1, \ldots, s$.

The dual graph of C is, by definition, the dual graph $G(\pi, \varphi)$ for a minimal resolution of C. It is known that $G(\pi, \varphi)$ has $s+1$ *dead arcs* (i.e. maximal arcs with a vertex α such that $\# st(\alpha) = 2$, i.e. with an *end*) $L_0, \ldots, L_s$ and s rupture vertices $\sigma_1, \ldots, \sigma_s$. We denote $\tau_0, \ldots, \tau_s$ the ends corresponding to $L_0, \ldots, L_s$ (see Figure 1).

Let $\gamma = \{\psi = 0\}$ be a branch defined by the irreducible analytic function ψ such that the strict transform $\tilde{\gamma}$ of γ by π does not intersect the one of C. It is known ([D], [ZT], [D2]) that

(1) $\bar{\beta}_i = (C, \gamma)_0$ if and only if the strict transform $\tilde{\gamma}$ of γ is a curvetta on E_{τ_i}. In particular $\bar{\beta}_i = m_\varphi(E_{\tau_i})$.
(2) If $\tilde{\gamma}$ intersects L_s (i.e. intersects $\bigcup_{\alpha \in L_s} E_\alpha$) then $(C, \gamma)_0$ is a multiple of $\bar{\beta}_s$.

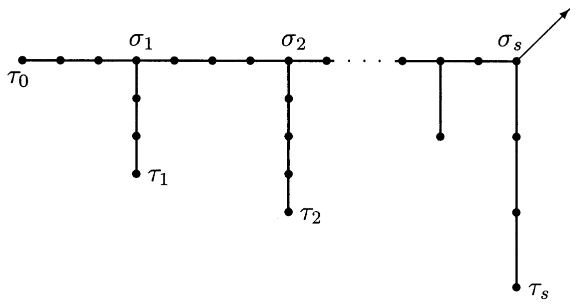

Figure 1. The dual resolution graph of the curve C.

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(3) If $\tilde{\gamma}$ does not intersect $L_s - \{\sigma_s\}$ then $(C, \gamma)_0 \in \langle \bar{\beta}_0, \ldots, \bar{\beta}_{s-1} \rangle$ (i.e. e_{s-1} divides $(C, \gamma)_0$).

A direct consequence of the above remarks is the following lemma:

LEMMA 1. *Let* $\psi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *be an analytic function and assume that* $(\varphi, \psi)_0 = m + l\bar{\beta}_s$ *with* $m \in \langle \bar{\beta}_0, \ldots, \bar{\beta}_{s-1} \rangle$ *and* $0 < l < N_s$. *Then there exists a branch* γ *of* $\{\psi = 0\}$ *such that the strict transform of* γ *by* π *intersects* $L_s - \{\sigma_s\}$.

2.5. PROOF OF THEOREM 1. PART (3) AND (4)

Let π: $(X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of Λ and let Δ be a connected component of $(\overline{E - \mathcal{D}})$. Let $\Phi_w = \{\phi_w = 0\}$ be the special fibre associated to Δ. Let π': $(X', E') \to (\mathbb{C}^2, 0)$ be the minimal resolution of $\{\phi\phi'\phi_w = 0\}$ for ϕ and ϕ' in Λ such that $\Phi = \{\phi = 0\}$ and $\Phi' = \{\phi' = 0\}$ are generic fibres of Λ and let $\mathcal{D}'$ be the dicritical locus in E'. We can factorize $\pi' = \pi \circ \sigma$ with $\sigma : (X', E') \to (X, E)$ and so $\mathcal{D}'$ just coincides with the strict transform of $\mathcal{D}$ by σ.

There exists Δ', a connected component of $\overline{E' - \mathcal{D}'}$, such that $\sigma(\Delta') = \Delta$. We will distinguish two different cases:

Case (1) There exists a rupture divisor E_α in Δ'.

For each vertex $\beta \in \Delta$ we have $q_\phi^{\phi'}(\beta) = 1$. Thus, as ϕ and ϕ' are generic, their strict transforms do not intersect Δ and as a consequence $q_\phi^{\phi'}(\beta) = 1$ for any $\beta \in \Delta'$. Then $\widetilde{f/g}$ is constant (and equal to $w \in \mathbb{CP}^1$) along the divisors corresponding to the vertices of Δ'. This implies that $q_\phi^{\phi_w}(\beta) > 1$ for any $\beta \in \Delta'$ and in particular for the rupture vertex α. The rupture zone R of $G(\pi', \phi_w\phi)$ corresponding to E_α is also contained in Δ' because $q_\phi^{\phi_w}(\beta) = 1$ for any β such that E_β is a dicritical divisor.

Now, from Theorem 5, there exists a branch γ of J such that its strict transform by π' intersects some divisor corresponding to a vertex of R and as a consequence intersects Δ'. Then it is obvious that the strict transform of γ by π intersects Δ.

Case (2) Δ' does not contain any rupture divisor.

This situation could only happen if the strict transform $\widetilde{\Phi_w}$ of Φ_w by π intersects Δ in a smooth point of a divisor E_τ with τ an end vertex of $G(\pi, \phi\phi')$ (in fact the strict transform of Φ_w must also be smooth and transversal at that point). Moreover, Δ must consist of the divisors corresponding to a dead arc L minus its rupture vertex σ and E_σ must be a dicritical divisor. As a consequence there exists ξ (resp. ξ') a branch of $\Phi = \{\phi = 0\}$ (resp. $\Phi' = \{\phi' = 0\}$) such that its strict transform by π intersects E_σ.

Assume first that $\mathbf{1} \neq \tau$. In this case the dead arc L is the last one of the dual graph corresponding to the minimal resolution of ξ. Let $S(\xi)$ be the semigroup of values of ξ and $\{\bar{\beta}_0, \ldots, \bar{\beta}_s\}$ its minimal set of generators. Now, being that L is also a dead arc in $G(\pi, \phi\phi')$, the strict transform of any branch γ, $\gamma \neq \xi$, of

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$\Phi \cup \Phi'$ does not intersect $L - \{\sigma\}$. Then $(\xi, \gamma)_0 \in \langle \bar\beta_0, \ldots, \bar\beta_{s-1} \rangle$ and, in particular, $(\xi, \xi')_0 = e_{s-1}\bar\beta_s \in \langle \bar\beta_0, \ldots, \bar\beta_{s-1} \rangle$ as e_{s-1} divides $e_{s-1}\bar\beta_s$.

Let $\mathcal{B}$ be the set of branches of $\Phi \cup \Phi'$ different from ξ. Then, following the computation of $(\xi, J)_0$ given in [D1], one reaches:

$$\begin{aligned}(\xi, J)_0 &= \sum_{i=1}^{s}(N_i - 1)\bar\beta_i - \bar\beta_0 + \sum_{\gamma\in\mathcal{B}}(\xi, \gamma)_0 \\ &= \sum_{i=1}^{s}(N_i - 1)\bar\beta_i - \bar\beta_0 + e_{s-1}\bar\beta_s + \sum_{\gamma\in\mathcal{B}-\{\xi'\}}(\xi, \gamma)_0.\end{aligned}$$

As

$$e_{s-1}\bar\beta_s \in \langle \bar\beta_0, \ldots, \bar\beta_{s-1} \rangle \quad \text{and} \quad e_{s-1}\bar\beta_s > \sum_{i=1}^{s}(N_i - 1)\bar\beta_i,$$

then the unique representation of $(\xi, \xi')_0 = e_{s-1}\bar\beta_s$ in the semigroup is $e_{s-1}\bar\beta_s = \ell_0\bar\beta_0 + \sum_1^{s-1} \ell_i\bar\beta_i$ with $\ell_i < N_i$ for $i = 1, \ldots, s-1$, and $\ell_0 > 0$. In particular, $e_{s-1}\bar\beta_s - \bar\beta_0 \in \langle \bar\beta_0, \ldots, \bar\beta_{s-1} \rangle$ and as a consequence

$$\sum_{i=1}^{s-1}(N_i - 1)\bar\beta_i - \bar\beta_0 + e_{s-1}\bar\beta_s + \sum_{\gamma\in\mathcal{B}-\{\xi'\}}(\xi, \gamma)_0 \in \langle \bar\beta_0, \ldots, \bar\beta_{s-1} \rangle.$$

Thus, one has $(\xi, J)_0 = m + (N_s - 1)\bar\beta_s$ for some $m \in \langle \bar\beta_0, \ldots, \bar\beta_{s-1} \rangle$. As ϕ, ϕ' are such that $\Phi = \{\phi = 0\}$ and $\Phi' = \{\phi' = 0\}$ are generic (hence ϕ and ϕ' are reduced), then $(\gamma, \phi)_0/(\gamma, \phi')_0 = 1$ for any branch γ of J. This implies that the strict transform of J by π does not meet either the one of Φ or the one of Φ'. So, by Lemma 1 there exists a branch γ of J whose strict transform intersects Δ.

It only remains to prove the theorem for Case 2 and when $\mathbf{1} = \tau$. In this case ξ and ξ' must be either smooth or, otherwise, have only one characteristic pair.

Assume that ξ is smooth. Then one has $(\xi, \xi') = k > 1$ and for any branch γ of $\Phi \cup \Phi'$, $\gamma \neq \xi$, $(\gamma, \xi)_0$ is a multiple of k. The computation of $(\xi, J)_0$ in this case gives:

$$(\xi, J)_0 = -1 + k + \sum_{\gamma\in\mathcal{B}-\{\xi'\}}(\xi, \gamma)_0 = (k - 1) + mk.$$

In particular, $(\xi, J)_0$ is not a multiple of k. Let γ be a branch of J and $\tilde\gamma$ its strict transform by π. If $\tilde\gamma \cap \Delta = \emptyset$, then $(\gamma, \xi)_0$ is a multiple of k. So there exists a branch γ of J such that $\tilde\gamma \cap \Delta \neq \emptyset$.

Assume that ξ (and ξ') has exactly one characteristic pair and let $S(\xi) = \langle \bar\beta_0, \beta_1 \rangle$ be its semigroup of values. Now we have $(\xi, \xi')_0 = \bar\beta_0\bar\beta_1$ and for any branch γ of $\Phi \cup \Phi'$, $\gamma \neq \xi$, $(\xi, \gamma)_0$ is a multiple of $\bar\beta_1$. As a consequence

$$(\xi, J)_0 = (\bar\beta_0 - 1)\bar\beta_1 - \bar\beta_0 + \bar\beta_0\bar\beta_1 + n\bar\beta_1 = m\bar\beta_1 + (\bar\beta_1 - 1)\bar\beta_0\,,$$

and, in particular, $(\xi, J)_0$ is not a multiple of $\bar\beta_1$. As above, if γ is a branch of J and $\tilde\gamma \cap \Delta = \emptyset$ then $(\xi, \gamma)_0$ is multiple of $\bar\beta_1$ and so there exists a branch γ of J whose strict transform meets Δ.

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Now let us show part (4): if γ is a branch of J whose strict transform intersects Δ then $\rho(\gamma) = \Phi_w$; that is $(\gamma, \phi_w)_0 > (\gamma, \phi)_0$ for any $\phi \neq \phi_w$.

Let π': $(X', E') \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\{\phi\phi' = 0\} \cup \gamma$ with ϕ, ϕ' generic elements of Λ. Then Δ is contained in a connected component Δ' of $\overline{E' - \mathcal{D}'}$ and the strict transform $\tilde{\gamma}$ of γ by π' is a curvetta in some divisor α of Δ'. Because the strict transform of $\Phi \cup \Phi'$ by π does not meet Δ then, as in the proof of Theorem 1 (3), Case 1, we have $q_{\phi}^{\phi_w}(\beta) > 1$ for any $\beta \in \Delta'$. In particular, for α we find

$$q_{\phi}^{\phi_w}(\alpha) = \frac{m_{\phi_w}(E_\alpha)}{m_\phi(E_\alpha)} = \frac{(\gamma, \phi_w)_0}{(\gamma, \phi)_0} > 1. \qquad \square$$

2.6. PROOF OF THEOREM 2

Let π: $(X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of Λ and let E_β and $E_{\beta'}$ be two dicritical divisors such that $\{P\} = E_\beta \cap E_{\beta'}$. If $\tilde{h}(P) = w \in \mathbb{CP}^1$ then it is known that $\Phi_w = \{\phi_w = 0\}$ is a special fibre of Λ ([LW]).

To prove (2), we make an additional blowing-up at P, and so we create a new exceptional divisor E_α. For this new resolution, π_1, we have $st(\alpha) = \{\beta < \alpha < \beta'\}$ with $q_{\phi}^{\phi_w}(\beta) = q_{\phi}^{\phi_w}(\beta') = 1$ and $q_{\phi}^{\phi_w}(\alpha) > 1$. From Proposition 1 there exists an irreducible component ξ of Φ_w whose strict transform by π_1 intersects E_α and so its strict transform by π intersects the exceptional divisor E at the point P.

Let π': $(X', E') \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\{\phi\phi'\phi_w = 0\}$ for ϕ and ϕ' generic elements of Λ. We can factorize $\pi' = \pi \circ \sigma$ with σ: $(X', E') \to (X, E)$. The dicritical locus $\mathcal{D}'$ in E' is nothing but the strict transform of $\mathcal{D}$ by σ. Thus, there exists a connected component Δ' of $\overline{E' - \mathcal{D}'}$ such that $\sigma(\Delta') = \{P\}$ and Δ' must have a rupture divisor with respect to ϕ_w, ϕ (note that Δ' contains at least the rupture divisor E_α produced by the blowing-up at P). So, we can conclude the proof of statement (3) as in Case (1) of 2.5.

Finally, the proof of part (4) is the same as the one for Theorem 1. $\square$

2.7. PROOF OF THEOREM 3

Let π: $(X, E) \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of Λ, D a dicritical divisor, $P \in D$ a smooth point of E and $\tilde{h}(P) = w$. It is known ([LW], [Ca]) that P is a critical point of $\tilde{h}|_D$: $D \to \mathbb{CP}^1$ if and only if the strict transform of Φ_w by π is singular or tangent to D at the point P. Then part (1) and (2) are clear.Notice that if ϕ_w is not reduced and is divisible by φ^r with $r > 1$, in such a way that the strict transform of $\{\varphi = 0\}$ intersects D in the point P, then $\{\varphi = 0\}$ is a branch of J and so statement 3 is evident. As a consequence we can assume that if φ divides ϕ_w and the strict transform of $\{\varphi = 0\}$ intersects D in the point P then φ^2 does not divide ϕ_w.

Let us assume that P is a critical point and let π': $(X', E') \to (\mathbb{C}^2, 0)$ be the minimal resolution of $\{\phi\phi'\phi_w = 0\}$ for some generic elements ϕ, ϕ'. As π is the minimal resolution of $\{\phi\phi' = 0\}$ then π' is obtained from π by a finite number of blowing-ups

$\sigma: (X', E') \to (X, E)$ and, being P a singular point of the total transform of $\{\phi_w = 0\}$ by π, some blowing-ups are centered in P. In particular, the dual graph $G(\pi')$ is obtained from $G(\pi)$ by adding (among other things) a connected tree $\mathcal{T}$ (the dual picture of $\sigma^{-1}(P)$) to the vertex corresponding to D. The tree $\mathcal{T}$ contains, at least, one rupture vertex in $G(\pi', \phi\phi_w)$, and so at least one rupture zone. Then, from Proposition 2, there exists a branch γ of J whose strict transform by π' intersects $\mathcal{T}$. As a consequence the strict transform of γ by π intersects D at P.

The fourth part of the Theorem is clear, because $(\gamma, \phi_w)_0 > (\gamma, \phi)_0$.

Finally, if P is not a critical point then the strict transform of $\{\phi_w = 0\}$ at P is smooth and transversal to D at P. But if there exists a branch γ of J whose strict transform goes by P then $\{\phi_w = 0\}$ and J are not separated in the minimal resolution of $\{\phi\phi_w = 0\}$ and this gives a contradiction. □

2.8. PROOF OF THEOREM 4

Let $\pi: (X, E) \to (\mathbb{C}^2, 0)$ (respectively $\pi': (X', E') \to (\mathbb{C}^2, 0)$) be the minimal resolution of Λ (respectively of $\{fg = 0\}$). Let $\pi'': (X'', E'') \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\{\phi\phi' fg = 0\}$. Then one has a commutative diagram:

$$\begin{array}{ccc} (X'', E'') & \xrightarrow{\sigma} & (X, E) \\ \sigma' \downarrow & & \downarrow \pi \\ (X', E') & \xrightarrow{\pi'} & (\mathbb{C}^2, 0) \end{array}$$

with $\pi'' = \pi' \circ \sigma' = \pi \circ \sigma$. Let $CR(\pi')$ be the set defined in the introduction for π'. Let us consider $CR(\pi)$ (resp. $CR(\pi'')$) the set consisting of the connected components of $\overline{E - \mathcal{D}}$ (resp. of $\overline{E'' - \mathcal{D}''}$), the intersection points of two dicritical divisors of $\mathcal{D}$ in X (resp. of $\mathcal{D}''$ in X'') and the critical points of $(f/g) \circ \pi$ (resp. $f/g \circ \pi''$) in the smooth part of $\mathcal{D}$ (resp. $\mathcal{D}''$).

As σ does not produce new dicritical divisors, the dicritical locus $\mathcal{D}''$ in E'' is the strict transform of $\mathcal{D}$ by σ. As a consequence for each $A \in CR(\pi)$ there exists a unique $A'' \in CR(\pi'')$ such that the strict transform of A by σ is contained in A'' (we say simply $A \subset A''$). This condition is equivalent to saying that $\sigma(A'') = A$. Thus, the correspondence $A \mapsto A''$, with $\sigma(A'') = A$, defines a bijective map from $CR(\pi)$ to $CR(\pi'')$. For an irreducible germ of curve γ in $\mathbb{C}^2$ it is clear that its strict transform by π meets A if and only if the corresponding one meets A''. On the other hand the (constant) value of $(f/g) \circ \pi''$ along A'' is equal to the one of $(f/g) \circ \pi$ along A. So, as a consequence one can translate Theorems 1, 2 and 3 to π'', X'', E''.

Now we shall establish a bijective map from $CR(\pi')$ and $CR(\pi'')$ exactly in the same way. More precisely, for $A' \in CR(\pi')$ we will show that there exists a unique $A'' \in CR(\pi'')$ such that $A' \subset A''$ (or, equivalently, $\sigma'(A'') = A'$). Notice that in that case we can translate Theorems 1, 2 and 3 from π'', X'', E'' to π', X', E' and the Theorem follows. Note that σ' produces new dicritical divisors, so, essentially, one needs to prove that a new dicritical divisor D does not break a connected component of $CR(\pi')$ and moreover, that $(f/g) \circ \pi''|_D$ does not have critical points in the smooth part.

The construction of π'' from π' by composite blowing-ups was described in the Remark before Proposition 2. Using the notations as in the above-mentioned Remark, it is clear that S_1 and S_2 are not contained in the same connected component of $CR(\pi')$. Thus, the dicritical divisor D such that $\sigma'(D) = P$ does not break either the connected component containing S_1 or the one of S_2. Note that a new dicritical divisor D breaks the connected component A if and only if $\sigma'(D) \in A - \overline{(E' - A)}$.

Thus, in order to finish the proof it suffices to prove the following proposition:

PROPOSITION 3 *Let $D \subset E''$ be a dicritical divisor that does not appear in π' and $\tilde{h} = (f/g) \circ \pi''$. Then, either $\tilde{h}|_D\colon D \longrightarrow \mathbb{CP}^1$ is an isomorphism or it has exactly two critical points that correspond to the critical values* 0 *and* ∞.

Proof. As the component D does not appear in π', it is obtained by blowing-up the intersection between two irreducible components, E_α and E_β, of the total transform of $\{fg = 0\}$ in the corresponding space and with $q_g^f(E_\alpha) > 1$ and $q_g^f(E_\beta) < 1$. We have

$$m_f(D) = m_f(E_\alpha) + m_f(E_\beta) \quad \text{and} \quad m_g(D) = m_g(E_\alpha) + m_g(E_\beta).$$

(Note that, being that π' is an embedded resolution of $\{fg = 0\}$, if E_α and E_β are irreducible components of the exceptional divisor then $E_\alpha \cap E_\beta$ is not an infinitely near point neither for f nor for g.)

The degree of $\tilde{h}|_D$ is equal to $n = m_f(E_\alpha) - m_g(E_\alpha) = m_g(E_\beta) - m_f(E_\beta)$ because $P_1 = E_\alpha \cap D$ (respectively, $P_2 = E_\beta \cap D$) is the unique zero (respectively pole) of $\tilde{h}|_D$. If $n = 1$ then $\tilde{h}|_D$ is an isomorphism.

If $n > 1$, by the Riemann–Hurwitz formula, we have

$$2(n-1) = \sum_{P \in D} (e_P - 1) = \sum_{P \notin \{P_1, P_2\}} (e_P - 1) + (n-1) + (n-1),$$

where e_P is the ramification index of P.

We conclude that $\{P_1, P_2\}$ is the set of ramification points of $\tilde{h}|_D$, and 0 and ∞ its corresponding values. □

Remark. From Proposition 3, we know that, to study the special fibres of Λ associated to the critical points of the dicritical components, we have to consider only the dicritical components that appear in the minimal resolution of every $\Phi_\omega \cup \Phi_{\omega'}$.

2.9 COMMENTS AND EXAMPLES

This paragraph consists of three examples, the first one shows a way in which the above informations could be used in order to obtain some results in different context and problems. The last two examples are two concrete applications of the results which show different avatars of the statements and of the map ρ.

EXAMPLE. The results could be used to give necessary conditions on $\Lambda = \{\lambda f + \mu g\}$ in order to have J irreducible (see [Ma3]). In that case, as a consequence of the results, one must have that $\# RZ = 1$ (see the Introduction for the definition of RZ). It implies, in particular, that among the base points of Λ some are smooth for any branch of any generic fibre. More precisely, for any pair of generic fibres $\Phi = \{\phi = 0\}, \Phi' = \{\phi' = 0\}$ and for any irreducible component φ of ϕ there exists an irreducible component φ' of ϕ' such that the dual graph of the resolution of $\{\varphi\varphi' = 0\}$ has a vertex α with $\# st(\alpha) = 4$ and $st(\alpha)$ contains the two arrows corresponding to the strict transforms of $\{\varphi = 0\}$ and $\{\varphi' = 0\}$ (otherwise the corresponding dicritical divisor E_α produces at least two connected components in $\overline{E - \mathcal{D}}$). Moreover, there is only one branch of $\{\phi = 0\}$ (in fact of any generic fibre) whose strict transform meets E_α (otherwise the degree of the corresponding meromorphic function is at least 2 and, by the Riemann–Hurwitz Theorem, there must exist some critical point in E_α).

Thus, if e: $(Y, F) \to (\mathbb{C}^2, 0)$ is the minimal embedded resolution of a generic fibre $\Phi = \{\phi = 0\}$ and $P \in F$ is a point with multiplicity 1 of the strict transform $\tilde{\Phi}$ of $\Phi = \{\phi = 0\}$ (i.e. is a point which meets the strict transform) then P is a point of multiplicity 2 of the strict transform $\widetilde{\Phi \cup \Phi'}$ for any generic fibre $\Phi' \neq \Phi$. As a consequence in order to obtain the minimal resolution of Λ one needs to make at least an additional blowing-up at each point P as above.

EXAMPLE. Let Λ be the pencil generated by

$$f = (x^3 + y^2)(y^3 - x^2) = x^3y^3 - x^5 + y^5 - x^2y^2 \text{ and}$$
$$g = (x^3 + 2y^2)(y^3 - 2x^2) = x^3y^3 - 2x^5 + 2y^5 - 4x^2y^2.$$

The dual graph of the minimal embedded resolution, π: $(X, E) \to (\mathbb{C}^2, 0)$, for $\{fg = 0\}$ (which coincides with the minimal resolution of Λ, being $\{f = 0\}$ and $\{g = 0\}$ generic fibres) is shown in Figure 2. In the pictures big vertices are used to represent dicritical divisors. We have three connected components for $\overline{E - \mathcal{D}}$: $\Delta_1 = \{E_{\mathbf{1}}\}$, $\Delta_2 = \{E_2\}$, $\Delta_3 = \{E_{2'}\}$. As $\{f = 0\}$ and $\{g = 0\}$ are generic fibres then $q_g^f(-) = 1$ for any vertex, in particular no critical points exist for $(f/g) \circ \pi$ either in E_3 or in $E_{3'}$. As a consequence $CR(\pi) = \{\Delta_1, \Delta_2, \Delta_3\}$ and so J must have at least three branches and there exists at most three special fibres in Λ.

In fact the computation of J provides:

$$J = \{5(3xy + 4)xy(x + y)(x^4 - yx^3 + y^2x^2 - xy^3 + y^4) = 0\},$$

and so it consists of 7 smooth and transversal branches. The branch $\{x = 0\}$ meets 2, $\{y = 0\}$ meets 2′ and the others meet **1**. It is easy (killing monomials in the Newton diagram, for example) to find two special fibres, namely the zero locus of

$$g - 2f = -x^2y^2(2 + xy) \quad \text{and} \quad g - 4f = -3x^3y^3 + 2x^5 - 2y^5.$$

The first one has repeated factors and corresponds to two connected components: Δ_2 and Δ_3. So, $\{g - 2f = 0\}$ and $\{g - 4f = 0\}$ are the only special fibres. Moreover,

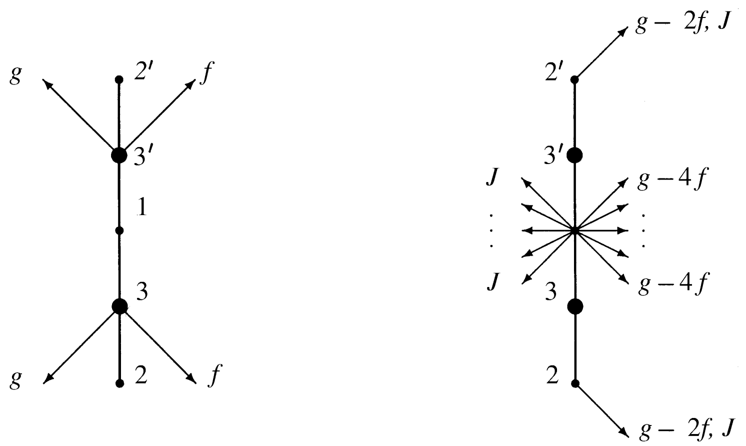

Figure 2. Minimal resolution and behaviour of critical locus and special fibres.

$\{g - 4f = 0\}$ consists of 5 smooth and transverse irreducible components which meet E_1 and so it corresponds to Δ_1 (the second picture in Figure 2 shows the behaviour of the special fibres as well as the one of the Jacobian J).

EXAMPLE. Let Λ be the pencil generated by

$$f = (x^3 + y^2)(y^3 - x^2)(x + y) \quad \text{and} \quad g = (x^3 + 2y^2)(y^3 - 2x^2)(x - y).$$

The dual graph of the minimal resolution of Λ (as in the above example f and g are generic) is represented in Figure 3. One has $\Delta_1 = E_1 \cap E_3$ and $\Delta_2 = E_1 \cap E_{3'}$ corresponding to the singular points of the dicritical locus and two connected components for $\overline{E - \mathcal{D}}$, $\Delta_3 = \{2\}$ and $\Delta_4 = \{2'\}$. As above, one can conclude that $CR(\pi) = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ and so the curve J has at least four branches and there are, at most, four special fibres.

The Jacobian determinant has the expression

$$\begin{aligned} j(f, g) = &- 14x^6y^6 + 18x^9y^2 + 39x^8y^3 - 17x^7y^4 + 17x^4y^7 - 39x^3y^8 - 18x^2y^9 - \\ &- 24x^{10} + 3x^6y^4 + 108x^5y^5 + 3x^4y^6 - 24y^{10} + 24x^8y - \\ &- 66x^7y^2 - 26x^6y^3 + 26x^3y^6 + 66x^2y^7 - 24xy^8 - 40x^4y^4. \end{aligned}$$

It consists of four branches, two of them tangent to $\{y = 0\}$ and two tangent to $\{x = 0\}$. Of the two branches tangent to $\{y = 0\}$ one is smooth (and meets the component $\{E_2\} = \Delta_3$) and the other has characteristic pair (4, 3) (it meets $\Delta_1 = E_1 \cap E_3$). A symmetric situation holds for the others.

The pencil Λ has four special fibres corresponding to the zero locus of the following functions:

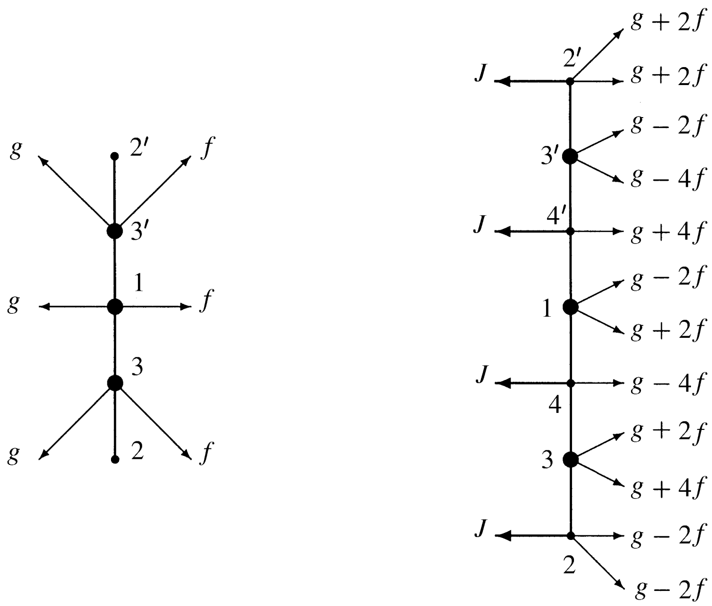

Figure 3. Minimal resolution and behaviour of critical locus and special fibres.

$$g-2f=-x^4y^3-3x^3y^4+4x^5y-4y^6-2x^3y^2+6x^2y^3\,,$$
$$g+2f=3x^4y^3+x^3y^4-4x^6+4xy^5-6x^3y^2+2x^2y^3\,,$$
$$g-4f=-3x^4y^3-5x^3y^4+2x^6+6x^5y-2xy^5-6y^6+8x^2y^3$$

and

$$g+4f=5x^4y^3+3x^3y^4-6x^6-2x^5y+6xy^5+2y^6-8x^3y^2.$$

The special fibre $\{g-2f=0\}$ consists of four branches, three of them are smooth. Namely, one has two smooth branches meeting $\Delta_3=\{E_2\}$ (one of them is $\{y=0\}$), one smooth branch transversal to $\{yx=0\}$ (and so meeting E_1) and finally a singular branch with only one Puiseux pair $(3,2)$ meeting $E_{3'}$. As a consequence, $\{g-2f=0\}$ is the special fibre corresponding to Δ_3. The behaviour of $\{g+2f=0\}$ is symmetric to the one of $\{g-2f=0\}$, it corresponds to the connected component Δ_4.

The fibre $\{g+4f=0\}$ has two branches. One of them has only one Puiseux pair $(4,3)$ tangent to $\{x=0\}$ (which meets Δ_2) and the other tangent to $\{y=0\}$ and has characteristic pair $(3,2)$ (which meets the dicritical divisor 3 as a curvetta). So, $\{g+4f=0\}$ is the fibre corresponding to Δ_2. For the fibre $\{g-4f=0\}$ it is the same situation by exchanging x and y; it corresponds to Δ_1. The right-hand part of Figure 3 shows the dual graph for the minimal resolution of the curve J and all the special fibres together.

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