

ON THE MODULI OF HYPERSURFACES IN TORIC ORBIFOLDS

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Abstract We construct and study the moduli of stable hypersurfaces in toric orbifolds. Let X be a projective toric orbifold and $\alpha \in \text{Cl}(X)$ an ample class. The moduli space is constructed as a quotient of the linear system $|\alpha|$ by $G = \text{Aut}(X)$. Since the group G is non-reductive in general, we use new techniques of non-reductive geometric invariant theory. Using the A -discriminant of Gelfand, Kapranov and Zelevinsky, we prove semistability for quasismooth hypersurfaces of toric orbifolds. Further, we prove the existence of a quasi-projective moduli space of quasismooth hypersurfaces in a weighted projective space when the weighted projective space satisfies a certain condition. We also discuss how to proceed when this condition is not satisfied. We prove that the automorphism group of a quasismooth hypersurface of weighted projective space is finite excluding some low degrees.

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1. Introduction

We study the moduli of hypersurfaces in toric orbifolds. The main tool we use to do this is geometric invariant theory (GIT), both reductive and non-reductive. We work over an algebraically closed field k of characteristic 0.

Suppose that X is a projective variety and $\alpha \in \text{Cl}(X)$ is an ample class. We are interested in the quotient stack:

$$\mathcal{M}(X, \alpha) = \left[|\alpha|/G \right],$$

which we define as the moduli stack of hypersurfaces of class α in X where $G = \text{Aut}(X, \alpha)$, the automorphisms of X fixing α . As it is, this stack has very little chance of possessing any desirable properties and, as with all moduli problems, we must define a notion of stability which will instruct us to remove certain hypersurfaces to obtain a stack with a more manageable geometry. In this paper, we apply a non-reductive GIT-stability condition to the case where X is a toric orbifold, which provides a coarse moduli space (which is a variety) for a substack and prove in a number of cases that quasismooth



hypersurfaces are semistable. Further we shall be able to prove stability when X falls into a class of certain weighted projective spaces. From this point on, when we refer to a moduli space of hypersurfaces, we mean a coarse moduli space of a substack of $\mathcal{M}(X, \alpha)$.

The moduli space of stable hypersurfaces in projective space was constructed by Mumford using reductive GIT. Consider the n -dimensional projective space \mathbb{P}^n and let $d > 2$ be an integer. The linear system $Y_d = |dH| = \mathbb{P}(k[x_0, \dots, x_n]_d)$ is a parameter space for all hypersurfaces of degree d in \mathbb{P}^n and the action of the reductive group $\text{Aut}(\mathbb{P}^n) = \text{PGL}_{n+1}$ on \mathbb{P}^n extends naturally to an action of PGL_{n+1} on Y_d . It turns out to be hard to describe the open set of stable points in Y_d , let alone the actual quotient variety. However, what Mumford does prove is that if $d > 2$ (and $d > 3$ if $n = 1$), then a smooth hypersurface is stable. Thus reductive GIT constructs a moduli space of smooth hypersurfaces.

By a toric orbifold we mean a projective toric variety with at worst orbifold singularities. One cannot mirror the construction of moduli spaces of hypersurfaces given above for toric orbifolds: the algebraic groups in question are in general non-reductive. Recent work of Kirwan, Bérczi, Doran and Hawes [4–6] develops a non-reductive GIT and allows one to construct such moduli spaces as non-reductive quotients.

There has been much work in search of GIT for non-reductive groups; see [16] for a comprehensive account of work undertaken in this area. We use the non-reductive GIT (NRGIT) developed in [4–6] where one requires that the unipotent radical of the group is graded (Definition 2.6). For these theorems to apply, we must assume that a property we call the (\mathfrak{C}) condition holds; see Definition 2.12.

In [10], Cox showed that the automorphism group of a complete simplicial toric variety can be calculated from graded automorphisms of the Cox ring. We use Cox’s construction and show that the automorphism group of a toric orbifold admits a graded unipotent radical (see Proposition 3.10 and [5, Section 4]) and thus the theory of NRGIT is applicable to the problem of moduli of hypersurfaces in toric orbifolds.

Let X be a well-formed weighted projective space where the condition (\mathfrak{C}) is satisfied for the action of $\text{Aut}(X)$. Recall that a weighted projective space is well formed if the weights satisfy a coprime property (see discussion above Lemma 3.5). Any weighted projective space is isomorphic (as a variety) to a well-formed weighted projective space. Theorem 5.18 proves that a Cartier quasismooth hypersurface in X is stable. Let \mathcal{Y}_d be the parameter space of degree d hypersurfaces. The group $\text{Aut}(X)$ acts on \mathcal{Y}_d and we denote the stable locus by \mathcal{Y}_d^s and the quasismooth locus by $\mathcal{Y}_d^{\text{QS}}$. We use NRGIT to construct a quotient space of such hypersurfaces which is a coarse moduli space. In particular, this coarse moduli space is a quasi-projective variety.

Theorem (Theorem 5.18). *Let $X = \mathbb{P}(a_0, \dots, a_n) = \text{Proj } k[x_0, \dots, x_n]$ be a well-formed weighted projective space and let $d \geq \max\{a_0, \dots, a_n\} + 2$. Suppose that the (\mathfrak{C}) condition holds for the action of $G = \text{Aut}(X)$ on $\mathcal{Y}_d = \mathbb{P}(k[x_0, \dots, x_n]_d)$. Then a quasismooth hypersurface of degree d is a stable hypersurface. In other words, there is an inclusion of open subsets:*

$$\mathcal{Y}_d^{\text{QS}} \subset \mathcal{Y}_d^s.$$

In particular, there exists a geometric quotient $\mathcal{Y}_d^{\text{QS}}/G$ and hence a coarse moduli space of quasismooth hypersurfaces of degree d in X . Moreover, the NRGIT quotient $\mathcal{Y}_d//G$ is a compactification of $\mathcal{Y}_d^{\text{QS}}/G$.

The proof of the theorem relies on a discrete-geometric version of the Hilbert–Mumford criterion for NRGIT. We also provide an explicit construction for quasismooth hypersurfaces in $X = \mathbb{P}(1, \dots, 1, r)$ which works without the heavy machinery of NRGIT.

To study the quasismooth locus in a given linear system, one uses the A -discriminant, defined and studied in [18] for general toric varieties and denoted by Δ_A . We show that $\mathcal{Y}_d^{\text{QS}} \subset (\mathcal{Y}_d)_{\Delta_A}$ and discuss the possible difference between these two sets. We prove in § 4 that the A -discriminant can be interpreted as an invariant section of an appropriate line bundle, just as for the classical discriminant.

Theorem (Corollary 4.12). *Let X be the toric variety associated to a polytope P and let A be the lattice points of P . The A -discriminant Δ_A is a semi-invariant section for the G -action on \mathcal{Y}_α and a true U -invariant, where $U \subset G$ is the unipotent radical of G .*

Restricting our attention to a weighted projective space X , we prove in § 3 that the stabiliser groups of the action of $G = \text{Aut}(X)$ on $\mathcal{Y}_d^{\text{QS}}$ is finite for $d \geq \max(a_0, \dots, a_n) + 2$. For a quasismooth hypersurface $Y \in \mathcal{Y}_d^{\text{QS}}$, denote the stabiliser group by $\text{Aut}(Y; X)$.

Theorem (Theorem 3.15). *A quasismooth hypersurface in $X = \mathbb{P}(a_0, \dots, a_n)$ of degree $d \geq \max\{a_0, \dots, a_n\} + 2$ has only finitely many automorphisms coming from the automorphisms of the ambient weighted projective space. That is, the group $\text{Aut}(Y; X)$ is finite for a quasismooth hypersurface $Y \subset \mathbb{P}(a_0, \dots, a_n)$.*

A corollary of this theorem is the existence of a moduli space as an algebraic space, a generalisation of a scheme, made up of affine schemes glued along étale neighbourhoods. This is a direct consequence of the Keel–Mori theorem [23]. However, Theorem 5.18 implies that this algebraic space is in fact a quasi-projective variety.

There are many different classes of varieties which present themselves as hypersurfaces in weighted projective spaces; for example, genus 2 curves are degree 6 curves in $\mathbb{P}(1, 1, 3)$, Petri special curves are degree 6 curves in $\mathbb{P}(1, 1, 2)$ and degree 2 del Pezzo surfaces are degree 4 surfaces in $\mathbb{P}(1, 1, 1, 2)$ to name a few. Hence we find constructions of new moduli spaces or new constructions of well-known moduli spaces.

There are many applications of the constructions we provide in this paper. The moduli spaces and notions of stability we present in this paper are related to K-stability and mirror symmetry and can be seen as a natural extension of work of Batyrev and Cox [8].

2. Non-reductive geometric invariant theory

In this section, we introduce non-reductive GIT and present some recent results due to Bérczi, Doran, Hawes and Kirwan [4, 5]. We call these results the \hat{U} -Theorems and they will be the main tools used in constructing the moduli spaces of hypersurfaces in complete simplicial toric varieties.

The method adopted in [4, 5] requires additional structure on the algebraic groups and the linearisations chosen. With this additional structure, many of the properties of reductive GIT can be recovered. In this section, we now introduce and explore this additional structure and state the resulting theorems. It must be noted that we introduce

definitions and state theorems in only as much generality as is required. The definitions and results hold in greater generality than is stated and we refer the reader to [4, 5] for statements in full generality.

We define notions of semistability and stability for linear algebraic group actions. One motivation of this definition is to have a quotient locally of finite type. Due to the presence of a global stabiliser for the actions we will be considering, we need our definition of stability to allow for a positive dimensional stabiliser, so we adopt a variant of the definition given in [4], in analogy to the construction of the moduli space of quiver representations [24]. We do this as it is easier to work a modified definition of stability rather than with than the group resulting from quotienting out by this global stabiliser.

Let X be a projective variety acted on by a linear algebraic group G with respect to a very ample linearisation \mathcal{L} , where G is not necessarily reductive. Let $G \simeq R \times U$ be the Levi decomposition. Associated to the inclusion of graded rings:

$$A(X, \mathcal{L})^G \subset A(X, \mathcal{L}),$$

we have the morphism of schemes:

$$q_G : X \dashrightarrow X //_{\mathcal{L}} G,$$

where $X //_{\mathcal{L}} G = \text{Proj } A(X, \mathcal{L})^G$. Note that X is a scheme, not necessarily of finite type.

Definition 2.1. [4, Definition 3.4] *Suppose additionally that $D \subset G$ is a torus acting trivially on X . We define:*

$$I^{fg} = \{ \sigma \in A(X, \mathcal{L})^G_+ \mid \mathcal{O}(X_\sigma) \text{ is finitely generated} \},$$

and the finitely generated semistable locus to be:

$$X^{ss} = \bigcup_{\sigma \in I^{fg}} X_\sigma.$$

The enveloping quotient is the image:

$$q_G(X^{ss}) \subset X //_{\mathcal{L}} G.$$

Further, we define $I^s \subset I^{fg}$ to be G -invariant sections satisfying the following conditions:

- the action of G on X_f is closed and for every $x \in X_f$ we have $D \subset \text{Stab}_G(x)$ with finite index; and
- the restriction of the U -enveloping quotient map:

$$q_U : X_\sigma \longrightarrow \text{Spec}(\mathcal{O}(X)_{(\sigma)}^U),$$

is a principal U -bundle for the action of U on X_σ .

Then, we define

$$X^s = \bigcup_{\sigma \in I^s} X_\sigma,$$

to be the stable locus.

Notation 2.2. When there is a possibility of confusion, we write $X^{s,G}$ and $X^{ss,G}$ for X^s and X^{ss} respectively when we want to emphasise the group.

Remark 2.3. When the torus $D \subset G$ is trivial, Definition 2.1 coincides with the original definition of [4, 5]. Alternatively, if the group G is reductive, so that $U = \{e\}$, the definition agrees with King’s notion of stability [24].

The following lemma details how we may study the quotient of an action of G on X in two stages. If we first deal with the action of U on X , then we may consider the action of R on X/U , provided it exists, using reductive GIT.

Lemma 2.4. [6, Lemma 3.3.1] *Suppose G is a linear algebraic group, N is a normal subgroup of G and X is a scheme with a G -action. Suppose all the stabilisers for the restricted action of N on X are finite and this action has a geometric quotient $\pi : X \rightarrow X/N$. Note that G/N acts canonically on X/N . Then the following statements hold.*

- (1) *For all the G/N -orbits in X/N to be closed, it is necessary and sufficient that all the G -orbits in X are closed;*
- (2) *given $y \in X/N$, the stabiliser $\text{Stab}_{G/N}(y)$ is finite if and only if $\text{Stab}_G(x)$ is finite for some (and hence all) $x \in \pi^{-1}(y)$; and*
- (3) *if G/N is reductive and X/N is affine, then X/N has a geometric G/N -quotient if and only if all G -orbits in X are closed.*

Using Lemma 2.4, we can construct a geometric quotient of the stable locus as defined in Definition 2.1.

Theorem 2.5. [6, Theorem 3.4.2] *Let X be a projective variety and G a linear algebraic group acting on X with respect to a very ample line bundle. There is a commutative diagram.*

$$\begin{array}{ccccc} X^s & \hookrightarrow & X^{ss} & \hookrightarrow & X \\ \downarrow & & \downarrow & & \swarrow \text{---} \\ X^s/G & \hookrightarrow & X //_{\mathcal{L}} G & & \end{array}$$

where the first arrow is a geometric quotient and all inclusions are open.

Note that the middle vertical morphism in the diagram in the above theorem may not be surjective and, moreover, the image may only be a constructible set. The question remains of how one can compute the stable and semistable locus. The following discussion aims to address this.

Let U be a unipotent group and $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(U)$ be a 1-parameter subgroup of automorphisms and let,

$$\hat{U}_\lambda = \mathbb{G}_m \ltimes_\lambda U,$$

be the semi-direct product, where multiplication is given as follows:

$$(u_1, t_1) \cdot (u_2, t_2) = (\lambda(t_2^{-1})(u_1) + u_2, t_1 t_2), \quad u_i \in U, t_i \in \mathbb{G}_m.$$

The pointwise derivation of λ defines a \mathbb{G}_m -action on $\text{Lie } U$. This action defines a grading $\text{Lie } U = \bigoplus_{i \in \mathbb{Z}} (\text{Lie } U)_i$ with respect to weights $i \in \mathbb{Z} = \text{Hom}(\mathbb{G}_m, \mathbb{G}_m)$.

Definition 2.6. [4, Definition 2.2] *We say that \hat{U}_λ is positively graded if the induced action of \mathbb{G}_m on $\text{Lie } U$ has strictly positive weights. That is $(\text{Lie } U)_i \neq 0$ implies that $i > 0$.*

Let $G \simeq R \ltimes U$ be a linear algebraic group. We say that G has a graded unipotent radical if there exists a central 1-parameter subgroup $\lambda_g : \mathbb{G}_m \rightarrow R$ such that the adjoint action on $\text{Lie } U$ has strictly positive weights. We often drop the grading 1-parameter subgroup from the subscript and write $\hat{U}_{\lambda_g} = \hat{U}$. Note that $\lambda_g(t)$ is an automorphism of U since U is a normal subgroup of G .

Let X be a projective variety and $\mathcal{L} \in \text{Pic}(X)$ be a very ample line bundle. Suppose that \hat{U} acts on X with respect to \mathcal{L} . By restricting the \hat{U} -action to \mathbb{G}_m , we have a \mathbb{G}_m -action on $V = H^0(X, \mathcal{L})^\vee$; let

$$\omega_{\min} = \text{minimal weight in } \mathbb{Z} = \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \text{ for the } \mathbb{G}_m\text{-action on } V,$$

and

$$V_{\min} = \{v \in V \mid t \cdot v = t^{\omega_{\min}} v \text{ for all } t \in \mathbb{G}_m\},$$

the associated weight space. Then $\mathbb{P}(V_{\min})$ is a linear subspace of $\mathbb{P}(V)$.

Definition 2.7. [4, Definition 2.1] *Suppose that X, \mathcal{L} and \hat{U} are as above. We define,*

$$Z_{\min} = X \cap \mathbb{P}(V_{\min}),$$

and

$$X_{\min}^0 = \{x \in X \mid \lim_{t \rightarrow 0} t \cdot x \in Z_{\min}\}, \quad \text{where } t \in \mathbb{G}_m \subset \hat{U}.$$

Remark 2.8. The subvarieties Z_{\min} and X_{\min}^0 are unaffected by replacing the linearisation \mathcal{L} by any element of the positive \mathbb{Q} -ray defined by \mathcal{L} in $\text{Pic}^{\hat{U}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Also note that X_{\min}^0 and the U -sweep $U \cdot Z_{\min}$ of Z_{\min} are \hat{U} -invariant subsets.

We may twist the linearisation \mathcal{L} by a character $\chi : \hat{U} \rightarrow \mathbb{G}_m$. We denote the twisted linearisation by \mathcal{L}^χ and the minimum \mathbb{G}_m -weight of $V = H^0(X, \mathcal{L}^\chi)^\vee = H^0(X, \mathcal{L})^\vee$ by ω_{\min}^χ .

Definition 2.9. We say that a linearisation $\mathcal{L} \in \text{Pic}^{\hat{U}}(X)$ is adapted if we have the following inequality:

$$\omega_{\min} < 0 < \omega_{\min+1},$$

where ω_{\min} is the lowest \mathbb{G}_m -weight and $\omega_{\min+1}$ is the second smallest weight.

Remark 2.10. Fix $\mathcal{L} \in \text{Pic}^{\hat{U}}(X)$ a linearisation and note that every character of $\hat{U} = \mathbb{G}_m \rtimes U$ is of the form:

$$\hat{U} \rightarrow \mathbb{G}_m; \quad (t, u) \mapsto t^r,$$

for some $r \in \mathbb{Z}$. We identify the characters of \hat{U} with \mathbb{Z} . Let $\epsilon > 0$ be a rational number and consider the rational character $\chi = -\omega_{\min} - \frac{\epsilon}{2}$. Twist \mathcal{L} by the character χ and denote this linearisation $\mathcal{L}^\chi \in \text{Pic}^{\hat{U}}(X)_{\mathbb{Q}}$. Then for $\epsilon > 0$ small enough, \mathcal{L}^χ is adapted: indeed, we have that:

$$\omega_{\min}^\chi = \omega_{\min} + \chi = -\frac{\epsilon}{2} < 0 < \omega_{\min}^\chi + \epsilon.$$

Remark 2.11. For the proofs of finite generation of invariants given in [4, 5] to work, we must twist the linearisation by a rational character so that it is not merely adapted but so that the weight ω_{\min} is within some sufficiently small $\epsilon > 0$ of the origin. That is, $\omega_{\min} < 0 < \omega_{\min} + \epsilon$. We say that a linearisation is *well-adapted* when this inequality holds for a sufficiently small $\epsilon > 0$. In the case of \hat{U} -actions, it follows from Remark 2.10, that an adapted linearisation can be twisted to a well-adapted linearisation. We refer the reader to [5, 7] for a more in depth discussion.

Before we state the \hat{U} -theorem, there is a technical condition which we require. Compare with the definition in the discussion preceding [4, Theorem 2.4] and [4, Remark 2.7].

Definition 2.12. The G -action is said to satisfy the semistability equals stability condition if:

$$\text{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}^{ss,R}, \tag{C}$$

where $Z_{\min}^{ss,R}$ is the semistable locus for the induced linear action of R on Z_{\min} .

We can now state the \hat{U} -theorem. The exact form we state is found in [5], however it coincides with [4, Theorem 2.16] when one takes the torus therein to be trivial.

Theorem 2.13. [5, Theorem 2] *Let X be a projective variety acted on by a graded unipotent group \hat{U} with respect to a very ample linearisation \mathcal{L} . Suppose that the action satisfies the condition (\mathfrak{C}) . Then the following statements hold.*

- (1) *The restriction to X_{\min}^0 of the enveloping quotient for the U -action:*

$$q_U : X_{\min}^0 \longrightarrow X_{\min}^0/U,$$

is a principal U -bundle, in particular, q_U is a geometric quotient.

Suppose furthermore that $X_{\min}^0 \neq U \cdot Z_{\min}$ and that the linearisation \mathcal{L} is well-adapted, then the following statements hold.

- (2) *There are equalities $X_{\min}^0 - U \cdot Z_{\min} = X^{s, \hat{U}}(\mathcal{L})$.*
- (3) *The enveloping quotient $X //_{\mathcal{L}} \hat{U}$ is a projective variety, and*

$$q_{\hat{U}} : X^{s, \hat{U}}(\mathcal{L}) \longrightarrow X //_{\mathcal{L}} \hat{U},$$

is a geometric quotient for the \hat{U} -action. In particular, the ring of \hat{U} -invariants is finitely generated.

Remark 2.14. It follows from the proof of the \hat{U} -hat theorem [21, Theorem 5.1.3] that if Z_{\min} is a point (so that $\dim V_{\min} = 1$), then $X_{\min}^0 = X_{\sigma}$ for some non-zero section $\sigma \in (V_{\min})^{\vee}$. Thus X_{\min}^0 is an affine open subscheme of X . Then a result of Asok and Doran [2, Theorem 3.12] gives us that affine unipotent principal bundles are trivial. Hence the U -quotient $q_U : X_{\min}^0 \rightarrow X_{\min}^0/U$ is a trivial U -bundle.

We now state the result for general linear algebraic groups. Let $G \cong R \ltimes U$ be a linear algebraic group with unipotent radical $U \subset G$ and Levi factor R . Suppose that there exists a 1-parameter subgroup $\lambda_g : \mathbb{G}_m \rightarrow R$ lying in the centre of R such that $\hat{U} = \lambda_g(\mathbb{G}_m) \ltimes U$ is a graded unipotent group.

Definition 2.15. [4, Definition 2.1] *A linearisation of the G -action is said to be (well-)adapted if its restriction to \hat{U} is a (well-)adapted in the sense of Definition 2.9.*

Theorem 2.16. [4, Theorem 2.4] *Let G be a linear algebraic group acting on a projective variety X with respect to \mathcal{L} . Assume that G has graded unipotent radical such that (\mathfrak{C}) holds. Further, assume that \mathcal{L} is well-adapted. Then the following statements hold.*

- (1) *The G -invariants are finitely generated and the enveloping quotient:*

$$X //_{\mathcal{L}} G = \text{Proj } A(X, \mathcal{L})^G$$

is a projective variety.

(2) The inclusion $A(X, \mathcal{L})^G \subset A(X, \mathcal{L})$ induces a categorical quotient of the semistable locus

$$X^{ss,G} \longrightarrow X //_{\mathcal{L}} G,$$

which restricts to a geometric quotient:

$$X^{s,G} \longrightarrow X^{s,G}/G.$$

We now state a Hilbert–Mumford criterion, whose proof is outlined in [4].

Theorem 2.17. [4, Theorem 2.20] *Keep the notation and assumptions as in Theorem 2.16. The following Hilbert–Mumford criterion holds.*

$$X^{(s)G} = \bigcap_{g \in G} gX^{(s)T},$$

where $T \subset G$ is a maximal torus of G containing the grading \mathbb{G}_m .

2.1. Weight polytopes

Using Theorem 2.17, we can reduce the study of stability to the study of stability of a maximal torus. The discrete-geometric version of the Hilbert–Mumford criterion for a torus described in [27] must be adapted to work in the presence of global stabilisers. Suppose that a torus T is acting on a projective space $X = \mathbb{P}^n$ with respect to a very ample linearisation $\mathcal{O}(1)$. Consider the canonical identification $X = \mathbb{P}(V)$, where $V = H^0(X, \mathcal{O}(1))^\vee$. Suppose further that there exists a 1-parameter subgroup:

$$\lambda_{\underline{a}} : \mathbb{G}_m \longrightarrow T,$$

such that $\lambda_{\underline{a}}(\mathbb{G}_m) \subset \text{Stab}_T(x)$ for every $x \in X$, where $\underline{a} \in \mathbb{Z}^{n+1}$ is the vector of weights of the action. Consider the weight space decomposition:

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi,$$

where $X^*(T) = \text{Hom}(T, k^*)$ is the character group and $V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \ \forall t \in T\}$. The 1-parameter subgroup $\lambda_{\underline{a}}$ defines a point in $W = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ given by $\underline{a} \in W$. Define the quotient vector space $H_{\underline{a}} = W/\mathbb{Q} \cdot \underline{a}$ and write $\bar{w} \in H_{\underline{a}}$ for the image of an element $w \in W$ in $H_{\underline{a}}$.

Let $\bar{T} = T/\lambda_{\underline{a}}(\mathbb{G}_m)$ and consider $x \in X$ and some $v \in V$ lying over x and write $v = \sum v_\chi$. Note that we can equivalently construct $H_{\underline{a}} = X^*(\bar{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We define the

\overline{T} -weight set of x to be:

$$\text{wt}_{\overline{T}}(x) = \{\overline{\chi} \mid v_{\chi} \neq 0\} \subset H_{\underline{a}},$$

and the associated weight polytope to be the convex hull of these weights:

$$\text{Conv}_{\overline{T}}(x) = \text{Conv}(\chi \mid \chi \in \text{wt}_T(x)) \subset H_{\underline{a}}.$$

We get the discrete-geometric Hilbert–Mumford Criterion for (semi)stability with respect to the torus in the presence of a global stabilising \mathbb{G}_m combining the numerical criteria of King [24, Lemma 2.2] and the well-known polytope formulation of the Hilbert–Mumford criterion, see for example [15, Theorem 9.2] or [4, Proposition 3.2].

Theorem 2.18. (Reductive Hilbert–Mumford criterion). *Let T be a torus acting on a projective scheme X with linearisation \mathcal{L} such that there is a global stabiliser $\lambda_{\underline{a}} : \mathbb{G}_m \rightarrow T$ acting trivially on X . Then*

$$\begin{aligned} x \in X^{ss,T}(\mathcal{L}) &\iff 0 \in \text{Conv}_{\overline{T}}(x), \\ x \in X^{s,T}(\mathcal{L}) &\iff 0 \in \text{Conv}_{\overline{T}}(x)^{\circ}, \end{aligned}$$

where $\text{Conv}_{\overline{T}}(x)^{\circ}$ is the interior of the polytope.

Combining this with Theorem 2.17, we have a non-reductive Hilbert–Mumford criterion.

Theorem 2.19. (Non-reductive Hilbert–Mumford criterion). *Let G be a linear algebraic group acting on a projective variety X with respect to \mathcal{L} . Assume that G has graded unipotent radical and assume that (\mathfrak{C}) holds for the linearised action. The following Hilbert–Mumford criterion holds.*

$$\begin{aligned} x \in X^{ss,G} &\iff 0 \in \text{Conv}_T(g \cdot x) \text{ for every } g \in G, \\ x \in X^{s,G} &\iff 0 \in \text{Conv}_T(g \cdot x)^{\circ} \text{ for every } g \in G. \end{aligned}$$

Example 2.20. We consider degree 4 curves in $\mathbb{P}(1, 1, 2)$, such curves are parametrised by the projective space of weighted forms $\mathcal{Y} = \mathbb{P}(k[x, y, z]_4)$, where $\deg x = \deg y = 1$ and $\deg z = 2$. Then GL_2 acts on (x, y) via matrix multiplication and \mathbb{G}_m acts on z via multiplication. This defines an action of $G = \text{GL}_2 \times \mathbb{G}_m$ on \mathcal{Y} . Consider the maximal torus of G defined by:

$$T = \{(\text{diag}(t_1, t_2), s) \in \text{GL}_2 \times \mathbb{G}_m \mid t_1, t_2, s \in k^*\}.$$

Consider the restricted action of T on \mathcal{Y} . Then for a general monomial $x^i y^j z^k \in k[x, y, z]_4 = V$ with $i + j + 2k = 4$, we have that:

$$(t_1, t_2, s) \cdot x^i y^j z^k = t_1^i t_2^j s^k x^i y^j z^k.$$

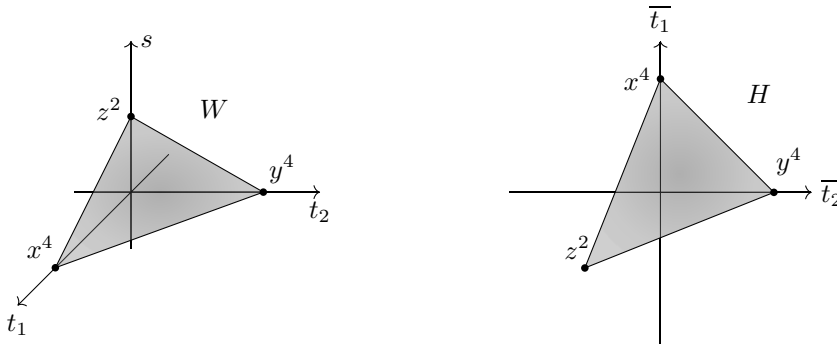


Figure 1. P_4 , the section polytope of $\mathcal{O}(4)$ in W and H .

Denote such a weight by $(i, j, k) \in X^*(T) \cong \mathbb{Z}^3$. Note that by collecting all possible weights as columns in a matrix, one gets the following matrix:

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

which is the matrix from Example 4.6.

Define P_4 to be the *section polytope* to be the convex hull of all torus weights in either W or $H = H_{\underline{a}}$ (see Figure 1). One sees directly that the 1-parameter subgroup $\lambda_{\underline{a}}$ with $\underline{a} = (1, 1, 2)$, normal to W , is contained inside the global stabiliser. The weight polytope of any element $[f] \in \mathcal{Y}_4$ will be a subpolytope of the section polytope. As shown in Figure 1, the section polytope considered in W is not full-dimensional, and is a 2-simplex in $W \cong \mathbb{Q}^3$ which does not contain the origin. We will always consider the section polytope in H , where it is full dimensional and contains the origin.

Explicitly, if e_1 and e_2 are a basis for H , then $P_4 = \text{Conv}(4e_1, 4e_2, -e_1 - e_2)$.

3. Automorphisms and toric varieties

In this section, we study the automorphism groups of toric varieties and prove that they carry the extra structure required by NRGIT; that is, that they admit graded unipotent radicals. We first recall some aspects of toric geometry; specifically toric orbifolds and their quasismooth hypersurfaces.

3.1. Toric orbifolds

Let X be a projective simplicial toric variety. These are precisely the projective toric varieties with at worst orbifold singularities and we refer to such varieties simple as toric orbifolds. The notion of quasismoothness was first introduced for subvarieties of weighted projective spaces by Dolgachev [14] and then generalised to toric varieties by Cox and Batyrev [8].

We recall the Cox ring and the quotient construction of a toric variety. We refer the reader to [1, 10] for details.

Definition 3.1. *Suppose that $X = X_\Sigma$ is a toric variety associated to a fan Σ . The 1-dimensional cones in Σ are called rays and the set of rays is denoted $\Sigma(1)$. Let*

$$S = k[x_\rho \mid \rho \in \Sigma(1)],$$

be the polynomial ring in $|\Sigma(1)|$ variables. Every monomial $\prod x_\rho^{a_\rho} \in S$ defines an effective torus-invariant divisor $D = \sum a_\rho D_\rho$, we write this monomial as x^D . In this way, we define the following notion of degree:

$$\text{deg}(x^D) = [D] \in \text{Cl}(X).$$

Thus, we have

$$S = \bigoplus_{\alpha \in \text{Cl}(X)} S_\alpha,$$

where $S_\alpha = \{f \in S \mid \text{all monomials of } f \text{ of degree } \alpha\}$. Then $S_\alpha \cdot S_\beta \subset S_{\alpha+\beta}$ and we define the Cox ring of X to be S with this grading by the class group $\text{Cl}(X) = \text{CH}_{n-1}(X)$ where $\dim X = n$.

Now when X is a projective orbifold, every Weil divisor is \mathbb{Q} -Cartier and for every $\alpha \in \text{Cl}(X)$ we denote the corresponding rank 1 reflexive sheaf by $\mathcal{O}_X(\alpha)$. By [10, Proposition 1.1], the Cox ring of X is the algebra of global sections of the rank 1 reflexive sheaves and denoted by

$$S = \bigoplus_{\alpha \in \text{Cl}(X)} H^0(X, \mathcal{O}_X(\alpha)).$$

Since X is toric, $\text{Cl}(X)$ is a finitely generated abelian group. The grading of S by $\text{Cl}(X)$ defines an action of $\mathbf{D} = \text{Hom}_{\mathbb{Z}}(\text{Cl}(X), k^*)$ on $\mathbb{A}^r = \text{Spec } S$ (for example, see [13]). Cox proved ([10, Theorem 2.1], although for the way we have formulated the action the proof can be found in [13, Theorem 2.12]) that $X = \mathbb{A}^r //_{\chi} \mathbf{D}$ as the affine GIT-quotient of this action twisted by any character $\chi \in X^*(\mathbf{D}) = \text{Cl}(X)$ corresponding to an ample line bundle. We denote the the quotient morphism restricted to the stable locus $q : (\mathbb{A}^r)^s \rightarrow X$.

Definition 3.2. *Let X be a toric orbifold and fix a class $\alpha \in \text{Cl}(X)$. Let $Y \subset X$ be a hypersurface defined by $f \in S_\alpha$. We say that Y is quasismooth if $q^{-1}(Y) \subset (\mathbb{A}^r)^s$ is smooth. Equivalently, Y is quasismooth if $\mathbf{V}(f) \subset \mathbb{A}^r$ is smooth in $(\mathbb{A}^r)^s$.*

Let $\mathcal{Y}_\alpha = \mathbb{P}(S_\alpha)$. We define the quasismooth locus to be the open set:

$$\mathcal{Y}_\alpha^{QS} = \{Y \subset X \mid Y \text{ is a quasismooth hypersurface of class } \alpha\} \subset \mathcal{Y}_\alpha.$$

We denote its complement, the non-quasismooth locus, by:

$$\mathcal{Y}_\alpha^{NQS} = \mathcal{Y}_\alpha - \mathcal{Y}_\alpha^{QS}.$$

Example 3.3. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space. Then $S = \bigoplus_{d \geq 0} k[x_0, \dots, x_n]_d$ where $\deg x_i = a_i$ and $q : (\mathbb{A}^{n+1} - \{0\}) \rightarrow \mathbb{P}(a_0, \dots, a_n)$ is the quotient morphism for the \mathbb{G}_m -action on \mathbb{A}^{n+1} defined by $t \cdot (x_0, \dots, x_n) = (t^{a_0}x_0, \dots, t^{a_n}x_n)$. Moreover, a hypersurface defined by $f \in k[x_0, \dots, x_n]_d$ is quasismooth if and only if $(\frac{\partial f}{\partial x_0}(\tilde{x}), \dots, \frac{\partial f}{\partial x_n}(\tilde{x})) \neq (0, \dots, 0)$ for every $\tilde{x} \in \mathbb{A}^{n+1} - \{0\}$.

Theorem 3.4. [22, Theorem 8.1] *The general hypersurface of degree d in $\mathbb{P}(a_0, \dots, a_n)$ is quasismooth if and only if either*

- (1) *there exists a variable x_i of degree d , or*
- (2) *for every non-empty subset $I = \{i_0, \dots, i_{k-1}\}$ of $\{0, \dots, n\}$ one of the following holds:*
 - (a) *there exists a monomial $x_I^M = x_{i_0}^{m_0} \dots x_{i_{k-1}}^{m_{k-1}}$ of degree d or*
 - (b) *for $\mu = 1, \dots, k$ there exists monomials:*

$$x_I^{M\mu} x_{e_\mu} = x_{i_0}^{m_{0,\mu}} \dots x_{i_{k-1}}^{m_{k-1,\mu}} x_{e_\mu}$$

of degree d , where $\{e_\mu\}$ are k distinct elements.

The following lemma will be needed for the proof of Theorem 5.18. It is a more explicit version of Theorem 3.4 and allows us to remove the general hypothesis. It follows from arguments used in the proof of Theorem 3.4 given in [22].

Recall that the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is *well formed* if for any i with $0 \leq i \leq n$ then $\gcd(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = 1$.

Lemma 3.5. *Suppose that $X = \mathbb{P}(a_0, \dots, a_n)$ is well formed and d a Cartier degree and denote $d_i = \frac{d}{a_i}$. Consider an $f \in k[x_0, \dots, x_n]_d$ which is quasismooth. For each variable x_i , either $x_i^{d_i}$ is a monomial of f or $x_i^{d_i - \frac{a_j}{a_i}} x_j$ is a monomial of f for some $j \neq i$ where $a_i | a_j$.*

Example 3.6. Let $X = \mathbb{P}(1, 1, 2)$. Consider the degree 6 hypersurfaces defined by $x^4y^2 + y^6 + z^3$ and $x^5y + x^4y^2 + y^6 + z^3$. The first has a unique singularity at $(1 : 0 : 0)$ and the second is quasismooth. One sees that on including the monomial x^5y the hypersurface satisfies the conditions of the lemma and becomes quasismooth.

Remark 3.7. Note that if for a fixed a_i , there exists no a_j such that $a_i | a_j$, then we must have that $x_i^{d_i}$ is a monomial of f .

3.2. Graded automorphisms of the Cox ring

The construction of the automorphism group of a complete simplicial toric variety X is a generalisation of the construction of the automorphism group of projective space. The generalisation is to be seen as follows. The Cox ring of projective space with the grading of the class group given in Definition 3.1 is the standard homogeneous coordinate ring; that is, the polynomial ring with the usual \mathbb{Z} -grading given by the total degree. The group of graded automorphisms of this ring is $\mathrm{GL}(n + 1)$ which fits into the following short exact sequence:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}(n + 1) \longrightarrow \mathrm{PGL}(n + 1) \longrightarrow 0,$$

where $\mathrm{PGL}(n+1) = \mathrm{Aut}(\mathbb{P}^n)$. More generally, let X be a complete toric variety associated to a fan Σ . Let $S = k[x_\rho \mid \rho \in \Sigma(1)]$ be the Cox ring of X . We denote the graded automorphisms of the Cox ring by $\mathrm{Aut}_g(S)$. In [10, Theorem 4.2 and Proposition 4.3], Cox provided a description of $\mathrm{Aut}(X)$. Central to this description is the following short exact sequence:

$$0 \longrightarrow \mathbf{D} \longrightarrow \mathrm{Aut}_g(S) \longrightarrow \mathrm{Aut}^0(X) \longrightarrow 0,$$

where $\mathrm{Aut}^0(X)$ is the connected component at the identity.

Theorem 3.8. *Let X be a complete toric variety and let $S = \mathrm{Cox}(X)$ be its Cox ring. Then the following statements hold.*

- (1) *The group of graded algebra automorphisms $\mathrm{Aut}_g(S)$ is a connected affine algebraic group of dimension $\sum_{i=1}^l |\Sigma_i| \dim_k S_i$.*
- (2) *The unipotent radical U of $\mathrm{Aut}_g(S)$ is of dimension $\sum_{i=1}^l |\Sigma_i| (\dim_k S_i - |\Sigma_i|)$.*
- (3) *We have the following isomorphism:*

$$\mathrm{Aut}_g(S) \cong \prod_{i=1}^l \mathrm{GL}(S'_i) \rtimes U,$$

where $S'_i \subset S_i$ is the vector subspace spanned by variables $x_\rho \in S_i$.

We refer the reader to [11, Theorem 4.2 and Proposition 4.3] for a proof and to [11, Lemma 4.4] for a more detailed description of the unipotent radical.

We consider the group of graded automorphisms as a matrix group via the following lemma, whose proof is taken from the proof of the corrected version of [11, Proposition 4.3]. We include the proof since we will need the explicit matrix description of the automorphism group. Note that when we refer to the degree of an element of S , we mean the degree with respect to the class group and by total degree we mean the degree with respect to the usual \mathbb{Z} -grading of the polynomial ring.

Lemma 3.9. *The endomorphism algebra of S is a linear algebraic monoid with unit group $\text{Aut}_g(S)$ and there is an inclusion of linear algebraic monoids:*

$$\begin{aligned} \text{End}_g(S) &\longrightarrow \prod_{i=1}^l \text{End}_k(S_i) \\ \phi &\longmapsto (\phi|_{S_i} : S_i \rightarrow S_i)_{i=1}^l. \end{aligned}$$

In particular, $\text{Aut}_g(S)$ is a linear algebraic group.

The definition of a linear algebraic monoid is identical to that of a linear algebraic group except that one does not require that every element has an inverse.

Proof. We show that the map:

$$\text{End}_g(S) \longrightarrow \prod_{i=1}^l \text{End}_k(S_i),$$

is a closed immersion and hence $\text{End}_g(S)$ is an affine submonoid. Since S is generated as an algebra by elements in S_1, \dots, S_l , an endomorphism is completely determined by the above restrictions and hence the map is injective. The fact that the map respects composition (and is well-defined) is immediate since we consider only graded endomorphisms. Thus $\text{End}_g(S)$ is a submonoid and it only remains to show that it is a closed subset; that is, cut out by polynomials.

To do this, we write down the corresponding collection of matrices with respect to the basis of each $S_i = S'_i \oplus S''_i$ given by monomials of degree α_i :

$$\phi \longleftrightarrow \left(\left(\begin{array}{c|c} A_i & 0 \\ \hline B_i & C_i \end{array} \right) \right)_{i=1}^l, \tag{*}$$

where $B_i \in \text{Hom}_k(S'_i, S''_i)$. We shall often suppress the brackets in this notation.

The matrices A_i and B_i come from evaluating the single variables in S'_i . The C_i come from evaluating monomials in S''_i which are products of 2 or more variables in S'_j with $j \neq i$ and hence C_i is completely determined by A_j and B_j for $j \neq i$. We claim that the elements of C_i are polynomials in elements of A_j and B_j for $j \neq i$.

Let us prove this claim. Consider monomials $x^D, x^E \in S''_i$, where D and E are effective non-prime divisors with class α_i . Both x^D and x^E are elements of the monomial basis of S''_i so that for $\phi \in \text{End}_g(S)$:

$$\phi(x^D) = \dots + c_i^{DE} x^E + \dots,$$

where c_i^{DE} is the corresponding entry in C_i . Then $x^D = x_{\rho_1} \dots x_{\rho_s}$ is a product of variables allowing duplications with $x_{\rho_i} \notin S'_i$. Thus,

$$\phi(x_{\rho_1}) \dots \phi(x_{\rho_s}) = \dots + c_i^{DE} x^E + \dots.$$

But each $\phi(x_{\rho_k})$ is a linear combination of monomials with coefficients given by elements of A_j and B_j with $j \neq i$. Thus the elements of the C_i are given by polynomials in the elements of A_j, B_j and we are done.

On the other hand, the A_i and B_i are chosen completely arbitrarily. In other words we have a bijection of sets:

$$\begin{aligned} \text{End}_{\mathfrak{g}}(S) &\longleftrightarrow \prod_{i=1}^l \text{Hom}_k(S'_i, S_i), \\ \phi &\longleftrightarrow \begin{pmatrix} A_i \\ B_i \end{pmatrix}. \end{aligned}$$

The 0 in the top right-hand corner of the matrices (*) comes from the fact that:

$$\phi(S''_i) \cap S'_i = 0,$$

since $S_0 = k$, and monomials in S''_i contain more than one variable.

It remains to remark that $\text{Aut}_{\mathfrak{g}}(S)$ is the group of invertible elements in a linear algebraic monoid. It follows from [29, Corollary 3.26] that $\text{Aut}_{\mathfrak{g}}(S)$ is a linear algebraic group. □

Proposition 3.10. *The unipotent radical U of $\text{Aut}_{\mathfrak{g}}(S)$ is given by matrices of the form:*

$$\left(\begin{array}{c|c} I_{n_i} & 0 \\ \hline B_i & C_i \end{array} \right),$$

under the correspondence in (*), where C_i are lower triangular matrices with 1's on the diagonal and I_{n_i} is the $(n_i \times n_i)$ -identity matrix with $n_i = \dim S'_i$.

Moreover, the 1-parameter subgroup given by:

$$\begin{aligned} \lambda_g : \mathbb{G}_m &\longrightarrow \text{Aut}_{\mathfrak{g}}(S), \\ t &\longmapsto (\phi_t : x_{\rho} \mapsto t^{-1}x_{\rho}), \end{aligned}$$

gives U a positive grading. We refer to λ_g as the distinguished \mathbb{G}_m .

Remark 3.11. Note that this result was already given in the paper [5]. The proof uses the original incorrect construction of the automorphism group given in the paper [10]. We present a proof using the corrected construction given in [11].

Proof. It is clear that the matrices above form a unipotent subgroup and we refer the reader to [11, Theorem 4.2] for a proof that it is in fact the unipotent radical. We prove that it is positively graded by the distinguished \mathbb{G}_m .

Under (*), we have

$$\lambda_g(t) \longleftrightarrow \left(\begin{array}{c|c} t^{-1}I_{n_i} & 0 \\ \hline 0 & Q_i(t) \end{array} \right),$$

where

$$Q_i(t) = \text{diag}(t^{-l_1}, \dots, t^{-l_k}),$$

are diagonal matrices with $l_j \geq 2$. To see this, consider $x^D = x_{\rho_1} \cdots x_{\rho_l} \in S''_i$ again allowing duplications. Then,

$$\lambda_g(t)(x^D) = \lambda_g(t)(x_{\rho_1}) \cdots \lambda_g(t)(x_{\rho_l}) = t^{-l}x^D$$

where l has to be greater than 2 since D was a non-prime divisor.

To calculate the weights on the Lie algebra of U consider the conjugation action:

$$\lambda_g(t^{-1}) \left(\begin{array}{cc} I_{n_i} & 0 \\ B_i & C_i \end{array} \right) \lambda_g(t) = \left(\begin{array}{c|c} I_{n_i} & 0 \\ \hline tQ_i(t^{-1})B_i & Q_i(t)C_iQ_i(t) \end{array} \right),$$

of an arbitrary element of U by $\lambda_g(t)$. Then the matrix in the bottom left hand corner is given by:

$$\left(\begin{array}{ccc} t^{l_1-1} & & \\ & \ddots & \\ & & t^{l_k-1} \end{array} \right) B_i$$

and since each $l_j \geq 2$, the exponents here are strictly positive. This suffices to show that the group is graded unipotent since the matrices B_i describe the Lie algebra. \square

Remark 3.12. For a general weighted projective space $X = \mathbb{P}(a_0, \dots, a_n) = \text{Proj } S$, where $S = k[x_0, \dots, x_n]$, we describe the Levi factor of the group $G = \text{Aut}_g(S)$ explicitly. First, partition the variables x_i into distinct weights $\Sigma_j = \{x_i \mid \text{deg } x_i = a_j\}$ and set $n_i = |\Sigma_i|$. Then the Levi factor of G is equal to:

$$\prod_{\Sigma_i} \text{GL}_{n_i} \subset G,$$

where the product is taken over the distinct Σ_i . Thus the Levi factor contains all linear automorphisms: that is, automorphisms which take variables to linear combinations of other variables. As an automorphism must respect the grading, these linear combinations only contain variables of the same weight.

The unipotent radical of G is given by ‘non-linear’ automorphisms: that is, automorphisms which involve a monomial of total degree higher than 1.

Let us illustrate this in an example.

Example 3.13. Consider $\mathbb{P}(1, 1, 2)$ so that $S = k[x, y, z]$ with $\deg x = \deg y = 1$ and $\deg z = 2$. Then, $S_1 = S'_1 = \{x, y\}$ and $S_2 = S'_2 \oplus S''_2 = \text{Span}(z) \oplus \text{Span}(x^2, xy, y^2)$. We see that the only unipotent automorphisms are given by $z \mapsto z + Ax^2 + Bxy + Cy^2$ for $A, B, C \in k$. Thus automorphisms are described by the map $S_2 = S'_2 \oplus S''_2 \mapsto S'_2 \oplus S''_2$ given by the matrix:

$$\begin{pmatrix} I_1 & 0 \\ M & I_3 \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} A \\ B \\ C \end{pmatrix}.$$

Let $X = \mathbb{P}(a_0, \dots, a_n)$. Assume the weights are in ascending order and label the distinct weights $a_0 = b_1 < b_2 < \dots < b_l = a_n$ so that the weight b_i occurs exactly n_i times (the n_i coincide with the n_i in Remark 3.12). Let $m = n_1 + \dots + n_{l-1}$ and let $x_1, \dots, x_m, y_1, \dots, y_{n_l}$ be coordinates on X where variables of the highest weight b_l are given by y_j and all others given by x_i . For weighted projective space we define another 1-parameter subgroup which grades the unipotent radical $U \subset G$ positively depending on a parameter $N \in \mathbb{Z}$.

Proposition 3.14. *Let $N > 0$ be a positive integer. The 1-parameter subgroup $\lambda_{g,N} : \mathbb{G}_m \rightarrow G$ defined by:*

$$\lambda_{g,N} : t \mapsto \left((t^{-N} I_{n_i})_{i=1}^{l-1}, t I_{n_l}, 0 \right) \in G = \left(\prod_{i=1}^{l-1} \text{GL}_{n_i} \times \text{GL}_{n_l} \right) \rtimes U,$$

gives $U \subset G$ a positive grading.

Proof. Let $X = \mathbb{P}(a_0, \dots, a_n) = \text{Proj } S$ where $S = k[x_1, \dots, x_m, y_1, \dots, y_{n_l}]$ so that the y_j have the maximum weight $b_l = a_n$. Then $\lambda_{g,N}(\mathbb{G}_m) \subset G = \text{Aut}_g(S)$ acts on X as follows:

$$\begin{aligned} \lambda_{g,N}(t) \cdot (0 : \dots : x_i : \dots : 0) &= (0 : \dots : t^{-N} x_i : 0 : \dots : 0) \\ \lambda_{g,N}(t) \cdot (0 : \dots : y_j : \dots : 0) &= (0 : \dots : 0 : t y_j : \dots : 0), \end{aligned}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n_l$. Let $u \in U \subset G$ be an element of the unipotent radical. By Remark 3.12, u acts on S as follows:

$$\begin{aligned} u \cdot x_i &= x_i + p_i(x_0, \dots, x_{n'}) \\ u \cdot y_j &= y_j + q_j(x_0, \dots, x_{n'}), \end{aligned}$$

for $0 \leq i \leq n'$ and $0 \leq j \leq n_l$, where $p_i, q_j \in k[x_1, \dots, x_m]$ are weighted homogeneous polynomials (possibly 0) of degree a_i and a_n respectively. Note that $p_i = 0$ for those i such that $a_i = b_1$ is the minimum weight and that p_i and q_j do not contain any factors of y_j , since the y_j all have the same maximal weight. In particular, if $p_i \neq 0$, then $\deg p_i > 1$.

Consider the action by conjugation of $\lambda_{g,N}(\mathbb{G}_m)$ on U , first on the x_i :

$$\begin{aligned} (\lambda_{g,N}(t) \cdot u \cdot \lambda_{g,N}(t^{-1})) \cdot x_i &= (\lambda_{g,N}(t) \cdot u) \cdot t^N x_i \\ &= \lambda_{g,N}(t) \cdot (t^N x_i + p_i(t^N x_0, \dots, t^N x_{n'})) \\ &= x_i + t^{-N} p_i(t^N x_0, \dots, t^N x_{n'}). \end{aligned}$$

Those p_i 's which are non-zero have degree $a_i > 1$ and hence if u is a weight vector for the $\lambda_{g,N}(\mathbb{G}_m)$ -action, it has weights $a_i N - N > 0$. The argument for the y_i is identical and is omitted. □

3.3. Finiteness of the stabilisers

Let $S = \text{Cox}(X) = k[x_0, \dots, x_n]$ be the Cox ring of the weighted projective space $X = \mathbb{P}(\underline{a})$, where $\underline{a} = (a_0, \dots, a_n)$ and assume that $a_0 \leq a_1 \leq \dots \leq a_n$. Label the distinct values of the a_i 's by b_1, \dots, b_l such that $b_1 < \dots < b_l$. Define numbers n_1, \dots, n_l such that each of the b_j occur exactly n_j times, so that the n_j sum to $n + 1$. Recall from Theorem 3.8 that:

$$\text{Aut}_{\mathfrak{g}}(S) = \prod_{j=1}^l \text{GL}_{n_j} \times U,$$

where U is the unipotent radical. Let $G = \text{Aut}_{\mathfrak{g}}(S)$ and denote the 1-parameter subgroup of G by:

$$\lambda_{\underline{a}} : t \mapsto ((t^{b_j} I_{n_j})_{j=1}^l, 0).$$

Note that the 0 is the unipotent part of the image. Since the automorphism group of weighted projective space is connected, we have that:

$$\text{Aut}(\mathbb{P}(a_0, \dots, a_n)) \cong \text{Aut}_{\mathfrak{g}}(S) / \lambda_{\underline{a}}(\mathbb{G}_m).$$

Theorem 3.15. *Let $S = k[x_0, \dots, x_n]$ be the polynomial ring with the weighted grading $\deg x_i = a_i$ and let $f \in k[x_0, \dots, x_n]_d$ define a quasismooth hypersurface $V(f) \subset \mathbb{P}(a_0, \dots, a_n)$ where $d \geq \max\{a_i\} + 2$. Define the subgroup $\text{Aut}(f) \subset \text{Aut}_{\mathfrak{g}}(S)$ as follows:*

$$\text{Aut}(f) = \{\phi \in \text{Aut}_{\mathfrak{g}}(S) \mid V(\phi(f)) = V(f)\}.$$

Then $\text{Aut}(f) = \mu \times \lambda_{\underline{a}}(\mathbb{G}_m)$, with $\lambda_{\underline{a}} : \mathbb{G}_m \rightarrow G$ defined as above and μ is a finite group.

Proof. We write $G = \text{Aut}_{\mathfrak{g}}(S) = \prod_{j=1}^l \text{GL}_{n_j} \times U$ and denote $\text{Lie } G$ by:

$$\mathfrak{g} = \prod_{j=1}^l \mathfrak{gl}_{n_j} \times \mathfrak{u}.$$

It is clear that $\lambda_{\underline{a}}(\mathbb{G}_m) \subset \text{Aut}(f)$. To obtain the desired result it suffices to show that the Lie algebras of $\text{Aut}(f)$ and $\lambda_{\underline{a}}(\mathbb{G}_m)$ agree as sub-Lie algebras of \mathfrak{g} .

The Lie algebra \mathfrak{g} acts on S by derivation: let $\xi \in \mathfrak{g}$ and $F \in S$ be arbitrary elements of \mathfrak{g} and S respectively, then

$$\xi(F) = \sum_{i=0}^n F_i \xi(x_i),$$

where $F_i = \frac{\partial F}{\partial x_i}$. Suppose that $\xi \in \text{Lie}(\text{Aut}(f)) \subset \mathfrak{g}$. Then since f is semi-invariant under the action of $\text{Aut}(f)$, it is also a semi-invariant for the action of $\text{Lie}(\text{Aut}(f))$; that is, $\xi(f) = \tilde{\alpha}f$ for some $\tilde{\alpha} \in k$. The weighted Euler formula tells us that $f = \frac{1}{d} \sum_{i=0}^n a_i f_i$ and so,

$$\sum_{i=0}^n f_i (\xi(x_i) - \alpha a_i x_i) = 0,$$

where $\alpha = \frac{\tilde{\alpha}}{d}$.

Rearranging, for each i we get an equation:

$$p_i f_i = -(p_0 f_0 + \dots + p_{i-1} f_{i-1} + p_{i+1} f_{i+1} + \dots + p_n f_n),$$

where $p_j = \xi(x_j) - \alpha a_j x_j$. Thus $p_i f_i \in (f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$.

Since f is quasismooth, its partial derivatives f_0, \dots, f_n form a regular sequence. Moreover, any permutation of the f_i is a regular sequence. Thus f_i is a non-zero divisor in the ring $S/(f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$ and hence $p_i \in (f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$. However, $\deg p_i = a_j$ and since we assumed $\deg f \geq \max\{a_j\} + 2$, this forces $p_i = 0$ and $\xi(x_i) = \alpha a_i x_i$. Thus α is the only parameter and we have shown that $\text{Lie}(\text{Aut}(f))$ is one dimensional and hence agrees with that of $\text{Lie} \lambda_{\underline{a}}(\mathbb{G}_m)$.

Moreover, we can see explicitly that:

$$\text{Lie}(\text{Aut}(f)) = \{((\alpha b_j I_{n_j})_{j=1}^l, 0) \mid \alpha \in k\} \subset \mathfrak{g},$$

which is precisely the Lie algebra of $\lambda_{\underline{a}}(\mathbb{G}_m)$. □

Remark 3.16. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space $d \geq \max(a_0, \dots, a_n) + 2$ be an integer. Then the algebraic stack $\mathcal{M}(X, d) = [\mathcal{Y}_d^{\text{QS}} / \text{Aut}(X)]$ admits a coarse moduli space as an algebraic space. This is an immediate consequence of Theorem 3.15 and the Keel-Mori Theorem [23, Corollary 1.2].

4. The A -discriminant of a toric variety

In the following section, we take our definitions and conventions from [18, Chapter 9].

4.1. The A -discriminant

Consider a torus $(k^*)^{r+1}$ with coordinates (x_0, \dots, x_r) and consider a matrix:

$$A = (\omega^{(0)} \mid \dots \mid \omega^{(N)}) \in \mathbb{Z}^{(r+1) \times (N+1)},$$

where $\omega^{(j)} \in \mathbb{Z}^{r+1}$ is a column vector for $0 \leq j \leq N$. Then, A defines a toric variety as in [12, Definition 2.1.1] and [18, Section 5.1.B]. Define the vector space of Laurent functions on $(k^*)^{r+1}$ associated to A by:

$$k^A := \left\{ \sum_{i=0}^N a_i x^{\omega^{(i)}} \mid a_i \in k \right\},$$

here $x^{\omega^{(i)}} = x_0^{\omega_0^{(i)}} \dots x_r^{\omega_r^{(i)}}$, where $\omega^{(i)}$ is a column vector defined to be the transpose of $(\omega_0^{(i)}, \dots, \omega_r^{(i)})$.

Definition 4.1. [18, Section 9.1.A] Consider the following subset of $\mathbb{P}(k^A)$ consisting of Laurent functions (up to scalar multiple) with a singular point on the torus:

$$\nabla_A^\circ = \left\{ f \in \mathbb{P}(k^A) \mid \exists x \in (k^*)^{r+1} \text{ s.t. } f(x) = \frac{\partial f}{\partial x_i}(x) = 0 \text{ for all } i = 0, \dots, r \right\}.$$

Then, define the A -discriminant locus to be:

$$\nabla_A = \overline{\nabla_A^\circ} \subset \mathbb{P}(k^A).$$

We define the defect of A to be:

$$\text{Def } A = \text{codim}_{\mathbb{P}(k^A)}(\nabla_A) - 1.$$

If $\text{Def } A = 0$, then define the A -discriminant Δ_A as the polynomial defining ∇_A which is well defined and unique up to a scalar multiple. If the codimension is greater than 1, we set $\Delta_A = 1$. We shall only work with embeddings where $\text{Def } A = 0$ and we shall always assume that this is the case. We refer to [18, Corollary 1.2] for a geometric characterisation of this property.

We now apply this theory in the context of toric varieties. The following definition is [18, Definition 1.2].

Definition 4.2. Let $X = X_\Sigma$ be a toric variety and $\alpha \in \text{Cl}(X)$ and $S = k[x_0, \dots, x_r]$ be the Cox ring of X . Let $N = \dim S_\alpha - 1$. We define a matrix $A_{\Sigma, \alpha} \in \mathbb{Z}^{(r+1) \times (N+1)}$ by collecting the exponents of the monomial basis of S_α as columns of this matrix with respect to some ordering of the monomials. We define the A -discriminant associated to X and α to be $\Delta_{A_{\Sigma, \alpha}}$. When it is clear from context, we shall drop the Σ and α from the subscript and write simply $A = A_{\Sigma, \alpha}$.

Remark 4.3. Let X and α be as above, then $k^A = S_\alpha$ and hence

$$\nabla_A \subset \mathbb{P}(S_\alpha).$$

Let X be a simplicial projective toric variety and suppose that α is a very ample class. The corresponding A -discriminant is a special case of the discriminant as defined in [18, Chapter 1] in terms of the projective dual of a variety, as the following proposition shows.

Proposition 4.4. [18, Proposition 1.1]. *Let X be a projective toric variety, $\alpha \in \text{Cl}(X)$ be a very ample class and $A = A_{\Sigma, \alpha} \in \mathbb{Z}^{(r+1) \times (N+1)}$ be the associated matrix of exponents of the monomial basis of S_α . Then*

$$\nabla_A = X^{\vee, \alpha},$$

where $X^{\vee, \alpha}$ denotes the projective dual of X with respect to the embedding given by α .

Proof. Since α is very ample and A corresponds to the monomial basis of S_α , the toric variety $X_A \subset \mathbb{P}(k^A)^\vee$ is the toric variety X with the embedding defined by α . Thus $X^{\vee, \alpha} = X_A^\vee$.

It remains to see that $X_A^\vee = \nabla_A$. To see this, we consider the map $\tilde{\Phi}_A : (k^*)^{n+1} \rightarrow k^A$ as local parameters on the torus in the cone $Y_A \subset (k^A)^\vee$ over X_A . Note that since α is very ample, Y_A is an affine toric variety and that $Y_A - \{0\} \rightarrow X_A$ is a toric morphism. We claim that $X_A^\vee = \nabla_A$.

For some $f \in (k^A)^\vee$, if $\mathbf{T}_y Y_A \subset V(f) \subset (k^A)$ for some $y \in T_{Y_A}$, where T_{Y_A} is the torus in Y_A , then $f \in Y_A^\vee$. However, if it holds that $\mathbf{T}_y Y_A \subset V(f) \subset (k^A)$ for some torus point $y \in T_Y$, then $f \in \nabla_A^\circ$ (see Definition 4.1), since the torus $T_{Y_A} \subset Y_A$ is contained in the smooth locus of A .

Thus we have identified ∇_A° with a non-empty open (and hence dense) subset of X_A^\vee given by hyperplanes containing the tangent space to points on the torus. Note that $\nabla_A^\circ \subset \nabla_A$ is a dense subset by definition. Since both ∇_A and X_A^\vee are irreducible hypersurfaces in $\mathbb{P}(k^A)$, we must have $X_A^\vee = \nabla_A$. □

Remark 4.5. This proposition has a very nice geometric meaning. It tells us that for projective toric varieties, the locus of non-quasismooth hypersurfaces in a given complete linear system associated to a very ample class contains (as an irreducible component) the dual to the variety, where the dual is taken with respect to the embedding defined by the very ample line bundle. That is,

$$X_{\Sigma}^{\vee, \alpha} = \nabla_{A_{\Sigma, \alpha}} \subset \mathcal{Y}_\alpha^{\text{NQS}} \subset \mathcal{Y}_\alpha = \mathbb{P}(S_\alpha) = \mathbb{P}(k^{A_{\Sigma, \alpha}}),$$

where the first containment is as an irreducible component and the second containment is closed.

Example 4.6. Let $X = \mathbb{P}(1, 1, 2)$ and $\alpha = 4 \in \text{Cl}(X) \simeq \mathbb{Z}$. Then $S = k[x, y, z]$ where $\deg x = \deg y = 1$ and $\deg z = 2$. The monomial basis of $S_\alpha = k[x, y, z]_4$ gives the

following matrix:

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

For example, the first and second column corresponds to the monomials x^4 and x^3y respectively. An element $F \in k^A$ is given by $F(x, y, z) = a_0x^4 + a_1x^3y + \dots + a_4y^4 + a_5x^2z + a_6xyz + a_7y^2z + a_8z^2$ where $a_i \in k$. Note that $S_\alpha = k^A$ is a parameter space for degree 4 hypersurfaces in $\mathbb{P}(1, 1, 2)$.

Then $X_A \simeq \mathbb{P}(1, 1, 2)$ and $\mathcal{O}_X(4)$ is very ample. Explicitly,

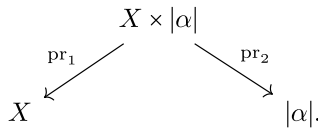
$$X_A = \overline{\{[x^4 : x^3y : x^2y^2 : xy^3 : y^3 : x^2z : xyz : y^2 : z^2] \mid x, y, z \in k^*\}} \subset \mathbb{P}^8.$$

4.2. Invariance of the A-discriminant

In this section, we prove that the A -discriminant of a toric variety X is a semi-invariant for the action of the automorphism group of X on $\mathbb{P}(S_\alpha)$. Since the unipotent radical U of $\text{Aut}(X)$ admits no characters, it follows that the discriminant is a true U -invariant. To prove this, and to put ourselves in a better position to study the moduli spaces we shall construct in § 5, we prove some results on the geometry of the discriminant locus.

Let $X = X_\Sigma$ be a projective toric variety and $\alpha \in \text{Cl}(X)$ an effective class. By effective class, we mean a class such that the linear system $|\alpha|$ is non-empty. Let us fix some notation: let $G = \text{Aut}_\alpha(X)$ and let $T \subset X$ be the torus in X . By the definition of toric varieties, the action of T on itself extends to an action on X . Thus, we have a map $T \hookrightarrow \text{Aut}(X)$, which is injective since T acts faithfully on itself. In fact we have a morphism $T \hookrightarrow \text{Aut}^0(X) \subset \text{Aut}_\alpha(X)$ since T is connected.

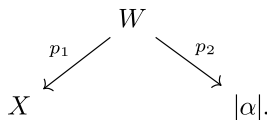
Recall that $|\alpha| = \mathbb{P}(S_\alpha)$. Consider the projection maps.



We define the closed set:

$$W = \{(x, [f]) \in X \times |\alpha| \mid f_i(x) = 0 \text{ for } 0 \leq i \leq r\} \subset X \times |\alpha|,$$

where $f_i = \frac{\partial f}{\partial x_i}$ and $r = |\Sigma(1)| - 1$. We define the restrictions of the projection maps.



Clearly, we have a G -action on $X \times |\alpha|$ given by $g \cdot (x, [f]) = (g \cdot x, g \cdot [f])$. With respect to this action W is an invariant subscheme. Indeed, for $g \in G$ and $(x, [f]) \in W$ we have $\frac{\partial(g \cdot f)}{\partial x_i}(g \cdot x) = f_i(x) = 0$ for every i , and hence $g \cdot (x, [f]) = (g \cdot x, g \cdot [f]) \in W$. We prove the following result describing the flattening stratification of the morphism p_1 .

Proposition 4.7. *For any point $x_0 \in X$, the fibre $p_1^{-1}(x_0) \subset |\alpha|$ is a linear subspace. Moreover, suppose that $x, y \in X$ are in the same G -orbit, then $p_1^{-1}(x) \cong p_1^{-1}(y)$.*

Proof. We can describe the fibre explicitly:

$$\begin{aligned}
 p_1^{-1}(x_0) &= \{(x_0, [f]) \mid \frac{\partial f}{\partial x_i}(x_0) = 0 \ 0 \leq i \leq r\} \\
 &= \bigcap_{i=0}^r \{(x_0, [f]) \mid \frac{\partial f}{\partial x_i}(x_0) = 0\}.
 \end{aligned}$$

Each of the sets $\{(x_0, [f]) \mid \frac{\partial f}{\partial x_i}(x_0) = 0\}$ is the vanishing of a linear polynomial in the coefficients of the polynomial f . It follows that $p_1^{-1}(x_0)$ is the intersection of hyperplanes and thus a linear subspace.

For $g \in G$, we have that:

$$g \cdot p_1^{-1}(x_0) = p_1^{-1}(g \cdot x_0),$$

as $g \cdot (x_0, [f]) = (g \cdot x_0, g \cdot [f])$. In particular, they are all linear subspaces of the same dimension. □

Proposition 4.7 implies that the map p_1 is flat when restricted to the the G -sweep of the torus.

Corollary 4.8. *Define $W' = p_1^{-1}(G \cdot T) \subset W$. Then the map $p_1|_{W'} : W' \rightarrow G \cdot T$ is flat.*

Proof. Since $B := G \cdot T = \bigcup_{g \in G} g \cdot T \subset X$ is an open subset of X , it is an integral noetherian scheme. Then

$$B \times |\alpha| = |\alpha|_B \subset |\alpha|_X = X \times |\alpha|,$$

is open and $W \subset |\alpha|_X$ is closed, so $W' \subset |\alpha|_B$ is a closed subscheme. To see that all fibres over points in B have the same Hilbert polynomial, we observe that, since the torus acts transitively on itself, $G \cdot x = G \cdot T = B$ for all $x \in T$. So applying Lemma 4.7, we have that all the fibres over B are linear subspaces of the same dimension and thus have the same Hilbert polynomial. Hence we can apply [20, Theorem III.9.9] and conclude that $p_1|_{W'}$ is a flat morphism. □

By [19, IV.2, Corollaire 2.3.5 (iii)], we know that a flat map to an irreducible variety with irreducible generic fibre has an irreducible source. The result holds more generally for open maps (see [30, Tag 004Z]).

Proposition 4.9. *Let W' be defined as in Corollary 4.8. Then W' is irreducible.*

Proof. Consider the map $p_1|_{W'} : W' \rightarrow G \cdot T$. Since X is irreducible and $G \cdot T$ is open, $G \cdot T$ is also irreducible. By Corollary 4.8, $p_1|_{W'}$ is flat and hence open. By Lemma 4.7, every fibre is isomorphic to the same projective space, and hence all fibres are irreducible.

Hence we can apply [19, IV.2, Corollaire 2.3.5 (iii)] to $p_1|_{W'}$ and conclude that W' is irreducible. □

We are now in position to prove the main theorem of this section.

Theorem 4.10. *Suppose that $X = X_\Sigma$ is a complete toric variety and that $\alpha \in \text{Cl}(X)$ is a class such that $|\alpha|$ is non-empty. Let $G = \text{Aut}_\alpha(X)$ be the automorphism group preserving α . Let $A = A_{\Sigma, \alpha} \in \mathbb{Z}^{r \times N}$ be defined as in Definition 4.2. Then the discriminant locus $\nabla_A \subset |\alpha|$ has the following description:*

$$\nabla_A = \overline{\{[f] \in |\alpha| \mid \exists x_0 \in G \cdot T \text{ such that } \frac{\partial f}{\partial x_i}(x_0) = 0 \text{ for all } i\}}.$$

In particular, ∇_A is a G -invariant subvariety of $|\alpha|$.

Proof. Note that by the definition of ∇

$$\nabla_A = \overline{\{[f] \in |\alpha| \mid \exists x \in T \text{ such that } \frac{\partial f}{\partial x_i}(x) = 0 \text{ for all } i\}} = \overline{p_2(p_1^{-1}(T))},$$

where $T \subset X$ is the torus. Then since $T \subset G \cdot T$, it holds that $p_1^{-1}(T) \subset p_1^{-1}(G \cdot T)$. Thus

$$\overline{p_2(p_1^{-1}(T))} \subset \overline{p_2(p_1^{-1}(G \cdot T))}.$$

Applying the definition of W' and the observation that $\overline{p_2(p_1^{-1}(T))} = \nabla_A$, we conclude:

$$\nabla_A \subset \overline{p_2(W')}.$$

Then since W' is irreducible, $\overline{p_2(W')}$ is irreducible. Also note that $\text{codim } p_2(W') \geq 1$, since the quasismooth locus in $|\alpha|$ is open and W' is disjoint from $|\alpha|^{\text{qs}}$. Hence $\overline{p_2(W')}$ is an irreducible closed subvariety of codimension 1. Then as ∇_A is an irreducible subvariety of codimension 1, we conclude that:

$$\nabla_A = \overline{p_2(W')},$$

which completes the first part of the theorem.

Now we prove that ∇_A is G -invariant. Note that both maps p_1 and p_2 are G -equivariant since they are restrictions of projections. Then as $G \cdot T$ is G -invariant it follows that $W' = p_1^{-1}(G \cdot T)$ is G -invariant and thus $\nabla_A = \overline{p_2(W')}$ is also G -invariant. □

Remark 4.11. This means that the A -discriminant will check for hypersurfaces with singularities on the G -sweep of the torus in X . Note that by Remark 4.5 we have the inclusion:

$$|\alpha|^{\text{QS}} \subseteq |\alpha|_{\Delta_A}.$$

In general these subvarieties do not coincide.

Corollary 4.12. *Keep the notation of Theorem 4.10. The A -discriminant Δ_A is a semi-invariant section for the G -action on $|\alpha|$ and a true U -invariant, where $U \subset G$ is the unipotent radical of G .*

Proof. By definition, $\nabla_A = \mathbf{V}(\Delta_A) \subset |\alpha|$. The automorphism group G acts on $H^0(|\alpha|, \mathcal{O}_{|\alpha|}(\text{deg } \Delta_A))$. Since ∇_A is G -invariant, for every $g \in G$ we have that $\mathbf{V}(g \cdot \Delta_A) = \mathbf{V}(\Delta_A)$. Thus $g \cdot \Delta_A = \chi(g)\Delta_A$ for some $\chi(g) \in k^*$. It follows from the group action laws that $\chi(g'g) = \chi(g')\chi(g)$ and thus $\chi : g \mapsto \chi(g)$ is a character. This proves the result. \square

Remark 4.13. The character for which the A -discriminant is a semi-invariant is denoted by χ_A and we denote the degree by $r = \text{deg } \Delta_A$. Thus if we consider the action of G on $|\alpha|$ linearised with respect to $(\mathcal{O}(r), \chi_A)$ (or any linearisation on the ray in $\text{Pic}^G(|\alpha|)_{\mathbb{Q}}$ defined by this linearisation), it follows that $|\alpha|^{\text{QS}}$ lies in the *naively semistable locus* defined in [6]. If this linearisation is also well-adapted (or if G is reductive), then it follows that $|\alpha|^{\text{QS}}$ lies in the semistable locus and thus there exists a categorical quotient. See Section 5.4 for an example.

Example 4.14. Let $X = \mathbb{P}^n$ be standard projective space and $d > 0$ a positive integer. In this case $G \cdot T = \mathbb{P}^n$, since the action of $G = \text{GL}_{n+1}$ on \mathbb{P}^n is transitive. In this case we have that $\nabla_A = \nabla$ is the classical discriminant and that quasismoothness is equivalent to smoothness since X is smooth. Thus,

$$|\mathcal{O}(d)|^{\text{QS}} = \mathbb{P}(k[x_0, \dots, x_n]_d)^{\text{SM}} = \mathbb{P}(k[x_0, \dots, x_n]_d) - \nabla.$$

This is the ideal situation. The quasismooth locus is given by the vanishing of one invariant section. In general this won't be true. However, we can generalise a little: for an arbitrary complete toric variety X , we have that $G \cdot T = X$ if and only if the action of G on X is transitive, and thus by [9] X is a product of projective spaces.

Example 4.15. Let $X = \mathbb{P}(1, \dots, 1, r) = \text{Proj } k[x_0, \dots, x_{n-1}, y]$ be the rational cone of dimension n , let G the automorphism group of X and let $d = d'r > 1$ an integer divisible by r . Then X has a single isolated singularity at $(0 : \dots : 0 : 1)$. Let $S_d = k[x_0, \dots, x_{n-1}, y]_d$, where $\text{deg } x_i = 1$ and $\text{deg } y = r$. Suppose that $F \in S_d$ is a weighted homogeneous polynomial; then,

$$F(x_0, \dots, x_{n-1}, y) = \sum_{j=0}^{d'} F_j(x_0, \dots, x_{n-1})y^j,$$

where the $F_j \in k[x_0, \dots, x_{n-1}]_{d-rj}$ are homogeneous (possibly 0) polynomials of degree $d - jr$. Note that $F_{d'} \in k$ is a constant, write $F_{d'} = c \in k$, then $F(0, \dots, 0, 1) = c$.

Thus $(0 : \dots : 0 : 1) \in \mathbf{V}(F)$ if and only if $c = 0$. Moreover, if $c = 0$ then $(0 : \dots : 0 : 1)$ is a singular point of $\mathbf{V}(F)$. Indeed, the derivatives are given by:

$$\frac{\partial F}{\partial x_i}(x_0, \dots, x_{n-1}, y) = \sum_{j=0}^{d'} \frac{\partial F_j}{\partial x_i}(x_0, \dots, x_{n-1})y^j,$$

$$\frac{\partial F}{\partial y}(x_0, \dots, x_{n-1}, y) = \sum_{j=1}^{d'} jF_j(x_0, \dots, x_{n-1})y^{j-1}.$$

Since $d > 1$, the $\frac{\partial F_j}{\partial x_i}$ are either 0 or non-constant homogeneous polynomials in the x_i . Thus $\frac{\partial F}{\partial x_i}(0, \dots, 0, 1) = 0$ for every i and $\frac{\partial F}{\partial y}(0, \dots, 0, 1) = d'c$. Thus the point $(0 : \dots : 0 : 1)$ is a singular point if and only if $c = 0$. Note that this means for hypersurfaces in X , quasismooth is equivalent to being smooth.

We can write down explicitly the non-quasismooth locus:

$$|\alpha|^{\text{NQS}} = |\alpha| - |\alpha|^{\text{QS}} = \nabla_A \cup \mathbf{V}(c) = \mathbf{V}(\Delta_A \cdot c),$$

where $\nabla_A = \mathbb{P}(1, \dots, 1, r)^{\vee, d}$ and we are considering c as a coordinate on $|\alpha|$. In this example $G \cdot T = X - \{(0 : \dots : 0 : 1)\}$. To see this, note that

$$G = \text{Aut}(X) = ((\mathbb{G}_m \times \text{GL}_n) \ltimes \mathbb{G}_a^M) / \mathbb{G}_m,$$

and that: $\text{GL}_n \hookrightarrow \text{Aut}(X)$ acts transitively on the set $\{(x_0 : \dots : x_{n-1} : 1) \mid x_i \neq 0 \text{ for some } i\} \subset X$. We prove in Proposition 5.12 that the unipotent radical is abelian and that:

$$M = \begin{pmatrix} n - 1 + r \\ r \end{pmatrix}.$$

5. Moduli of quasismooth hypersurfaces

In this section, we construct coarse moduli spaces of quasismooth hypersurfaces of fixed degree in certain of toric orbifolds. We prove that quasismooth hypersurfaces of weighted projective space (excluding some low degrees) are stable when the (\mathfrak{C}) condition is satisfied for the action of a grading of the unipotent radical of the automorphism group of this weighted projective space. Once stability is established, we apply the non-reductive GIT Theorem (Theorem 2.16) to conclude that a coarse of moduli space of quasismooth hypersurfaces exists as a quasi-projective variety. Moreover, Theorem 2.16 provides a compactification of this moduli space. We also discuss the (\mathfrak{C}) condition and show that it holds for certain weighted projective spaces. We give examples when it does not hold; in this case, one should be able to construct moduli spaces of quasismooth hypersurfaces using the blow-up procedure in [4].

We also consider smooth hypersurfaces in products of projective spaces and prove that smoothness implies semistability. If we suppose further that the degree is such that the

hypersurfaces are of general type, then we prove that smoothness implies stability. Hence we construct a coarse moduli space of such hypersurfaces.

5.1. \hat{U} -stability for quasismooth hypersurfaces in weighted projective space

Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space and assume that $a_0 \leq \dots \leq a_n$. Let us give coordinates on X as in §3. Label the distinct weights $b_1 < \dots < b_l$ so that $a_0 = b_1$ and $a_n = b_l$. Let

$$X = \text{Proj } k[x_1, \dots, x_m, y_1, \dots, y_{n_l}],$$

such that variables x_1, \dots, x_m have weights less than a_n and variables y_1, \dots, y_{n_l} have maximum weight a_n . If all the weights coincide then $X = \mathbb{P}^n$, so we disregard this case.

Let d be a positive integer such that $a_j \mid d$ for $0 \leq i \leq n$ so that hypersurfaces of degree d of X are Cartier divisors; recall that we call such an integer a *Cartier degree*.

Notation 5.1. We give the parameter space:

$$\mathcal{Y}_d = \text{Div}_X^d = \mathbb{P}(k[x_1, \dots, y_{n_l}]_d),$$

the following coordinates of the coefficients of the monomials: $(u_0 : \dots : u_L : v_0 : \dots : v_M) \in \mathcal{Y}_d$, where the v_j correspond to monomials in the y_j and all have the same total degree, and the u_i are the coefficients of monomials containing an x_i for some $1 \leq i \leq n'$. The integer M is defined by:

$$M = \binom{n_l + d_n}{d_n},$$

where $d_n = \frac{d}{a_n}$ and the integer L is computed in terms of the a_i 's, but its exact value is not required for the subsequent discussion.

Recall $G = \text{Aut}_g(S)$ and that:

$$G \simeq \prod_{i=1}^l \text{GL}_{n_i} \rtimes U,$$

where U is the unipotent radical and that $b_1 < \dots < b_l$ are the distinct values of a_0, \dots, a_n with each b_i occurring with multiplicity n_i . By Proposition 3.14, the 1-parameter subgroup of G given by:

$$\lambda_{g,N} : t \mapsto ((t^{-N} \text{Id}_{n_i})_{i=1}^{l-1}, t \text{Id}_{n_l}, 0),$$

for $N > 0$ defines a positive grading of the unipotent radical of G and we define the graded unipotent group $\hat{U}_N = \lambda_{g,N}(\mathbb{G}_m) \rtimes U$.

Remark 5.2. Note that \hat{U}_N depends on the integer $N > 0$: that is, for different values of N , the subgroups $\lambda_{g,N}(\mathbb{G}_m) \times U$ are different. However, we shall see in Remark 5.4 that the semistable and stable locus for \hat{U}_N is the same for all $N \gg 0$.

Let G act on \mathcal{Y}_d with respect to the linearisation $\mathcal{O}(1)$. Suppose that \underline{x}^I is a monomial in $k[x_1, \dots, x_{n'}, y_1, \dots, y_{n_l}]_d$. Then

$$\lambda_{g,N}(t) \cdot \underline{x}^I = t^{r(N,I)} \underline{x}^I,$$

where $r(N, I) \in \mathbb{Z}$ is an integer depending on N and the monomial. Note that for $\underline{y}^I \in k[y_1, \dots, y_{n_l}]_d$ we have that $r(N, I) = -d_n$ is independent of N , and

$$\lambda_{g,N}(t) \cdot \underline{y}^I = t^{-d_n} \underline{y}^I.$$

Recall the definition of Z_{\min} and $(\mathcal{Y}_d)_{\min}^0$ from Definition 2.7. Both subsets are defined with respect to a \hat{U}_N -action.

Lemma 5.3. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ and $d \in \text{Pic}(X)$ be a Cartier degree. Fix $\hat{U}_N = \lambda_{g,N}(\mathbb{G}_m) \times U$. Then*

$$Z_{\min} = \{(0 : \dots : 0 : v_0 : \dots : v_M) \mid \exists j : v_j \neq 0\} \subset \mathcal{Y}_d,$$

and the minimum weight of the $\lambda_{g,N}(\mathbb{G}_m)$ -action on $V = H^0(\mathcal{Y}_d, \mathcal{O}(1))^\vee$ is $\omega_{\min} = -d_n$.

Note that both Z_{\min} and ω_{\min} are independent of N .

Proof. As noted above, $\lambda_{g,N}$ acts on monomials containing only variables y_i with weight $-d_n$. Suppose that $\underline{x}^I \in V = k[x_1, \dots, y_{n_l}]_d$ is another monomial containing at least one x_i variable. Then $\lambda_{g,N}(t) \cdot \underline{x}^I = t^{r(N,I)} \underline{x}^I$ and since $\lambda_{g,N}(t) \cdot x_i = t^N x_i$ we have that $r(N, I) > -d_n$. Hence $V_{\min} = k[y_1, \dots, y_{n_l}]_d$ and thus

$$Z_{\min} = \mathbb{P}(V_{\min}) = \{(0 : \dots : 0 : v_0 : \dots : v_M) \mid \exists j : v_j \neq 0\},$$

using Notation 5.1. □

Remark 5.4. It follows from the lemma that:

$$(\mathcal{Y}_d)_{\min}^0 = \{(u_0 : \dots : u_{M'} : v_0 : \dots : v_M) \mid \exists j : v_j \neq 0\} \subset \mathcal{Y}_d.$$

The following lemma follows immediately from Lemma 5.3.

Corollary 5.5. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ such that $a_n > a_{n-1} \geq \dots \geq a_0$. Then for every Cartier degree $d \in \mathbb{Z}$, we have*

$$Z_{\min} = \{(0 : \dots : 0 : 1)\} \subset \mathcal{Y}_d,$$

is a point.

Example 5.6. Let $X = \mathbb{P}(1, 1, 2) = \text{Proj } k[x_1, x_2, y]$ and $d = 2$. Then the monomial basis for $V = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(1))^\vee$ is $(x_1^2, x_1x_2, x_2^2, y)$. Writing down an arbitrary polynomial

$$[f(x_1, x_2, y)] = [u_1x_1^2 + u_2x_1x_2 + u_3x_2^2 + vy] \in \mathcal{Y}_2,$$

in coordinates gives $(u_1 : u_2 : u_3 : v) \in \mathcal{Y}_2$. Now let us calculate the weights for the grading \mathbb{G}_m -action defined by $\lambda_{g,N}$, with $N > 0$. For positive integers i and j such that $i + j = 2$ we have:

$$\lambda_{g,N}(t) \cdot x_1^i x_2^j = (t^N x_1)^i (t^N x_2)^j = t^{2N} x_1^i x_2^j \quad \text{and} \quad \lambda_{g,N}(t) \cdot y = t^{-1}y.$$

Hence, we have two distinct weights $2N$ and 1 and the decomposition into weight spaces is given by:

$$V = V_{2N} \oplus V_{-1} = \text{Span}(x_1^2, x_1x_2, x_2^2) \oplus \text{Span}(y).$$

Thus, $Z_{\min} = \mathbb{P}(V_{-1}) = \{(0 : 0 : 0 : 1)\}$.

Notation 5.7. For the rest of this section, we fix $N > 0$ and $\hat{U} = \lambda_{g,N} \times U$.

Proposition 5.8. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space and d be a Cartier degree. Denote $\mathcal{Y} = \mathcal{Y}_d$. Then we have the following inclusion:

$$\mathcal{Y}^{QS} \subset \mathcal{Y}_{\min}^0 - U \cdot Z_{\min}.$$

Proof. We begin by observing that

$$\mathcal{Y} - \mathcal{Y}_{\min}^0 = \{(u_0 : \dots : u_{M'} : 0 : \dots : 0)\}.$$

Take some $f \in \mathcal{Y} - \mathcal{Y}_{\min}^0$. We know that f contains no monomials made up of only the y_i . Thus $(0 : \dots : 0 : 1) \in X$ will be a common zero for all $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial y_i}$ since d is a Cartier degree (as monomials of the form $y_{n_l}^{d_n-1} x_i$ can never be homogeneous of degree d). It follows that f is not quasismooth and hence

$$\mathcal{Y}^{QS} \subset \mathcal{Y}_{\min}^0.$$

Suppose that $f \in Z_{\min}$. Then f is a polynomial in the y_i and so $(1 : 0 : \dots : 0) \in X$ will be a common zero for all $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial y_i}$. Thus f is not quasismooth and we have that:

$$Z_{\min} \subset \mathcal{Y}^{NQS}.$$

Since \mathcal{Y}^{NQS} is a G -invariant subset, it follows that:

$$U \cdot Z_{\min} \subset \mathcal{Y}^{NQS}$$

and so we conclude that:

$$\mathcal{Y}^{\text{QS}} \subset \mathcal{Y}_{\min}^0 - U \cdot Z_{\min}.$$

□

Suppose that $T \subset G = \text{Aut}_g(S)$ is the maximal torus of G given by diagonal matrices and define $\bar{T} = T/\lambda_{\underline{a}}(\mathbb{G}_m)$ to be the quotient by the 1-parameter subgroup $\lambda_{\underline{a}}$. Recall from § 2.1 that the stability of a hypersurface with respect to T is determined by its weight polytope considered inside the character space $H = X^*(\bar{T}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{n+1}/\mathbb{Q} \cdot \underline{a}$. The weight polytope is a subpolytope of the section polytope and in the case of hypersurfaces we refer to the weight polytope for the canonical maximal torus as the Newton polytope (see Definition 5.10 below).

Definition 5.9. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space and $d \in \mathbb{Z}_{>0}$ be a Cartier degree. Consider $H^0(X, \mathcal{O}_X(d)) = k[x_0, \dots, x_n]_d$ and define:

$$A = \{(i_0, \dots, i_n) \in \mathbb{Z}^{n+1} \mid i_j \geq 0 \text{ and } \sum a_j i_j = d\}$$

to be the set of exponent vectors of the monomials. The set A is precisely the set of torus weights and let $\tilde{P}_d = \text{Conv}(A) \subset \mathbb{Q}^{n+1}$. We define the section polytope to be $P_d \subset H = \mathbb{Q}^{n+1}/\mathbb{Q} \cdot \underline{a}$, the image of \tilde{P}_d in H . Note that P_d contains the origin, indeed, the origin is its barycentre. One can also work with $\tilde{P}_d \subset \mathbb{Q}^{n+1}$ as the section polytope of X ; both produce the same toric variety, however, \tilde{P}_d is not full-dimensional, so for explicit computations it is often simpler to work in H .

Definition 5.10. For a degree d hypersurface $Y \subset X$, we define the Newton polytope of Y by:

$$\text{NP}(Y) = \text{Conv}(\text{wt}_{\bar{T}}(Y)) \subset H = X^*(\bar{T}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Example 5.11. Let $X = \mathbb{P}(1, 1, 2)$ and consider Figure 2 where we have the section polytope of $\mathcal{O}_X(4)$ and the Newton polytope of $f = xy^3 + x^2z + y^2z + z^2$. Note that by \hat{U} -stability, for a point to be semistable we must have that the circled vertex corresponding to $Z_{\min} = \{z^2\}$ appear in the weight space and thus also in the Newton polytope.

We now show that the condition (C) (see Definition 2.12) holds in the case for weighted projective spaces of the form $\mathbb{P}(a, \dots, a, b, \dots, b)$.

Proposition 5.12. Suppose $X = \mathbb{P}(a, \dots, a, b, \dots, b) = \text{Proj } k[x_1, \dots, x_n, y_1, \dots, y_m]$ is a weighted projective space such that $b > a$ and let $d > b + 1$ be a Cartier degree (so that ab divides d).

Then the graded automorphism group of $S = k[x_1, \dots, x_n, y_1, \dots, y_m]$ is of the following form:

$$\text{Aut}_g(S) = (\text{GL}_n \times \text{GL}_m) \ltimes \mathbb{G}_a^L,$$

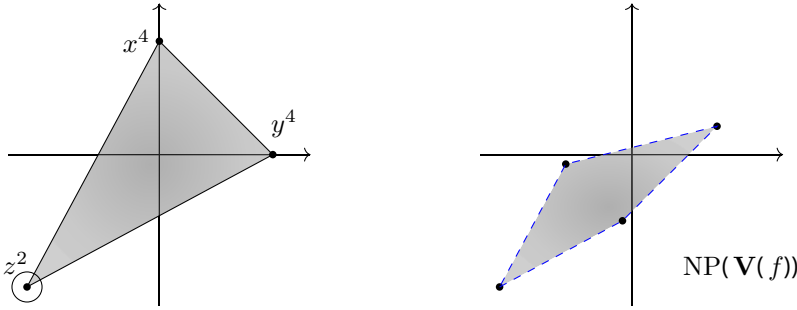


Figure 2. On the left is the section polytope P of $\mathcal{O}(4)$. On the right is the Newton polytope of $\mathbf{V}(f)$ where $f = xy^3 + x^2z + y^2z + z^2$.

where $L \geq 0$. In particular, the unipotent radical $U = \mathbb{G}_a^L$ is abelian. Moreover, the action of G on \mathcal{Y}_d with respect to $\mathcal{O}(1)$ satisfies the condition (\mathfrak{C}) ; that is, the stabiliser group is trivial:

$$\text{Stab}_U([f]) = \{e\},$$

for every $[f] \in Z_{\min}^{ss,R} \subset \mathcal{Y}_d$, where $R \cong \text{GL}_n \times \text{GL}_m$.

Proof. Let $G = \text{Aut}_g(S)$; then a general automorphism in the unipotent radical $\phi \in U \subset G$ is given by:

$$\phi : \begin{cases} x_i \mapsto x_i \\ y_j \mapsto y_j + p_{\phi,j}(x_1, \dots, x_n), \end{cases}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$ and with $p_{\phi,j} \in k[x_1, \dots, x_n]_r$. Composing two such elements $\phi, \psi \in U$ gives

$$\phi \circ \psi : \begin{cases} x_i \mapsto x_i \\ y_j \mapsto y_j + p_{\phi,j}(x_1, \dots, x_n) + p_{\psi,j}(x_1, \dots, x_n). \end{cases}$$

It follows that any two automorphisms commute and hence U is abelian and thus

$$U \simeq \mathbb{G}_a^L.$$

Let us prove the second statement. Let $d' = \frac{d}{b}$ and recall that $Z_{\min} = \mathbb{P}(k[y_1, \dots, y_m]_{d'})$. Take some $[f] \in Z_{\min}^{ss,R}$ and $\phi \in \text{Stab}_U([f])$. Assume that ϕ is non-trivial.

Since the GL_n acts trivially on Z_{\min} it follows that $Z_{\min}^{ss,R} = Z_{\min}^{ss, \text{GL}_m}$. Note that this only holds for semistability. Then considering the standard maximal torus in GL_m , the reductive Hilbert–Mumford criteria imply that the barycentre of the section polytope of $\mathcal{O}_{Z_{\min}}(d')$ is contained in the Newton polytope of $[f]$. Since $d > b + 1$, it follows that each variable y_j appears in $[f]$ and thus $\phi \cdot [f]$ must contain at least one x_i . Hence $\phi \cdot [f] \neq [f]$ and we have a contradiction. \square

This encompasses many interesting examples of weighted projective hypersurfaces. For example smooth degree 4 hypersurfaces in $\mathbb{P}(1, \dots, 1, 2)$ from a large family of Fano varieties of Fano index 2. Moreover, degree 2 smooth del Pezzo surfaces are completely described by degree 4 surfaces in $\mathbb{P}(1, 1, 1, 2)$ and a blow-up of the corresponding moduli space has been shown to describe the K-stable degree 2 del Pezzo surfaces [28]. Other examples of interest [25] are the K-stable del Pezzo surfaces of Fano index 2 which present themselves as degree $2a$ hypersurfaces in $\mathbb{P}(1, 1, a, a)$ for $a \geq 2$.

Remark 5.13. The condition (\mathfrak{C}) is not satisfied for every weighted projective space; for example, consider $X = \mathbb{P}(1, 2, 3) = \text{Proj } k[x, y, z]$ and $d = 6$. Then

$$Z_{\min} = \{(0 : \dots : 0 : 1)\} = \{[z^2]\},$$

is a point by Corollary 5.5 and corresponds to the hypersurface defined by z^2 . However, the additive 1-parameter subgroup of U

$$a(u) : y \mapsto y + ux^2,$$

acts trivially on Z_{\min} .

There are other examples of weighted projective space for which the condition (\mathfrak{C}) is satisfied. For example, let $X = \mathbb{P}(2, 2, 3, 3, 5)$ with coordinates x_1, x_2, y_1, y_2, z such that $\deg x_i = 2$, $\deg y_i = 3$ and $\deg z = 5$. Let $d = 20$ and note that:

$$\text{Aut}(X) = ((\text{GL}_2 \times \text{GL}_2 \times \mathbb{G}_m) \times (\mathbb{G}_a)^4) / \lambda_{\underline{a}}(\mathbb{G}_m).$$

Again we have that Z_{\min} is a point corresponding to the hypersurface z^4 . Then the action of $(\mathbb{G}_a)^4$ is trivial on coordinates x_1, x_2, y_1, y_2 and on z the action is defined by

$$(A_1, A_2, A_3, A_4) \cdot z = z + A_1x_1y_1 + A_2x_1y_2 + A_3x_2y_1 + A_4x_2y_2,$$

where $(A_1, A_2, A_3, A_4) \in (\mathbb{G}_a)^4$. It follows that the $(\mathbb{G}_a)^4$ -stabiliser of $[z^4]$ is trivial.

Remark 5.14. Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space and $d = \text{lcm}(a_0, \dots, a_n)$. If the condition (\mathfrak{C}) holds for the action of $\hat{U} \subset G$ on \mathcal{Y}_d with respect to $\mathcal{O}(1)$, an induction argument on l shows it also holds for the G -action on \mathcal{Y}_{ld} for every $l > 0$. Hence the condition (\mathfrak{C}) for \hat{U} is an intrinsic property of X . Thus we may talk about X satisfying the condition (\mathfrak{C}) .

Remark 5.15. Let us revisit the definition of a well-adapted linearisation from Remark 2.11. The well-adapted requirement is needed in the proof of the \hat{U} -theorem [4, 5] and insures that semistability and stability coincide for the grading \mathbb{G}_m once a high tensor power of the linearisation is taken. However, this high power is determined only by the U -invariants. Hence if we take $N > d'$ then we have the weight diagram shown in Figure 3, where $\omega_{\min+1}$ is the next biggest weight whose value we denote by $r(N)$, which grows linearly with N , see Example 5.6.¹ Thus, we can choose some $N \gg 0$ such

¹ This choice of lower bound $N > d'$ is not optimal; there could exist a smaller N for which we would have the required distribution of ω_{\min} and $\omega_{\min+1}$.

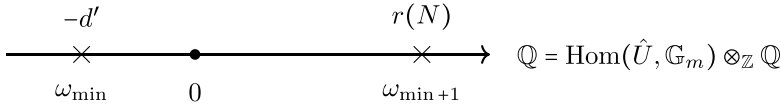


Figure 3. The weight diagram for $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \times U$ for $N \gg 0$.

that the linearisation $\mathcal{O}(1)$ is well-adapted and we can readily apply the \hat{U} -theorem. For the details of the proof we refer to [4, 5] and for a discussion on the variation of the grading 1-parameter subgroup we refer to [7]. From now on we shall assume that we fixed $N \gg 0$.

Corollary 5.16. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a weighted projective space and d be a Cartier degree. Assume that X satisfies the condition (\mathfrak{C}) for the action of \hat{U} on \mathcal{Y}_d with respect to $\mathcal{O}(1)$, where $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \times U \subset G = \text{Aut}_g(S)$ for $N \gg 0$. Then the following statements hold.*

(a) *The quotient morphism:*

$$q_U : \mathcal{Y}_{\min}^0 \rightarrow \mathcal{Y}_{\min}^0/U,$$

is a principal U -bundle.

(b) *The quotient morphism:*

$$q_{\hat{U}} : \mathcal{Y}_d^{s,\hat{U}} \rightarrow \mathcal{Y}_d // \hat{U},$$

is a projective geometric quotient, where $\mathcal{Y}_d^{s,\hat{U}} = \mathcal{Y}_{\min}^0 - U \cdot Z_{\min}$.

(c) *The subset:*

$$\mathcal{Y}^{QS}/\hat{U} = q_{\hat{U}}(\mathcal{Y}^{QS}) \subseteq \mathcal{Y}_d // \hat{U},$$

is open, and thus \mathcal{Y}^{QS}/\hat{U} is quasi-projective.

Proof. The first statement follows directly from the \hat{U} -Theorem (Theorem 2.13) and the second statement follows from the fact that a geometric quotient is an open map (since it is a topological quotient) and that Proposition 5.8 we that $\mathcal{Y}^{QS} \subset \mathcal{Y}^{s,\hat{U}}$ is an open subset. □

5.2. Stability of quasismooth hypersurfaces in weighted projective space

We present a proof that quasismooth hypersurfaces are stable using the non-reductive Hilbert–Mumford criteria of Theorem 2.19. The Hilbert–Mumford criterion says that if G is a linear algebraic group with graded unipotent radical acting on a projective variety Y , then a point $y \in Y$ is stable if and only if every G translate $g \cdot y$ is stable for a maximal torus $T \subset G$ containing the grading \mathbb{G}_m . We shall prove stability of quasismooth hypersurfaces for a maximal torus T and then use the fact that the quasismooth locus is

invariant under the action of the automorphism group and the NRGIT Hilbert–Mumford criterion to deduce stability for G . The proof of T -stability uses the Newton polytope of a hypersurface.

Lemma 5.17. *Suppose that $Y \subset X = \mathbb{P}(a_0, \dots, a_n)$ is a quasismooth hypersurface of degree $d \geq \max\{a_0, \dots, a_n\} + 2$ defined by a weighted polynomial $f \in k[x_0, \dots, x_n]_d$. Then, $NP(Y)^\circ$ contains the barycentre of the section polytope of X .*

This result follows directly from the corresponding result for smooth hypersurfaces in straight projective space.

Proof. Consider the geometric quotient map $q : \mathbb{P}^n \rightarrow X$ defined by the injective map of graded rings $q^* : k[x_0, \dots, x_n] \rightarrow k[\bar{x}_0, \dots, \bar{x}_n]$ sending $x_i \mapsto \bar{x}_i^{a_i}$. Let $\tilde{Y} = q^{-1}(Y) \subset \mathbb{P}^n$, then \tilde{Y} is the hypersurface defined by the polynomial $f(x_0^{a_0}, \dots, x_n^{a_n})$ and is of degree $d \geq \max\{a_0, \dots, a_n\} + 2 \geq 3$. As Y is quasismooth, we have that \tilde{Y} is smooth (see for example [8, Proposition 3.5]). This implies that the barycentre of the section polytope of \mathbb{P}^n is contained in the interior of the Newton polytope of \tilde{Y} by the reductive moduli problem for hypersurfaces of degree $d \geq 3$ [27, Proposition 4.2].

Let $P = \text{Conv}(de_0, \dots, de_n)$ denote the section polytope of \mathbb{P}^n and $P_X = \text{Conv}(d_0e_0, \dots, d_n e_n)$ denote the section polytope of X , where $d_i = \frac{d}{a_i}$. Then the respective barycentres are given by $B = \frac{1}{n+1} \sum_{i=0}^n de_i$ and $B_X = \frac{1}{n+1} \sum_{i=0}^n d_i e_i$.

Consider the map $\varphi : \mathbb{Q}^{n+1} \rightarrow \mathbb{Q}^{n+1}$ given by $\varphi(e_i) = \frac{1}{a_i} e_i$ for $i = 0, \dots, n$, where the e_i are the unit vectors. Note that $\varphi(B) = B_X$. Under this linear map, the Newton polytope of \tilde{Y} is sent to the Newton polytope of Y . Moreover, $B \in NP(\tilde{Y})^\circ$ implies that $B_X \in NP(Y)^\circ$. □

We are now in a position to prove the main theorem of the section.

Theorem 5.18. *Let $X = \mathbb{P}(a_0, \dots, a_n)$ be a well-formed weighted projective space such that $n > 1$ which satisfies the condition (C) and let $d \geq \max\{a_0, \dots, a_n\} + 2$ be a Cartier degree. Let $G = \text{Aut}_g(S)$ be the graded automorphism group of the Cox ring and consider the action of G on $\mathcal{Y} = \mathcal{Y}_d$ linearised with respect to $\mathcal{O}(1)$. Define the graded unipotent radical $\hat{U} = \lambda_{g,N}(\mathbb{G}_m) \rtimes U \subset G$ for some fixed $N \gg d$. We have the inclusion:*

$$\mathcal{Y}^{QS} \subset \mathcal{Y}^{s,G}.$$

In particular, there exists a geometric quotient \mathcal{Y}^{QS}/G and hence a coarse moduli space of quasismooth hypersurfaces as a quasi-projective variety. Moreover, $\mathcal{Y} // G$ is a compactification of \mathcal{Y}^{QS}/G .

Proof. We drop the linearisation from the notation. Note that it suffices to prove that $\mathcal{Y}^{QS} \subset \mathcal{Y}^{s,T}$, since by the non-reductive Hilbert–Mumford criterion of Theorem 2.19

we get $\mathcal{Y}^{\text{QS}} \subset \mathcal{Y}^{\text{s},G}$. Indeed, as \mathcal{Y}^{QS} is G -invariant, we have that $\mathcal{Y}^{\text{QS}} \subset g \cdot \mathcal{Y}^{\text{s},T}$ and hence

$$\mathcal{Y}^{\text{QS}} \subset \bigcap_{g \in G} g \cdot \mathcal{Y}^{\text{s},T} = \mathcal{Y}^{\text{s},G},$$

by Theorem 2.19.

Let us prove that $\mathcal{Y}^{\text{QS}} \subset \mathcal{Y}^{\text{s},T}$. Suppose that $Y \subset X$ is a quasismooth hypersurface of degree d . Then by Lemma 5.17, the polytope $\text{NP}(Y)$ contains the origin O . Thus by the Hilbert–Mumford criterion of Theorem 2.18 quasismooth hypersurfaces are T -stable for the twisted linearisation $\mathcal{O}(1)$. □

Remark 5.19. In the case where the condition (\mathfrak{C}) is not satisfied, there is a blow-up procedure outlined in [4] where one performs a sequence of blow-ups of the locus in \mathcal{Y} where there is a positive dimensional U -stabiliser. Using this procedure it is expected that we can remove the requirement that the (\mathfrak{C}) condition holds. This will be pursued in further work.

5.3. Explicit construction for $\mathbb{P}(1, \dots, 1, r)$

We provide an explicit construction of a coarse moduli spaces of quasismooth hypersurfaces in the case where $X = \mathbb{P}(1, \dots, 1, r) = \text{Proj } k[x_1, \dots, x_n, y]$ and $d = d' \cdot r$ with $d' > 0$ and $n > 1$. This example illustrates the ‘quotienting in stages’ procedure.

We give a direct construction of these coarse moduli space using Lemma 2.4. Recall that a projective over affine variety X is a variety admitting a projective morphism $X \rightarrow A$, where A is an affine variety.

Theorem 5.20. *Let $X = \mathbb{P}(1, \dots, 1, r)$ and $d = d' \cdot r$ be a Cartier degree such that $d \geq r + 2$. Let $\mathcal{Y} = \mathbb{P}(k[x_1, \dots, x_n, y]_d)$ be the parameter space of degree d hypersurfaces. Then there exists a geometric quotient for the G -action on \mathcal{Y}^{QS} :*

$$\mathcal{Y}^{\text{QS}} \longrightarrow \mathcal{Y}^{\text{QS}}/G,$$

which is coarse moduli space and a projective over affine variety.

Proof. Let $c \in H^0(\mathcal{Y}, \mathcal{O}(1)) = (k[x_1, \dots, x_n, y]_d)^\vee$ be the section corresponding to the coefficient of the monomial $y^{d'}$. By Example 4.15, we have that $\mathcal{Y}^{\text{QS}} = \mathcal{Y}_{c, \Delta_A} \subset \mathcal{Y}_c$ and hence \mathcal{Y}^{QS} is an affine variety. In this case, $Z_{\min} = \{(0 : \dots : 0 : c) \mid c \neq 0\}$ is a point, and so by Remark 2.14 the quotient:

$$q_U : \mathcal{Y}_{\min}^0 \longrightarrow \mathcal{Y}_{\min}^0/U,$$

from Corollary 5.16 is a trivial U -bundle. Hence \mathcal{Y}_{\min}^0/U is affine by [3, Theorem 3.14]. Thus $\mathcal{Y}^{\text{QS}} \rightarrow \mathcal{Y}^{\text{QS}}/U$ is a trivial bundle and $Q = \mathcal{Y}^{\text{QS}}/U$ is an affine variety.

Consider the action of $R = G/U$ on Q . Since \mathcal{Y}^{QS} is affine, Lemma 2.4 implies that Q admits a geometric quotient by R if and only if all the G -orbits are closed in \mathcal{Y}^{QS} .

Then as $d \geq r + 2$, Theorem 3.15 implies that all stabiliser groups are finite giving that the action on \mathcal{Y}^{QS} is closed. Hence we have a geometric quotient:

$$\mathcal{Y}^{\text{QS}}/G = Q/R.$$

Since Q is an affine variety and Q/R is a reductive quotient, we conclude that $\mathcal{Y}^{\text{QS}}/G$ is a projective over affine variety. □

Example 5.21. Suppose that $X = \mathbb{P}(1, 1, 2)$ and $d = 6$. Then quasismooth hypersurfaces are exactly Petri special curves of genus 4 in X . Thus $(\mathcal{Y}_6)^{\text{QS}}/G$ is an projective over affine coarse moduli space of Petri special curves. This moduli space is a divisor on the moduli space of genus 4 curves (see [31]).

Example 5.22. Let $X = \mathbb{P}(1, 1, 1, 2)$ and consider $d = 4$. In this case quasismoothness coincides with smoothness. The smooth hypersurfaces are exactly degree 2 del Pezzo surfaces. Hence $(\mathcal{Y}_4)^{\text{QS}}/G$ is a projective over affine coarse moduli space of degree 2 del Pezzo surfaces.

5.4. Stability of hypersurfaces in products of projective space

Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and $(d, e) \in \mathbb{Z}^2 \simeq \text{Pic}(X)$. Then $\text{Aut}_g(S) = \text{GL}_{n+1} \times \text{GL}_{m+1}$ and we consider the action of G on the projective space $\mathcal{Y} = \mathbb{P}(k[x_0, \dots, x_n; y_0, \dots, y_m]_{(d,e)})$. Note that every stabiliser contains the subgroup:

$$\begin{aligned} \lambda : \mathbb{G}_m^2 &\longrightarrow \text{Aut}_g(S) \\ (t_1, t_2) &\longmapsto (t_1 I_{n+1}, t_2 I_{m+1}). \end{aligned}$$

We may replace the action of $\text{Aut}_g(S)$ with the action of the subgroup $G = \text{SL}_{n+1} \times \text{SL}_{m+1} \subset \text{Aut}_g(S)$, as the orbits are the same. By doing this, we remove the global stabiliser and moreover, we are now in the situation where G has no non-trivial characters and so Δ_A is a true invariant for the G -action.

Proposition 5.23. *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ be a product of projective spaces and $Y \subset X$ be a smooth hypersurface of degree $(d, e) \in \mathbb{Z}^2$. If $d > n + 1$ and $e > m + 1$, then $\dim \text{Stab}_G(Y) = 0$.*

Proof. Since Y is a proper algebraic scheme, by [26, Lemma 3.4], we can identify the Lie algebra of the automorphism group of Y with the vector space $H^0(Y, \mathcal{T}_Y)$ of global vector fields on Y , where \mathcal{T}_Y is the tangent sheaf. If we show that $H^0(Y, \mathcal{T}_Y) = 0$, then we can conclude that $\dim \text{Aut}(Y) = 0$ and since $\text{Stab}_G(Y) \subset \text{Aut}(Y)$ we have that $\dim \text{Stab}_G(Y) = 0$.

Let us prove that $H^0(Y, \mathcal{T}_Y) = 0$. Let $N = \dim Y = n + m - 1$; then by Serre duality:

$$H^0(Y, \mathcal{T}_Y) \simeq H^N(Y, \Omega_Y \otimes \omega_Y)^\vee,$$

where ω_Y is the canonical line bundle of Y . Let $\mathcal{O}_Y(1, 1)$ be the restriction of $\mathcal{O}_X(1, 1) = \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(1)$ to Y . By the adjunction formula we have that $\omega_Y \cong \mathcal{O}_Y(d - n - 1, e -$

$m - 1$) and hence by our assumption we have that ω_Y is very ample. Then by Kodaira-Nakano vanishing ([17, Theorem 1.3]) we have that $H^N(Y, \Omega_Y(d - n - 1, e - m - 1)) = 0$. Hence

$$H^0(Y, \mathcal{T}_Y) \simeq H^N(Y, \Omega_Y(d - n - 1, e - m - 1))^\vee = 0.$$

We conclude that $\dim \text{Aut}(Y) = 0$. □

Theorem 5.24. *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ be a product of projective spaces and $Y \subset X$ be a smooth hypersurface of degree $(d, e) \in \mathbb{Z}^2$. Consider the action of $G = \text{SL}_{n+1} \times \text{SL}_{m+1}$ on $\mathcal{Y} = \mathbb{P}(k[x_0, \dots, x_n; y_0, \dots, y_m]_{(d,e)})$ with linearisation given by $\mathcal{O}_{\mathcal{Y}}(1)$. Then, we have the open inclusion:*

$$\mathcal{Y}^{SM} \subset \mathcal{Y}^{ss}(\mathcal{O}(1)),$$

where \mathcal{Y}^{SM} is the of smooth hypersurfaces. If $d > n + 1$ and $e > m + 1$ then we have the open inclusion:

$$\mathcal{Y}^{SM} \subset \mathcal{Y}^s(\mathcal{O}(1)).$$

In particular, there exists a coarse moduli space of smooth hypersurfaces of degree (d, e) .

Proof. First, note that the discriminant Δ_A is a true invariant for the G -action since G has no non-trivial characters. Then by Theorem 4.10, we have that:

$$\mathcal{Y}^{SM} = \mathcal{Y}_{\Delta_A},$$

since G acts transitively on X and hence $\mathcal{Y}^{SM} \subset \mathcal{Y}^{ss}$. Finally, if $d > n + 1$ and $e > m + 1$, by Proposition 5.23, we have that the stabiliser for every point in \mathcal{Y}^{SM} is finite and hence all the orbits are closed. It follows that $\mathcal{Y}^{SM} \subset \mathcal{Y}^s$. □

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