ON EXTREMAL PROPERTIES OF THE DERIVATIVES OF POLYNOMIALS AND RATIONAL FUNCTIONS

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Introduction. Let $p(z)$ be a polynomial of degree n, i.e. a finite sum of the form $\sum_{\nu=0}^{n} c_\nu z^\nu$ where c_ν are any given numbers and $z=x+iy$ is a complex variable. To answer a question raised by the chemist Mendelieff, A. Markoff [3] proved the following theorem.

THEOREM A. *If* $p(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ *is a polynomial of degree* n, *and* $|p(x)| \leq 1$ *in the interval* $-1 \leq x \leq 1$ *then in the same interval*

$$|p'(x)| \leq n^2 .$$

The result is best possible, but for every subinterval of $[-1, 1]$ the following result of Bernstein provides a much better estimate.

THEOREM B. *Under the conditions of Theorem A*

$$|p'(x)| \leq n(1 - x^2)^{-1/2} \qquad (-1 < x < 1) .$$

For real-valued polynomials having no roots in the interior of the unit circle, Erdős [2] proved the following sharper theorem.

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THEOREM C. Let $p(z)$ be a real-valued polynomial of degree n having no roots in the interior of the unit circle. If $|p(x)| \leqq 1$ in the interval $-1 \leqq x \leqq 1$ and $0 < c < 1$, then for $-1 + c < x < 1 - c$

$$|p'(x)| < \frac{4}{c^2} \sqrt{n}$$

for $n > n_0$.

He also showed that in theorem C, $\sqrt{n}$ can not be replaced by any function tending to infinity more slowly.

We give an improvement on theorem C, and prove the following.

THEOREM. Let $f(z)$ be a rational function which is the quotient of two polynomials of degrees m and n respectively. If $f(z)$ has neither zeros nor poles inside the unit circle and $|f(x)| \leqq 1$ for $-1 \leqq x \leqq 1$, then for $0 < c < 1$ and $-1 + c \leqq x \leqq 1 - c$

$$(1) \qquad |f'(x)| < \frac{[2(m+n)]^{1/2}}{c}$$

and

$$(2) \qquad |f''(x)| < \frac{[2(m+n)]^{5/4}}{c^2},$$

for $m > m_0$ and $n > n_0$.

Proof of the Theorem. Let $f(x)$ be positive for $-1 < x < 1$. Suppose that for a certain x_0 in $(-1+c, 1-c)$, $|f'(x_0)| > \frac{[2(m+n)]^{1/2}}{c}$. Take $|x - x_0| < \frac{\sqrt{2}\, c}{\sqrt{m+n}}$; denote the roots of $p(z)$ and $q(z)$ by $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_n$ respectively. By hypothesis $|\alpha_\nu| \geqq 1$ for $1 \leqq \nu \leqq m$, and

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$|\beta_\nu| \geqq 1$ for $1 \leqq \nu \leqq n$. (We give a proof for $f'(x_0) > 0$; the proof for $f'(x_0) < 0$ is similar.) Hence

$$(3) \qquad \frac{f'(x_0)}{f(x_0)} = \sum_{i=1}^{m} \frac{1}{x_0 - \alpha_i} - \sum_{i=1}^{n} \frac{1}{x_0 - \beta_i} \geqq \frac{[2(m+n)]^{1/2}}{c} .$$

Divide the interval $(x_0, x_0 + \frac{\sqrt{2}\, c}{\sqrt{m+n}})$ into k equal parts. Then for $x_0 + \frac{\sqrt{2}\, c}{\sqrt{m+n}} \cdot \frac{\lambda - 1}{k} < x < x_0 + \frac{\sqrt{2}\, c}{\sqrt{m+n}} \cdot \frac{\lambda}{k}$, where $1 \leqq \lambda \leqq k$, we have

$$(4) \qquad \left| \frac{1}{x - \alpha_i} - \frac{1}{x_0 - \alpha_i} \right| = \frac{|x - x_0|}{|x - \alpha_i|\,|x_0 - \alpha_i|} < \frac{\sqrt{2}}{c\sqrt{m+n}} \cdot \frac{\lambda}{k} ,$$

$$(5) \qquad \left| \frac{1}{x_0 - \beta_i} - \frac{1}{x - \beta_i} \right| = \frac{|x - x_0|}{|x - \beta_i|\,|x_0 - \beta_i|} < \frac{\sqrt{2}}{c\sqrt{m+n}} \cdot \frac{\lambda}{k}$$

so that

$$(6) \qquad \left| \sum_{i=1}^{m} \left(\frac{1}{x - \alpha_i} - \frac{1}{x_0 - \alpha_i} \right) + \sum_{i=1}^{n} \left(\frac{1}{x_0 - \beta_i} - \frac{1}{x - \beta_i} \right) \right| < \frac{[2(m+n)]^{1/2}}{c} \cdot \frac{\lambda}{k}$$

and

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{m} \frac{1}{x - \alpha_i} - \sum_{i=1}^{n} \frac{1}{x - \beta_i}$$

$$= \frac{f'(x_0)}{f(x_0)} + \sum_{i=1}^{m} \left(\frac{1}{x - \alpha_i} - \frac{1}{x_0 - \alpha_i} \right) + \sum_{i=1}^{n} \left(\frac{1}{x_0 - \beta_i} - \frac{1}{x - \beta_i} \right)$$

$$> \frac{f'(x_0)}{f(x_0)} - \frac{[2(m+n)]^{1/2}}{c} \cdot \frac{\lambda}{k} > \frac{[2(m+n)]^{1/2}}{c} \left(1 - \frac{\lambda}{k}\right) .$$

Hence $f'(x) > f(x) \dfrac{[2(m+n)]^{1/2}}{c} \left(1 - \dfrac{\lambda}{k}\right)$.

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Therefore $f(x)$ increases in this interval and $f(x) > f(x_0)$.

Since $f'(x_0) > \frac{[2(m+n)]^{1/2}}{c}$ by hypothesis, we have for

$$x_0 + \frac{\sqrt{2}\,c}{\sqrt{m+n}} \cdot \frac{\lambda - 1}{k} < x < x_0 + \frac{\sqrt{2}\,c}{\sqrt{m+n}} \cdot \frac{\lambda}{k},$$

$$(7) \qquad f'(x) > f(x)\frac{f'(x_0)}{f(x_0)} - \frac{[2(m+n)]^{1/2}}{c} \cdot \frac{\lambda}{k}$$

$$> f'(x_0) - \frac{[2(m+n)]^{1/2}}{c} \cdot \frac{\lambda}{k}$$

$$> \frac{[2(m+n)]^{1/2}}{c} \left(1 - \frac{\lambda}{k}\right).$$

But then,

$$1 > f\left(x_0 + \frac{\sqrt{2}\,c}{\sqrt{m+n}}\right) - f(x_0)$$

$$= \int_{x_0}^{y_0} f'(x)\,dx \qquad \left(y_0 = x_0 + \frac{\sqrt{2}\,c}{\sqrt{m+n}}\right)$$

$$= \sum_{\lambda=1}^{k} \int_{y_1}^{y_2} f'(x)\,dx \quad \left(y_1 = x_0 + \frac{\sqrt{2}\,c}{\sqrt{m+n}} \cdot \frac{\lambda-1}{k}; \quad y_2 = x_0 + \frac{\sqrt{2}\,c}{\sqrt{m+n}} \cdot \frac{\lambda}{k}\right)$$

$$> \sum_{\lambda=1}^{k} \frac{[2(m+n)]^{1/2}}{c} \left(1 - \frac{\lambda}{k}\right) \frac{\sqrt{2}\,c}{\sqrt{m+n}} \cdot \frac{1}{k}$$

$$= \sum_{\lambda=1}^{k} 2\left(1 - \frac{\lambda}{k}\right)\frac{1}{k} \to 1 \quad \text{as} \quad k \to \infty .$$

This contradiction proves (1).

If $f(z) = \frac{p(z)}{q(z)}$ is typically real and all the conditions of our theorem are satisfied, then for $\varepsilon > 0$ the function

$$F(z) = 1 + \varepsilon - f(-z) = \frac{P(z)}{Q(z)}$$

is the quotient of two polynomials $P(z)$ and $Q(z)$ of degrees $\max(m, n) = m$ (say) and n respectively, $P(z)$ and $Q(z)$ do not have zeroes in the unit circle, and $|f(x)| \leq 1 + \varepsilon$ for $-1 \leq x \leq 1$. If $f(x) > 0$ for $-1 < x < 1$, then $F(x) > 0$ for $-1 < x < 1$. Besides, $F'(x) = f'(-x)$. Hence if $f'(x_0) \geqq \frac{(1+\varepsilon)\sqrt{m+n}}{c}$, then $F'(-x_0) \geqq \frac{\sqrt{m+n}}{c}$. As for (7), it follows that for $-x_0 + \frac{c}{\sqrt{m+n}} \cdot \frac{\lambda-1}{k} < x < -x_0 + \frac{c}{\sqrt{m+n}} \cdot \frac{\lambda}{k}$,

$$|F'(x)| > \frac{(1+\varepsilon)\sqrt{m+n}}{c}\left(1 - \frac{\lambda}{k}\right),$$

i.e.,
$$|f'(x)| > \frac{(1+\varepsilon)\sqrt{m+n}}{c}\left(1 - \frac{\lambda}{k}\right)$$

for $x_0 - \frac{c}{\sqrt{m+n}} \cdot \frac{\lambda}{k} < x < x_0 - \frac{c}{\sqrt{m+n}} \cdot \frac{\lambda-1}{k}$.

Also (see the proof (7)) $|f'(x)| > \frac{(1+\varepsilon)\sqrt{m+n}}{c}\left(1 - \frac{\lambda}{k}\right)$ for

$x_0 + \frac{c}{\sqrt{m+n}} \cdot \frac{\lambda-1}{k} < x < x_0 + \frac{c}{\sqrt{m+n}} \cdot \frac{\lambda}{k}$ and then

$$1 > \int_{y_3}^{y_4} f'(x)dx \qquad \left(y_3 = x_0 - \frac{c}{\sqrt{m+n}}\ ;\ y_4 = x_0 + \frac{c}{\sqrt{m+n}}\right)$$

$$> 2 \sum_{\lambda=1}^{k} \frac{(1+\varepsilon)\sqrt{m+n}}{c}\left(1 - \frac{\lambda}{k}\right)\frac{c}{\sqrt{m+n}} \cdot \frac{1}{k} \rightarrow 1 + \varepsilon \text{ as } k \rightarrow \infty .$$

Thus we get a contradiction.

Since ε is arbitrary we can state the following corollary.

<u>If</u> $f(z)$ <u>is typically real and satisfies the conditions of the</u>

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theorem, then for $-1 + c < x < 1 - c$

$$|f'(x)| < \frac{\sqrt{m+n}}{c},$$

provided $m \geqq n$.

Suppose now that for a certain x_0 in $(-1+c, 1-c)$, $f''(x_0) \geq \frac{[2(m+n)]^{5/4}}{c^2}$. Take $|x - x_0| < 2^{1/4} c[(m+n)]^{-3/4}$, and let $f(x_0) > 0$, $f'(x_0) > 0$. We give a proof for $f''(x_0) > 0$; the proof for $f''(x_0) < 0$ is similar. Now

$$(8) \quad \frac{f''(x_0)}{f(x_0)} = \left\{\frac{f'(x_0)}{f(x_0)}\right\}^2 - \left[\frac{d}{dx}\left\{\frac{f'(x)}{f(x)}\right\}\right]_{x = x_0}$$

$$= \left\{\frac{f'(x_0)}{f(x_0)}\right\}^2 - \sum_{i=1}^{m} \frac{1}{(x_0 - \alpha_i)^2} + \sum_{i=1}^{n} \frac{1}{(x_0 - \alpha_i)^2}$$

$$= \sum_{\mu \neq \nu}\sum \frac{1}{(x_0 - \alpha_\mu)(x_0 - \alpha_\nu)} + \sum_{\mu \neq \nu}\sum \frac{1}{(x_0 - \beta_\mu)(x_0 - \beta_\nu)}$$

$$+ 2\sum \frac{1}{(x_0 - \beta_i)^2} - 2\sum \frac{1}{x_0 - \alpha_i} \sum \frac{1}{x_0 - \beta_i}$$

$$\geqq \frac{[2(m+n)]^{5/4}}{c^2}$$

Divide the interval $(x_0, x_0 + 2^{1/4} c(m+n)^{-3/4})$ into k equal parts. Then for $x_0 + 2^{1/4} c(m+n)^{-3/4} \frac{\lambda - 1}{k} < x < x_0 + 2^{1/4} c(m+n)^{-3/4} \frac{\lambda}{k}$, where $1 \leqq \lambda \leqq k$ and $\mu \neq \nu$, we have

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(9) $$\left|\frac{1}{(x-\alpha_\mu)(x-\alpha_\nu)} - \frac{1}{(x_0-\alpha_\mu)(x_0-\alpha_\nu)}\right|$$

$$= \left|\frac{1}{(x-\alpha_\mu)}\left[\frac{1}{x-\alpha_\nu} - \frac{1}{x_0-\alpha_\nu}\right] - \frac{1}{(x_0-\alpha_\nu)}\left[\frac{1}{x-\alpha_\mu} - \frac{1}{x_0-\alpha_\mu}\right]\right|$$

$$< 2^{5/4}(m+n)^{3/4} c^2 \cdot \frac{\lambda}{k} ,$$

(10) $$\left|\frac{1}{(x-\beta_\mu)(x-\beta_\nu)} - \frac{1}{(x_0-\beta_\mu)(x_0-\beta_\nu)}\right| < 2^{5/4} (m+n)^{3/4} c^2 \cdot \frac{\lambda}{k} ,$$

(11) $$\left|\sum_{i=1}^{m} \frac{1}{x-\alpha_i} \sum_{i=1}^{n} \frac{1}{x-\beta_i} - \sum_{i=1}^{m} \frac{1}{x_0-\alpha_i} \sum_{i=1}^{n} \frac{1}{x_0-\beta_i}\right|$$

$$= \left|\sum_{i=1}^{m} \frac{1}{x-\alpha_i} \sum_{i=1}^{n} \left(\frac{1}{x-\beta_i} - \frac{1}{x_0-\beta_i}\right)\right.$$

$$\left. + \sum_{i=1}^{n} \frac{1}{x_0-\beta_i} \sum_{i=1}^{m} \left(\frac{1}{x-\alpha_i} - \frac{1}{x_0-\alpha_i}\right)\right|$$

$$< 2^{1/4}(m.n + n.m) (m+n)^{-3/4} c^{-2} \cdot \frac{\lambda}{k} ,$$

(12) $$\left|\sum_{i=1}^{n} \left\{\frac{1}{(x-\beta_i)^2} - \frac{1}{(x_0-\beta_i)^2}\right\}\right|$$

$$= \left|\sum_{i=1}^{n} \left[\frac{1}{x-\beta_i} - \frac{1}{x_0-\beta_i}\right] \cdot \left[\frac{1}{x_0-\beta_i} + \frac{1}{x-\beta_i}\right]\right|$$

$$< 2^{5/4} n (m+n)^{-3/4} c^{-2} \cdot \frac{\lambda}{k} ;$$

From (9), (10), (11) and (12) we deduce that

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$$\left|\frac{f''(x)}{f(x)} - \frac{f''(x_0)}{f(x_0)}\right| = \left| \sum_{\mu\neq\nu}\sum \left(\frac{1}{(x-\alpha_\mu)(x-\alpha_\nu)} - \frac{1}{(x_0-\alpha_\mu)(x_0-\alpha_\nu)}\right)\right.$$

$$+ \sum_{\mu\neq\nu}\sum \left(\frac{1}{(x-\beta_\mu)(x-\beta_\nu)} - \frac{1}{(x_0-\beta_\mu)(x_0-\beta_\nu)}\right)$$

$$+ 2\left\{\frac{1}{(x-\beta_i)^2} - \frac{1}{(x_0-\beta_i)^2}\right\}$$

$$\left. - 2\sum \frac{1}{x-\beta_i}\sum\frac{1}{x-\alpha_i} + \sum\frac{1}{x_0-\alpha_i}\sum\frac{1}{x_0-\beta_i}\right|$$

$$< \frac{[2(m+n)]^{5/4}}{c^2} \cdot \frac{\lambda}{k} .$$

It follows that

$$\frac{f''(x)}{f(x)} = \frac{f''(x_0)}{f(x_0)} + \left[\frac{f''(x)}{f(x)} - \frac{f''(x_0)}{f(x_0)}\right] > \frac{f''(x_0)}{f(x_0)} - \left|\frac{f''(x)}{f(x)} - \frac{f''(x_0)}{f(x_0)}\right|$$

$$> \frac{f''(x_0)}{f(x_0)} - \frac{[2(m+n)]^{5/4}}{c^2}\frac{\lambda}{k} .$$

Hence

$$f''(x) > f(x)\frac{f''(x_0)}{f(x_0)} - \frac{2(m+n)^{5/4}}{c^2} \cdot \frac{\lambda}{k} .$$

For

$$x_0 + 2^{1/4}c\,(m+n)^{-3/4}\,\frac{\lambda-1}{k} < x < x_0 + 2^{1/4}c\,(m+n)^{-3/4}\,\frac{\lambda}{k} ,$$

$f(x)$ increases and $f(x) > f(x_0)$. If $f''(x) > 0$, this is obvious. If $f''(x) < 0$ we can prove that $f'(x)$ can never be negative in this interval. To prove this suppose $f'(x) = 0$ for a certain x in this interval. Then

$$\left|\frac{f'(x_0)}{f(x_0)}\right| = \left|\frac{f'(x_0)}{f(x_0)} - \frac{f'(x)}{f(x)}\right|$$

$$= \left|\sum_{i=1}^{m}\left(\frac{1}{x_0-\alpha_i} - \frac{1}{x-\alpha_i}\right) + \sum_{i=1}^{n}\left(\frac{1}{x-\beta_i} - \frac{1}{x_0-\beta_i}\right)\right|$$

$$< \frac{[2(m+n)]^{1/4}}{c} ,$$

so that

$$(13) \qquad |f'(x_0)| < \frac{[2(m+n)]^{1/4}}{c} .$$

On differentiating

$$f'(x) = f(x)\left[\sum_{i=1}^{m} \frac{1}{x-\alpha_i} - \sum_{i=1}^{n} \frac{1}{x-\beta_i}\right]$$

with respect to x and using (13), we have

$$|f''(x_0)| < \frac{[2(m+n)]^{5/4}}{c^2} .$$

This contradicts the hypothesis that $|f''(x_0)| \geqq \dfrac{[2(m+n)]^{5/4}}{c^2}$.

Therefore we have

$$f''(x) > f''(x_0) - \frac{[2(m+n)]^{5/4}}{c^2} > \frac{[2(m+n)]^{5/4}}{c^2} \left(1 - \frac{\lambda}{k}\right) .$$

But then,

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$$\frac{[2(m+n)]^{1/2}}{c} > f'(x_0 + 2^{1/4}c(m+n)^{-3/4}) - f'(x_0)$$

$$= \int_{y_5}^{y_6} f''(x)\,dx \qquad (y_5 = x_0 \; ; \; y_6 = x_0 + 2^{1/4}c(m+n)^{-3/4})$$

$$= \sum_{\lambda=1}^{k} \int_{y_7}^{y_8} f''(x)\,dx \qquad (y_7 = x_0 + 2^{1/4}c(m+n)^{-3/4}\frac{\lambda-1}{k} ;$$

$$y_8 = x_0 + 2^{1/4}c(m+n)^{-3/4}\frac{\lambda}{k})$$

$$> \sum_{\lambda=1}^{k} \frac{[2(m+n)]^{5/4}}{c^2}(1-\frac{\lambda}{k})\, 2^{1/4}c\,(m+n)^{-3/4} \cdot \frac{1}{k}$$

$$= \sum_{\lambda=1}^{k} \frac{2[2(m+n)]^{1/2}}{c}(1-\frac{\lambda}{k}) \cdot \frac{1}{k}$$

$$\to \frac{[2(m+n)]^{1/2}}{c} \quad \text{as } k \to \infty .$$

This contradiction proves (2).

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REFERENCES

1. S. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné, Mémoires de l'Académie Royale de Belgique, (2), 4(1912), 1-103.

2. P. Erdős, On extremal properties of the derivative of the polynomials, Annals of Math. 41 (1940), 310-313.

3. A. Markoff, On a certain problem of D. I. Mendelieff, Utcheniya Zapiski Imperatorskoi Akademie Nauk. (Russia) 62 (1889), 1-24.

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